Do not turn this page until instructed.

Math 1100 Core Algebra I

Term Test

University of Toronto, October 25, 2011

Solve the 4 problems on the other side of this page.
Each problem is worth 25 points.
You have an hour and fifty minutes to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Advance apology: It may take us a while to grade this exam; sorry.

Good Luck!

Solve the following 4 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1. Let G be a finite group, let p be a prime number, and let α be the largest natural number such that $p^\alpha \mid |G|$.

1. Prove that there is a subgroup P of G whose order is p^α. (You are not allowed to use the Sylow theorems, of course).
2. Suppose that $x \in G$ is an element whose order is a power of p, and suppose that x normalizes P. Show that $x \in P$.

Problem 2. A group G is said to be "torsion free" if every non-trivial element thereof has infinite order.

1. Prove that a semi-direct of two torsion free groups is again torsion free.
2. Let β be a pure braid on n strands. Prove that if $\beta^7 = e$ then $\beta = e$.

Problem 3. Let H_1 and H_2 be subgroups of some group G. Prove that the left G-sets G/H_1 and G/H_2 are isomorphic (as left G-sets) iff the subgroups H_1 and H_2 are conjugate.

Problem 4.

1. Let G be a subgroup of S_n that contains both the transposition (12) and the n-cycle $(123\dots n)$. Prove that $G = S_n$. (Hint: Conjugate your way up, do not use non commutative Gaussian elimination).
2. Let n be odd and let G be a subgroup of S_n that contains both the 3-cycle (123) and the n-cycle $(123\dots n)$. Prove that $G = A_n$. (Hint: For the lower bound, conjugate your way up, do not use non commutative Gaussian elimination).
3. In the previous part, what if n is even?

Good Luck!