December 6 Debts

November-30-11
7:50 PM

Abelian groups & the mult. groups of finite fields

$$A \cong \mathbb{Z}^k \oplus \bigoplus \mathbb{Z}/p_i^{s_i} \cong \mathbb{Z}^k \oplus \mathbb{Z}/a_1 \oplus \mathbb{Z}/a_2 \oplus \cdots$$

$$a_1 \mid a_2 \mid a_3 \cdots$$

<u>Theorem</u> If F is finite, F^* is cyclic.

<u>Proof</u> otherwise, $x^{a_1}-1$ has too many roots.

(Aside: λ is a root of $f \in F[x] \iff x-\lambda \mid f$, so f may have at most $\deg(f)$ roots)

Theorem. The universal property for tensor products.

$$\begin{tikzcd} M \times N \arrow[r, "\iota"', "\text{bilinear}"'] \arrow[dr, "\text{bilinear}"'] & M \otimes N \arrow[d, dashed, "\exists!"] \\ & P \end{tikzcd}$$

Cayley-Hamilton. Let R be any commutative ring, let $A \in M_{n\times n}(R)$, let $\chi_A(t) = \det(tI-A) \in R[t]$. Then $\chi_A(A) = 0$.

Proof I. Substitute $t = A$, so

$\chi_A(A) = \det(A \cdot I - A) = \det(0) = 0.$

[$tr(tI-A) = nt - tr\,A$
so $nA - (tr\,A)I = 0$
so all matrices are diagonal?!]

Proof II. Recall that every matrix B has an "adjoint" B^* s.t. $B^*B = BB^* = \det(B)\cdot I$. Then

$$\underbrace{(tI-A)^*}_{= \sum B_k t^k}(tI-A) = \chi_A(t) I$$

as elements of $M_n R[t]$ & even $C_A[t]$, where $C_A = \{B : AB = BA\}$

There is a well-defined & multiplicative $ev_A : C_A[t] \to C_A[t]$. Applying to both sides, get

$$\left(\sum B_k A^k\right) \cdot (A - A) = \chi_A(A) I$$

□