Plan. UFD blunder, JCF abstractly & in practice.

I said "I think in a UFD every prime ideal is maximal"

JCF. V a f.d.v.s, $A: V \to V$ linear, makes V a module over $F[x]$ via $xu = Au$. Then

$V \cong \bigoplus F[x]/(x-\lambda_i)^{s_i}$. What's $F[x]/(x-\lambda_i)^{s_i}$?

UFD Blunder. The above statement is nonsense.

In $\mathbb{Q}[x,y] = \mathbb{Q}[x][y]$, $\langle x \rangle$ is prime but not maximal.

Basis: $1, x-\lambda, (x-\lambda)^2, \ldots, (x-\lambda)^{s-1}$

$A-\lambda$ acts by "shift to the right" $\begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ \vdots & 1 & \ddots \end{pmatrix}$

So A acts by $\begin{pmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{pmatrix}$

Corollary 2. *Over an algebraically closed field $\mathbb{F}$, every square matrix A is conjugate to a block diagonal matrix* $B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$, *where each B_i is either a 1×1 matrix (λ_1) for some $\lambda_i \in \mathbb{F}$, or an $s_i \times s_i$ matrix with λ_i's on the diagonals, 1's right below the diagonal, and 0's elsewhere,*

$$\begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 & 0 \\ 1 & \lambda_i & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \lambda_i & 0 \\ 0 & 0 & \cdots & 0 & 1 & \lambda_i \end{pmatrix},$$

for some $\lambda_i \in \mathbb{F}$ and for some $s_i \geq 2$. Furthermore, B is unique up to a permutation of its blocks B_i.
(Corollary: good old diagonalization.)

Now lets do that in practice....

step 1. Find a presentation matrix for $V \in R\text{-mod}$.

w.l.o.g $V = F^n$ and $A \in M_{n \times n}(F)$.

$\ker \pi = ?$

$r_i := xe_i - Ae_i \in \ker \pi$

claim $\langle r_i \rangle = \ker \pi$

pf Consider

$$R^n \xrightarrow{xI - A} R^n \xrightarrow{\pi} F^n$$
$$e_i \longrightarrow e_i$$
$$x^k e_i \longrightarrow A^k e_i$$

$$\begin{tikzcd} F^n \arrow[r, "\text{onto?}", "\beta\ \text{v.s. map}"'] \arrow[rrr, bend right, "I"'] & R^n/\langle r_i \rangle \arrow[r, "\text{1-1?}", "\alpha\ \text{well-def?}"'] & R^n/\ker \pi \arrow[r, "\sim"] & F^n \end{tikzcd}$$

We want to know if α is 1-1; it is enough to show that β is onto; i.e., that any $x^k e_i$ can be written, modulo $\langle r_i \rangle$, as a combination of e_i's. Indeed,

$$x^k e_i = x^{k-1}(x e_i) = x^{k-1} A e_i = \dots = A^k e_i$$

Go over handout, first in the distinct-eigval's case:

Row and Column Operations

Row operations are performed by left-multiplying N by some properly-positioned 2×2 matrix and at the same time left-multiplying the "tracking matrix" P by the same 2×2 matrix. Column operations are similar, with left replaced by right and P by Q.

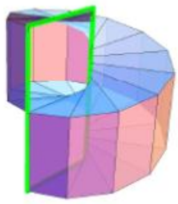
```
RowOp[i_, j_, mat_] := Module[{TT = II},
   TT[[{i, j}, {i, j}]] = mat;
   NN = Simplify[TT.NN]; PP = Simplify[TT.PP];
  ];
ColOp[i_, j_, mat_] := Module[{TT = II},
   TT[[{i, j}, {i, j}]] = mat;
   NN = Simplify[NN.TT]; QQ = Simplify[QQ.TT];
  ];
```

Swapping Rows and Columns

`SwapRows[i_, j_] := RowOp`$\left[i, j, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right]$;

`SwapColumns[i_, j_] := ColOp`$\left[i, j, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right]$;

`SwapBoth[i_, j_] := (SwapRows[i, j]; SwapColumns[i, j];)`



?

The "GCD" Trick

If $q = \gcd(a, b) = s a + t b$, the equality $\begin{pmatrix} s & t \\ -b/q & a/q \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix}$ allows us to replace pairs of entries in the same column by their greatest common divisoir (and a zero!), using invertible row operations. A similar trick works for rows.

`GCDTrick[{i_, j_}, k_] := Module[{a, b, q, s, t},`
`   {q, {s, t}} = PolynomialExtendedGCD[a = NN[[i, k]], b = NN[[j, k]], x];`
`   RowOp`$\left[i, j, \begin{pmatrix} s & t \\ -b/q & a/q \end{pmatrix}\right]$
`  ];`

`GCDTrick[k_, {i_, j_}] := Module[{a, b, q, s, t},`
`   {q, {s, t}} = PolynomialExtendedGCD[a = NN[[k, i]], b = NN[[k, j]], x];`
`   ColOp`$\left[i, j, \begin{pmatrix} s & -b/q \\ t & a/q \end{pmatrix}\right]$
`  ];`

Factoring Diagonal Entries

If $1 = \gcd(a, b) = s a + t b$, the equality $\begin{pmatrix} s a & 1 \\ -t b & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & a b \end{pmatrix}\begin{pmatrix} a & -b \\ t & s \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is an invertible row-column-operations proof of the isomorphism $\frac{R}{\langle a \rangle} \oplus \frac{R}{\langle b \rangle} \simeq \frac{R}{\langle ab \rangle}$.

`SplitToSum[i_, j_, a_, b_] := Module[`
`   {q, s, t, T1, T2},`
`   {q, {s, t}} = PolynomialExtendedGCD[a, b, x];`
`   If[q == 1,`
`    RowOp`$\left[i, j, \begin{pmatrix} s a & 1 \\ -t b & 1 \end{pmatrix}\right]$; `ColOp`$\left[i, j, \begin{pmatrix} a & -b \\ t & s \end{pmatrix}\right]$;
`   ]`
`  ];`

Recovering C from P?

$$\begin{tikzcd} R^n \arrow[r, "Ix-A"', "M"] & R^n \arrow[r, "\pi_A"] \arrow[d, "P"] & F^n \arrow[d, "C"] \\ R^n \arrow[u, "Q"] \arrow[r, "Ix-B"] & R^n \arrow[r, "\pi_B"] & F^n \end{tikzcd}$$

$$C e_i = \pi_B(P e_i)$$
$$= \pi_B\left(\sum x^k P_k e_i\right)$$
$$= \sum x^k \pi_B(P_k e_i)$$
$$= \sum B^k P_k e_i$$

$\Longrightarrow C = \sum B^k P_k$ ... complete run 1

The "Jordan Trick":

A repeated application of the identity $\begin{pmatrix} p^{k-1} & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & p^k \end{pmatrix} \cdot \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{-1+k} & 0 \\ 1 & p \end{pmatrix}$ will bring a matrix like $\begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&p^4 \end{pmatrix}$ to the "Jordan" form of $\begin{pmatrix} p&0&0&0 \\ 1&p&0&0 \\ 0&1&p&0 \\ 0&0&1&p \end{pmatrix}$, using invertible row and column operations.

`JordanTrick[i_, j_, p_, s_] := (RowOp`$\left[i, j, \begin{pmatrix} p^{s-1} & -1 \\ 1 & 0 \end{pmatrix}\right]$; `ColOp`$\left[i, j, \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\right]$`);`

Then go through run 2 & run 3 .....

done line

(debt) Theorem. The universal property for tensor products

1. Holds 2. Determines $M\otimes N$ up to a unique isomorphism.

$$\begin{tikzcd} M\times N \arrow[r, "\tau", "\text{bilinear}"'] \arrow[dr, "\text{bilinear}"'] & M\otimes N \arrow[d, dashed, "\exists!"] \\ & P \end{tikzcd}$$