September 21, hours 4-5: Quotients and The First Isomorphism Theorem

September-20-10
9:42 AM

Read Along: Selick's notes 1.1, 1.2.1, 1.4.
Lang's book I 1-3.

1. Go over the "about" handout.
2. Group homomorphisms, the "category" of groups, images and kernels. Example: S_3 is an image of S_4, but not a kernel.
3. Normal subgroups, kernels are normal.
4. Question: Is there a normal subgroup of S_4 which is isomorphic to S_3?

Add examples!

Question Is every normal subgroup the kernel of a homomorphism? Given $N \lhd G$, can we find a surjective homomorphism $\phi: G \to H$, with $\ker \phi = N$?

Set theoretic aside: Surjections are the same as equivalence relations.

(def'n, explanation ...)

Sol'n Suppose we had ϕ, consider the resulting equiv:

$g_1 \sim g_1 n$ or $g_1 \sim g_2$ iff $g_1^{-1} g_2 \in N$.

Let $H = G/\sim = \{[g]\}$ where $[g] = gN$

with $\phi: G \to H$ being $\phi(g) = [g]$

define $[g_1][g_2] = [g_1 g_2]$
$[g]^{-1} = [g^{-1}]$ (well defined!)

Claim $H = G/\sim$ is a group & ϕ is a morphism whose kernel is N ... we write $H = G/N$.

Theorem (The First Isomorphism Theorem) Given any morphism $\phi: G \to H$, $G/\ker\phi \cong \operatorname{im}\phi$.

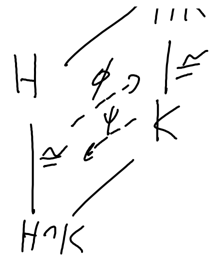
done line

(Should have used "N" for the "normal" one below)

Second Isomorphism Theorem. $H, K < G$, $H < N_G(K)$
$\Rightarrow H \cap K \lhd H$, $K \lhd HK$ and $HK/K \cong H/H\cap K$.

. HK To do: Define "Normalizer"

In a diagram:



2. $H \cap K$ is a subgroup.
3. $H \cap K \trianglelefteq H$
4. HK is a subgroup
5. $K \trianglelefteq HK$.
6. The isomorphism.

Now the isomorphism:

$\phi : h(H \cap K) \longmapsto hK$ clearly well defined.

$\psi : hkK \longmapsto h(H \cap K)$

$h_1 k_1 K = h_2 k_2 K$ iff $h_1 K = h_2 K$ iff $h_1 h_2^{-1} \in K \cap H$.

Third Isomorphism Theorem. $N \trianglelefteq G$, $H \trianglelefteq G$, $N \leq H$

$\Longrightarrow H/N \trianglelefteq G/N$ and $(G/N)/(H/N) \cong G/H$.

Proof. $\phi : G/N / H/N \longrightarrow G/H$ by $[[g]_N]_{H/N} \longrightarrow [g]_H$

$\psi : G/H \to G/N / H/N$ by $[g]_H \longrightarrow [[g]_N]_{H/N}$

Fourth Isomorphism Theorem. If $N \trianglelefteq G$ there's a bijection between subgroups of G/N and subgroups of G that contain N. This bijection preserves inclusions, indices, intersections, and normality of inclusions.

If time: A word about cosets and Lagrange's Theorem.