October 19, hours 16-17: Semi-Direct Products, braids

October-14-10
11:42 AM

Agenda. 1. Semi-direct products as on Oct 14.
2. Examples: D_n, $\{ax+b\}$, $\{Ax+b\}$, $ISO(3)$.
3. Example. PB_n [w/ aside on free groups, gens & rels].
4. Groups of order pq.
5. Groups of order 12.

The Final. Last class: Tue Dec 7. Final: Mon-Wed Dec 13-15.

The Term Test. Remind me at 11:50. "The usual" Decision: Tue 10-1.

Read Along. Selick 1.8, 1.10.

Riddle Along. 1. What about the potato?

2. Can you find uncountably many nearly-disjoint $[\forall \alpha,\beta\ |A_\alpha \cap A_\beta| < \infty]$ subsets of $\mathbb{N}$?

3. Can you find an uncountable chain $[\forall \alpha,\beta, (A_\alpha \subset A_\beta) \vee (A_\beta \subset A_\alpha)]$ of subsets of $\mathbb{N}$?

Semi-Direct Products. If $N < G$, $H < G$, compare $H \times N$ with HN.

There's always $\mu: H \times N \to HN$ by $(h,n) \mapsto hn$.

In general, nothing to say.

If $N \cap H = \{e\}$, injective, but the image might not be a group.

If $N \cap H = \{e\}$ & $N \triangleleft G$ & $H \triangleleft G$, then $[N,H] = \{e\}$ &

$$HN \cong H \times N.$$

The interesting case is when $N \cap H = \{e\}$, $N \triangleleft G$, H may not.

Get $H \xrightarrow{\phi} Aut(N)$ by $h \mapsto (n \mapsto n^h = h^{-1}nh)$

or $\phi_h(n) = h^{-1}nh$ } That's an anti-automorphism!

$$h_1n_1h_2n_2 = h_1h_2h_2^{-1}n_1h_2n_2 = h_1h_2\phi_{h_2}(n_1)n_2$$

Definition. Given abstract N, H & $\phi: H \to Aut(N)$,

the semi-direct product $H \ltimes N$ ...

Prop. 1. In the above case, $\mu: H \ltimes N \to HN$ is an isomorphism.

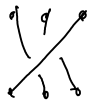
2. $N \triangleleft (H \ltimes N)$ and $H \ltimes N / N \cong H$.

Small Examples. 1. $D_{2n} = \mathbb{Z}_n \rtimes \{\pm 1\}$

2. $\{ax+b\} = \mathbb{R}^+_b \rtimes \mathbb{R}^\times_a$

3. $\{Ax+b : A \in GL(V), b \in V\} = V_b \rtimes GL(V)_A$

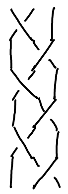
4 "The Poincare Relativity Group" $= \mathbb{R}^4 \rtimes SO(3,1)$

Big Example. $B_n = \pi_1\left((\mathbb{C}^2 \setminus \{\text{diags}\})/S_n\right) =$ 

done but review needed

$B_n = \left\langle \sigma_1, \dots \sigma_{n-1} : \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \end{array} \right\rangle$ } an aside on free groups, generators & relations.

$\pi : B_n \longrightarrow S_n \qquad PB_n = \ker \pi$



Two reasons why I like this one:
1. knotted \$20's.
2. Borromean.

$PB_n \triangleleft B_n$ yet not $B_n = S_n \ltimes PB_n$

$\rho : PB_n \to PB_{n-1} \qquad \ker \rho = F_{n-1}$ and

$PB_n = PB_{n-1} \ltimes F_{n-1} = ((\dots(F_1 \ltimes F_2) \ltimes F_3 \dots \ltimes F_{n-2}) \ltimes F_{n-1})$

not done

Groups of order 21. $\mathbb{Z}/21$, $\mathbb{Z}/7 \rtimes \mathbb{Z}/3 = \langle x \rangle \rtimes \langle y \rangle$

$\mathrm{Aut}(\mathbb{Z}/7) = \mathbb{Z}/6 = \langle \phi_3 \rangle$; $\phi_3(x) = x^3$; $x^y = x$ or x^2 or x^4 (isomorphic)

(iso: if $x^y = x^2$ & $\bar{y} = y^2$ then $x^{\bar{y}} = x^4$)

Groups of order 12. If $|G| = 12$, $P_4 = \mathbb{Z}/4$ or $(\mathbb{Z}/2)^2$, $P_3 = \mathbb{Z}/3$, and at least one of those is normal, for there's not enough room for 4 P_3 & 3 P_4's. So G is a semi-direct product.

$\mathbb{Z}/12$, $\mathbb{Z}/2 \times \mathbb{Z}/6$, A_4, D_{12}, $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$

The most fun case is $\mathbb{Z}/3 \circlearrowright (\mathbb{Z}/2)^2$, giving A_4.