Do not turn this page until instructed.

Math 1100 Core Algebra I

Final Examination

University of Toronto, December 9, 2011

Solve the 5 of the 6 problems on the other side of this page.
Each problem is worth 20 points.
You have three hours to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Good Luck!

Solve 5 of the following 6 problems. Each problem is worth 20 points. You have three hours. **Neatness counts! Language counts!**

Problem 1. Let G be a group, K a subgroup of G, and H a subgroup of the normalizer $N_G(K)$ of K in G. Prove that KH is a subgroup of G, that K is normal in KH, that $H \cap K$ is normal in H, and that KH/K is isomorphic to $H/(H \cap K)$.

Problem 2. Let S_n denote the symmetric group on n elements.

1. How many Sylow-2 subgroups does S_4 have? Describe one of those explicitly.
2. Let p be a prime. How many Sylow-p subgroups does S_p have? Describe one of those explicitly.

Problem 3. If S is a set, denote by $F(S)$ the $\mathbb{Z}$-module of integer-valued functions on S, with pointwise operations. Let X and Y be two sets.

1. Construct an injection $\iota : F(X) \otimes_{\mathbb{Z}} F(Y) \to F(X \times Y)$.
2. Show that if X and Y are finite then ι is an isomorphism.
3. By means of an example, show that ι need not be an isomorphism if X and Y are both infinite.

Problem 4. Define a "Principal Ideal Domain (PID)" and a "Unique Factorization Domain (UFD)" and show that every PID is a UFD. If you need to use the lemma that an increasing chain of ideals in a PID must become constant at some point (i.e., that a PID is "Noetherian"), prove it.

Problem 5. Prove the following simplified version of the structure theorem for finitely generated modules over a PID:

Let R be a PID and let M be the R-module $R^n/\langle r_1, \ldots, r_m \rangle$, where n and m are natural numbers and $r_1, \ldots, r_m \in R^n$. Then there exists a natural number k and elements $a_1, \ldots, a_l$ of R so that $M \cong R^k \oplus \bigoplus_{i=1}^l R/\langle a_i \rangle$.

Problem 6. Let p and q be primes in a PID R such that $p \not\sim q$, let $\hat{p}$ denote the operation of "multiplication by p", acting on any R-module M, and let s and t be positive integers. For each of the R-modules R, $R/\langle q^t \rangle$, and $R/\langle p^t \rangle$, determine $\ker \hat{p}^s$ and $(R/\langle p \rangle) \otimes \ker \hat{p}^s$.

Good Luck!