October 12, hours 13-14: The Sylow Theorems

October-07-10
12:02 PM

New OH: Thursdays $11^{30}-12^{30}$, but next Thu, $11^{30}-12^{00}$.

Read Along. D&F 4.5

HW1 due, HW2 will be on web by midnight.

THE SYLOW THEOREMS.

Def G finite, a p-group, a Sylow-p group, $Syl_p(G)$

Theorem. 1. Sylow p-groups always exist; $Syl_p(G) \neq \emptyset$.

2. Every p-group is contained in a Sylow-p group.

3. All Sylow-p subgroups of G are conjugate, and

$$n_p(G) := |Syl_p(G)| = 1 \mod p.$$

Groups of order 15.

P_5 is normal in G, P_3 is normal in G. Any $y \in P_3$ commutes with P_5 [otherwise, $|y| \mid |Aut\, P_5| = 4$],

Preliminary Lemma. A group of order p is $\mathbb{Z}/p$.

Aside. $n_p \mid pq \Rightarrow n_p \mid q$, (or $n_p = 1$)
$n_p = 1 \mod p \Rightarrow q = 1 \mod p$
$= p \mid q-1$

(Aside. $Aut(\mathbb{Z}/p) = (\mathbb{Z}/p)^*$ so $|Aut(\mathbb{Z}/p)| = p-1$)

So $G = x^i y^j = y^j x^i$ for generators $x \in P_5$, $y \in P_3$.

Aside. If $G = G_1 \cdot G_2$, $G_1 \cap G_2 = \langle e \rangle$, $[G_1, G_2] = \{e\}$, then

$G = G_1 \times G_2$

Aside. $\mathbb{Z}/p \times \mathbb{Z}/q = \mathbb{Z}_{pq}$

So $G_{15} = \mathbb{Z}/15$.

This also works for order pq, $p<q$ primes, $p \nmid q-1$.

done

Groups of order 21. P_7 is normal, P_3 might not be done P_3 may act on P_7. If $P_7 = \langle x \rangle$, $P_3 = \langle y \rangle$, we

skipped have $x^y = x$, or x^2, or x^4

Aside. $\mathrm{Aut}(\mathbb{Z}/p)$ is cyclic;
$\mathrm{Aut}(\mathbb{Z}/7) = \langle x \mapsto x^3 \rangle$
1 3 2 6 4 5

done **Def.** What does this mean?

This also works for order pq, $p<q$ primes, $p \mid q-1$.

done but merged up **Claim 1.** $\mathrm{Syl}_p(G) \neq \emptyset$. [Let α be s.t. $p^\alpha \mid |G|$, $p^{\alpha+1} \nmid |G|$]

Proof. Induct on $|G|$. By the class eq'n,

$$|G| = |Z(G)| + \sum_i (G : C_G(y_i))$$

If $p \mid |Z(G)|$, then G has a normal subgroup N of order divisible by p. Use induction on G/N.

Otherwise, for some y_i $p^\alpha \mid |C_G(y_i)|$. A Sylow-p subgroup of $C_G(y_i)$ is a Sylow-p subgroup of G.

barely touched **Claim 2.** If $P \in \mathrm{Syl}_p(G)$, then $|\text{conjugates of } P| \equiv 1 \bmod p$.
(and $n_p \mid |G|$, of course)

Proof. P acts on the set of its conjugates by conjugation. The orbit $\{P\}$ is a singleton; by lemma, the sizes of all other orbits are divisible by p.

Lemma. If a p-element normalizes P, it is in P. If a p-group H is in $N_G(P)$, then $H \subset P$.

Proof. $|PH| = \frac{|P||H|}{|P \cap H|}$ can't be bigger than $|P|$.

Claim 3. If H is a p-subgroup & $P \in \mathrm{Syl}_p(G)$, then H is contained is a conjugate of P. [In particular, all Sylow-p subgroups are conjugates]

Proof. H acts on the set of conjugates of P by conjugation. There must be a singleton orbit — a P' s.t. $H < N_G(P')$.