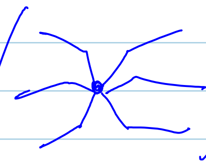
November 1, hours 22-23: Ideals, isomorphism theorems, prime and maximal ideals

October-28-11
3:10 PM

Agenda. Quotients, isomorphism thms, "better rings".

Read Along. Selick 2.1–2.3.

HW3 on web.

$6\Big($  $\Big) = ?$

Def. $I \subset R$ is an ideal...

Claim. If $\phi : R \to S$ is a morphism of rings, then $\ker(\phi)$ is an ideal in R.

Q. Is every ideal a quotient?

Ans. Define R/I.

Example. $\mathbb{R}[x]/\langle x^2+1\rangle = R_1$

The Isomorphism theorems. 1. $f : R \to S \Rightarrow R/\ker(f) = \operatorname{im} f$.

(Example: $ev_i : \mathbb{R}[x] \to \mathbb{C} \Rightarrow R_1 \cong \mathbb{C}$)

2. $\dfrac{A+I}{I} \cong A/A\cap I$ $\quad$ $A \subset R$ subring, $I \subset R$ ideal.

3. $I \subset J \subset R$ ideals $\Rightarrow \dfrac{R/I}{J/I} \cong R/J$

4. Given an ideal I of R, there's a bijection between ideals $I \subset J \subset R$ & ideals of R/I.

Better Rings. 1. The ultimate:

Field [commutative, $F^\times$ of a group]

[almost all of high-school & freshman algebra carries through]

("division ring", if not commutative

Example: $\mathbb{H} = \{a+bi+cj+dk\} \Big/ \begin{array}{l} i^2=j^2=k^2=-1 \\ ij=k \end{array}$

useful for 3D rotations, etc...)

2. (Integral) domains: commutative, has no 0-divisors.

How make? For ideals which, R/I is a field or a domain?

.... From now on, R is commutative.

Maximal Ideals. 1. Definition.

2. $I \subset R$ is maximal $\iff$ R/I is a field.

Fishy proof: Use the 4th isomorphism theorem.

Honest proof: $\Rightarrow$: $x \notin I \Rightarrow Rx + I = R \Rightarrow \exists y \in R \;\; yx + I = 1 + I$

$\Leftarrow$ $J \supsetneq I$, $x \in J \setminus I \Rightarrow [x]_I \neq 0 \Rightarrow \exists y \;\; xy - 1 \in I \Rightarrow 1 \in J$

Examples. 1. $p\mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$.

2. $S = \ell^\infty = \left\{ \begin{matrix} \text{bndd seq's} \\ \text{in } \mathbb{R} \end{matrix} \right\}$ $\quad A_n = \{(a_i) : a_n = 0\}$

done line

Fishy

Theorem. Every ideal is contained in a maximal ideal.

Proof using Zorn's Lemma.

Theorem There exists a function

$$\mathrm{Lim} : \left\{ \begin{matrix} \text{bndd seq's} \\ \text{in } \mathbb{R} \end{matrix} \right\} \longrightarrow \mathbb{R} \quad \text{s.t.}$$

1. If (a_n) is convergent, $\lim a_n = \mathrm{Lim}\, a_n$.
2. $\mathrm{Lim}\,(a_n + b_n) = \mathrm{Lim}\,(a) + \mathrm{Lim}\,(b_n)$
3. $\mathrm{Lim}\,(a_n b_n) = \mathrm{Lim}(a_n) \cdot \mathrm{Lim}\,(b_n)$

+ more....

Proof. $S = \{\text{bndd seq's in } \mathbb{R}\}$ $\quad I = \{(a_n) : a_n \neq 0 \text{ for finitely many } n\text{'s}\}$

J - a maximal ideal containing I.

$$\mathrm{Lim} : S \longrightarrow S/J \overset{?}{=} \mathbb{R}$$

Prime Ideals. 1. Definition $P \subset R$ is prime if $ab \in P$
$\Rightarrow a \in P$ or $b \in P$.

2. Theorem. R/P is a domain iff P is prime.

Proof. $\Rightarrow$ $ab \in P \Rightarrow [ab] = 0 \Rightarrow [a][b] = 0 \Rightarrow \begin{array}{l} [a] = 0 \Rightarrow a \in P \\ \text{or} \\ [b] = 0 \Rightarrow b \in P. \end{array}$

$\Leftarrow$ $[a][b] = 0 \Rightarrow [ab] = 0 \Rightarrow ab \in P \Rightarrow \begin{array}{l} a \in P \Rightarrow [a] = 0 \\ \text{or} \\ b \in P \Rightarrow [b] = 0 \end{array}$

Theorem. A maximal ideal is prime.