September 23, hour 6: Isomorphism Theorems 2-4

September-21-10
12:42 PM

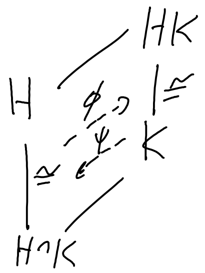
Read Along: Selick's notes 1.1, 1.2.1, 1.4. Lang's book I 1-3.

Quick review of quotients & the first isomorphism theorem, a word about cosets and Lagrange's theorem, then:

Second Isomorphism Theorem. $H, K < G$, $H < N_G(K)$

$\Rightarrow H \cap K \triangleleft H$, $K \triangleleft HK$ and $HK/K = H/H\cap K$.

In a diagram:



To do: 1. Define "Normalizer"
2. $H\cap K$ is a subgroup.
3. $H\cap K \triangleleft H$
4. HK is a subgroup
5. $K \triangleleft HK$.
6. The isomorphism.

Now the isomorphism:

$\phi : h(H\cap K) \longmapsto hK$ clearly well defined.

$\psi : hkK \longmapsto h(H\cap K)$

$h_1k_1K = h_2k_2K$ iff $h_1K = h_2K$ iff $h_1h_2^{-1} \in K \cap H$.

Third Isomorphism Theorem. $N \triangleleft G$, $H \triangleleft G$, $N < H$

$\Rightarrow H/N \triangleleft G/N$ and $(G/N)/(H/N) \cong G/H$.

Proof. $\phi : G/N/H/N \to G/H$ by $[[g]_N]_{H/N} \longrightarrow [g]_H$

$\psi : G/H \to G/N/H/N$ by $[g]_H \longrightarrow [[g]_N]_{H/N}$

Fourth Isomorphism Theorem. If $N \triangleleft G$ there's a bijection between subgroups of G/N and subgroups of G that contain N. This bijection preserves inclusions, indices, intersections, and normality of inclusions.

Done line

The Butterfly. $1 < a \triangleleft A < G$, $1 < b \triangleleft B < G$

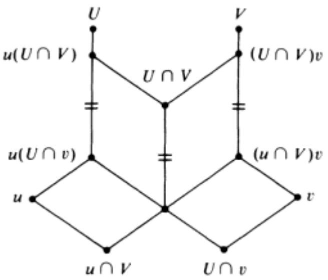
$\Rightarrow a(A\cap b) \triangleleft a(A\cap B)$. (Following Lang)

$\Rightarrow \quad a(A\cap b) \lhd a(A\cap B),$ (Following Lang)

$(a\cap B)b \lhd (A\cap B)b,$

and $\quad a(A\cap B)/a(A\cap b) \cong (A\cap B)b/(a\cap B)b$

"The B quotient in the A scale" $\cong$ "The A quotient in the B scale" (A catchy phrasing is a memory aid, not a proof & not even an intuition)



(likewise for pictures of insects)

(Picture from Lang's book)

Jordan-Hölder now follows....

Proof of Butterfly. Normality is obvious.

$(A\cap B)\cdot a(A\cap b) = a(A\cap b)(A\cap B) = a(A\cap B)$, so

$a(A\cap B)/a(A\cap b) = \overset{H}{\boxed{(A\cap B)}}\overset{K}{\boxed{a(A\cap b)}}/\overset{K}{\boxed{a(A\cap b)}} \cong$

$\cong \underset{H}{\boxed{A\cap B}}/\underset{H}{\boxed{(A\cap B)}}\cap\underset{K}{\boxed{a(A\cap b)}} = \underbrace{(A\cap B)/(a\cap B)b\cap a(A\cap b)}_{\text{symmetric.}}$

as $(A\cap B)\cap a(A\cap b) = (a\cap B)b\cap a(A\cap b)$

$\quad \| \qquad \supseteq$ easy $\quad \subseteq \downarrow$

$\quad B\cap a(A\cap b)$

Given $\alpha\beta$, $\alpha\in a$, $\beta\in A\cap b$, $\alpha\beta\in B \Rightarrow \alpha\in B \Rightarrow \alpha\beta\in(a\cap B)\cap b$

Exercise. Use the same principle to show that any two finite bases of a vector space have the same cardinality.