November 11, hours 24-25: Tychonoff, Stone-Cech

November-10-10 12:37 PM

Note. I'll vanish at 4PM.

Goal. Tychonoff & a taste of Stone-Cech. [hardest class of this course?]

HW. HW5 due today, HW6 on web by midnight.

Read Along. Munkres 37, 38

Riddle Along. $\mathcal{C}\left(\underset{-1}{\times}\!\!-\!\!\overset{z_1}{\bullet}\!\!-\!\!\underset{z_2}{\bullet}\!\!-\!\!\underset{1}{\times}\right) = \left\{(z_1,z_2)\in\mathbb{C}^2 : \begin{array}{l} d(-1,z_1)=d(z_1,z_2) \\ =d(z_2,1)=1 \end{array}\right\} = ?$

Reminder. X is compact iff every family of closed sets having the FIP has a common point. I.e., $\{F_\alpha\}_{\alpha\in I}$ closed, $\forall\{\alpha_1,\dots\alpha_n\}\subset I \quad \bigcap_{i=1}^n F_{\alpha_i}\neq\emptyset \Rightarrow \bigcap_{\alpha\in I} F_\alpha\neq\emptyset$.

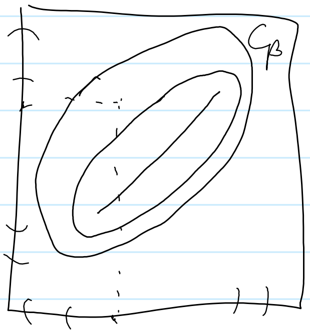
Thm. (Tychonoff) If X_α is compact for every α, then so is $\prod X_\alpha$.

Example $\{0,1\}^\omega$ is compact

EX. $\{0,1\}^\omega \simeq C$, the Cantor set.

Aside on Zorn's Lemma In a partially-ordered set in which every chain has a bound, there is a maximal element.

Proof. Given $\mathcal{C}$, choose a maximal collection $\mathcal{A}$ having the FIP and containing $\mathcal{C}$, of not-necessarily-closed sets (using Zorn).



Claim $\mathcal{A}$ is closed under finite intersections.

Claim If $B\cap A\neq 0$ for all $A\in\mathcal{A}$, then $B\in\mathcal{A}$.

Now $\forall\alpha$, $\{\overline{\pi_\alpha(A)} : A\in\mathcal{A}\}$ has the FIP,

so choose $x_\alpha\in\bigcap_{A\in\mathcal{A}}\overline{\pi_\alpha(A)}$. Want: $(x_\alpha)\in\bigcap_{C\in\mathcal{C}} C$

Claim If U is a nbd of x_α in X_α, then $\pi^{-1}(U)\in\mathcal{A}$.

Pf $\forall A\in\mathcal{A}$, $U\cap\overline{\pi_\alpha(A)}\neq\emptyset$, so $U\cap\pi_\alpha(A)\neq\emptyset$,
so $\pi^{-1}(U)\cap A\neq\emptyset$.

claim Every basic nbd of (x_α) is in $\mathcal{A}$.

claim $(x_\alpha)\in\bigcap_{C\in\mathcal{C}} C$ $\square$

Let $A=\left\{\begin{matrix}\text{bounded}\\ \text{sequences}\end{matrix}\right\}=\left\{a=(a_j)_{j=0}^{\infty} : \begin{matrix}\exists M \text{ s.t.}\\ \forall j\ |a_j|<M\end{matrix}\right\}\subset\mathbb{R}^{\mathbb{N}}$

For each $(a_j)\in A$, let $R_{(a_j)}$ be $[\inf(a_j),\sup(a_j)]\subset\mathbb{R}$

then $C=\prod_{\tilde a\in A}R_a$. C is compact! (and huge)

Let $\phi:\mathbb{N}\to C$ be given by $\phi(k)_a=a_k$; (it's 1-1)

let $\beta\mathbb{N}:=\overline{\phi(\mathbb{N})}\subset C$ It is compact, and has $\mathbb{N}$ as a dense subset.

Claim Every cont. bndd function b has a unique cont. extension $\tilde b$ to $\beta\mathbb{N}$.

proof for $b:\mathbb{N}\to\mathbb{R}$, set $\tilde b=\pi_b|_{\beta\mathbb{N}}$.

done line

Claim If $\mu\in\beta\mathbb{N}-\mathbb{N}$ and $b=(b_j)$ is a seq. whose limit $\lim_{j\to\infty}b_j$ exists, then $\tilde b(\mu)=\lim_{j\to\infty}b_j$.

Pf Enough to know that $\mu\in\overline{[n,\infty)}$, for any n.

In a T_2 space, $\mu\in\overline{D}-D$ iff every nbd of μ contains infinitely many elements of D.

Now fix μ and define $Lim(b_i) = \bar{b}(\mu)$. Then Lim is a "generalized limit".

This contradicts the principle of diminishing values.