October 14, hour 15: The Sylow Theorems, Semi-Direct Products

October-12-10
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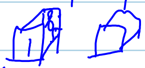
Read Along: D&F 4.5, Selick 1.8.

A & B play as follows: 1. A writes 1...18 on the faces of 3 blank dice:  ..

2. B chooses one, then A chooses another.

3. They play "war" a 1000 times, on money.

Whom would you rather be?

<u>Def</u> G finite, a p-group, a Sylow-p group, $Syl_p(G)$

Theorem. 1. Sylow p-groups always exist; $Syl_p(G) \neq \emptyset$.

2. Every p-group is contained in a Sylow-p group.

3. All Sylow-p subgroups of G are conjugate, and

$$n_p(G) := |Syl_p(G)| = 1 \mod p. \quad \& \quad n_p(G) \,\big|\, |G|$$

Tools. 1. Every G-set is a union of orbits, each one has order dividing the order of $|G|$.

2. Cauchy's theorem. In an Abelian group of order divisible by p, there's an element of order p.

Proof: Pick $x \in G$. If $p \mid |x|$, done. Else $p \mid G/\langle x \rangle$, so $\exists y \in G$ with $|\bar{y}| = p$ in $G/\langle x \rangle$. So $y^p = x^\alpha$. Write $|y| = pk + r$, with $0 \le r < p$. Get $e = y^{|y|} = x^{\alpha k} y^r \Rightarrow y^r \in \langle x \rangle \Rightarrow r = 0$, so $p \mid |y|$.

3. If a p-element x, or a p-group H, normalizes $P \in Syl_p(G)$, then $x \in P$, or $H \subset P$.

Step 1. $Syl_p(G) \neq \emptyset$

Step 2. If $P \in Syl_p(G)$, $|\text{conjugates of } P| = 1 \mod p$.

Step 3. If $P \in Syl_p(G)$ & $H < G$ is a p-group, then H is contained in a conjugate of P (and the rest follows).

Semi-Direct Products. If $N < G$, $H < G$, compare $H \times N$ with HN.

There's always $\mu : H \times N \to HN$ by $(h, n) \mapsto hn$.

In general, nothing to say.

1st image might not be a group.

If $N \cap H = \{e\}$, injective, ~~might not be surjective~~

done line.

If $N \cap H = \{e\}$ & $N \triangleleft G$ & $H \triangleleft G$, then $[N, H] = \{e\}$ &

$$HN \cong H \times N.$$

The interesting case is when $N \cap H = \{e\}$, $N \triangleleft G$, H may not.

Get $H \xrightarrow{\phi} Aut(N)$ by $h \mapsto (n \mapsto n^h = h^{-1} n h)$

$$\text{or} \qquad \phi_h(n) = h^{-1} n h$$

$$h_1 n_1 h_2 n_2 = h_1 h_2 h_2^{-1} n_1 h_2 n_2 = h_1 h_2 \phi_{h_2}(n_1) n_2$$

Definition. Given abstract N, H & $\phi : H \to Aut(N)$,

the semi-direct product $H \ltimes N$ ...

Prop. 1. In the above case, $\mu : H \ltimes N \to HN$ is an isomorphism.

2. $N \triangleleft (H \ltimes N)$ and $H \ltimes N / N \cong H$.