Lims

November-03-10

9:13 PM

Continues 10-1100/Lims.

$S = \{\text{bndd seq's in } \mathbb{R}\}$ $I = \{(a_n) : a_n \neq 0 \text{ for finitely many } n\text{'s}\}$

$J$ - a maximal ideal containing $I$.

Thm. $\mathrm{Lim}: S \to S/J \underset{!}{=} \mathbb{R}$ extends lim.

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Definition. Say that $A \subset \mathbb{N}$ is "essential" if $1_A \notin J$.

Claim. $\{A : A \text{ is essential}\} = \mu$ is a non-principal ultrafilter on $\mathbb{N}$.

Proof. $J$ is prime $\Rightarrow (A, B \in \mu \Rightarrow A \cap B \in \mu)$

$\mathbb{N} \in \mu$ because $1_S = 1_{\mathbb{N}}$ is not in $J$.

$A \in \mu \Leftrightarrow 1_A \notin J \Leftrightarrow (1_{\mathbb{N}} - 1_A) \in J \Leftrightarrow 1_{A^c} \in J \Leftrightarrow A^c \notin \mu$

Monotonicity because $J$ is an ideal: $A \subset B, B \notin \mu$

$\Rightarrow 1_B \in J \Rightarrow 1_A = 1_B \cdot 1_A \in J \Rightarrow A \notin \mu$.

Principality from the definition of $I$.

Definition. $\hat{J} = \{(a_n) : \forall \epsilon > 0 \ \{n : |a_n| < \epsilon\} \text{ is essential}\}$

Claim. $J \subset \hat{J}$

Proof. Suppose $(a_n) \in J$, and $\epsilon > 0$ is such that $\{n : |a_n| \geq \epsilon\}$ is essential.

Let $b_n = \begin{cases} \frac{1}{a_n} & |a_n| \geq \epsilon, \\ 0 & \text{otherwise.} \end{cases}$

Then $a_n \cdot b_n = 1$ on an essential set,

so $\overline{a_n}\,\overline{b_n} \neq 0$, so $\overline{a_n} \neq 0$ so $a_n \notin J$ $\Rightarrow\!\Leftarrow$.

Now by the maximallity of $J$, $J = \hat{J}$.

Claim. For every $(a_n) \in S$ there is some $\alpha \in \mathbb{R}$ s.t. $a_n - \alpha T \in \hat{J}$

(follows from convergence on ultrafilters)

$\Rightarrow$ $\mathrm{Lim}(a_n) = \mathrm{Lim}(\alpha T)$

Claim. The map $\mathbb{R} \longrightarrow S/J$ via $\alpha \longmapsto \alpha T$ is injective and surjective.

Proof. surjectivity was just shown. Injectivity is because any morphism of fields is injective, as field have no ideals to serve as kernels.

$\Rightarrow$ Using $\alpha \longmapsto \alpha T$ to identify $S/J$ with $\mathbb{R}$, the resulting Lim has all the required properties. $\square$