November 30, hours 32-33: Uniqueness part for modules over a PID, applications

November-28-10
11:10 AM

Balls & Boxes, December schedule.

HW. HW4 due, HW5 on web by midnight.

IT2C2W: $[M \text{ f.g.}/R \text{ PID} \Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle]$

$\Rightarrow$ structure of f.g. Abelian groups, J.C.F.

Reminder. $\frac{R}{\langle a\rangle} \otimes \frac{R}{\langle b\rangle} = \frac{R}{\langle \gcd(a,b)\rangle}$ "on board" in blue!

Theorem. $(M,N) \longmapsto M\otimes N$ is a "bifunctor"

Prop. For any domain R there is a unique field $Q(R)$ "the field of fractions"

s.t.

$$\begin{tikzcd} R \arrow[r, "1-1"] \arrow[dr] & Q(R) \arrow[d, dashed, "\exists!"] \\ & F \end{tikzcd}$$

Proof later.

<u>Claim</u> If M is torsion then $M_{Q(R)} = 0$.

<u>Prop</u> If $M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle$, then

1. $\dim_{Q(R)} M_{Q(R)} = k$
2. $\dim_{R/\langle p\rangle} M_{R/\langle p\rangle} = k + |\{i : p_i \sim p\}|$
3. $\dim_{R/\langle p\rangle} \operatorname{im}(m \mapsto p^s m) = k + |\{i : p_i \sim p \ \& \ s < s_i\}|$

as

$$\operatorname{im}(m \mapsto p^s m) \cong \begin{cases} p^s R & \text{on } R \\ R/\langle q^t\rangle & \text{on } R/\langle q^t\rangle \quad q \not\sim p \\ 0 & \text{on } R/\langle p^t\rangle \quad s \geq t \\ R/\langle p^{t-s}\rangle & \text{on } R/\langle p^t\rangle \quad s < t \end{cases}$$

and

$$\ker(m \mapsto p^s m) \cong \begin{cases} 0 & \text{on } R \\ 0 & \text{on } R/\langle q^t\rangle \quad q \not\sim p \\ R/\langle p^t\rangle & \text{on } R/\langle p^t\rangle \quad s \geq t \\ R/\langle p^s\rangle & \text{on } R/\langle p^t\rangle \quad s < t \end{cases}$$

$R/\langle p^s\rangle \mapsto \ker$ by $[r]_{p^s} \mapsto [p^{t-s} r]_{p^t}$

Localization & Fields of Fractions. Let R be a commutative domain

Def A multiplicative subset S of $R\setminus\{0\}$. (contains 1, closed under $\times$) domain

Examples $R\setminus\{0\}$, $R\setminus P$ (P prime), Powers of $a\neq 0$.

Definition $S^{-1}R = \{\frac{r}{s}\} \big/ \frac{r_1}{s_1}\sim\frac{r_2}{s_2}$ if $r_1 s_2 = r_2 s_1$

$\Big[\frac{r_1}{s_1}\sim\frac{r_2}{s_2},\ \frac{r_2}{s_2}\sim\frac{r_3}{s_3} \Rightarrow r_1 s_2 = r_2 s_1,\ r_2 s_3 = r_3 s_2 \Rightarrow$

$r_1 s_2 s_3 = r_2 s_1 s_3 = s_1 r_3 s_2 \Rightarrow r_1 s_3 = r_3 s_1 \Big]$

$\frac{r_1}{s_1}+\frac{r_2}{s_2}=\dots$

$\frac{r_1}{s_1}\cdot\frac{r_2}{s_2}=\dots$

$R\setminus\{0\}$ — "Field of fractions $Q(R)$"

$R\setminus P$ — "localization at P"

$\{2^n\}$ — "dyadic rationals".

$R\to S^{-1}R$ is injective

Abelian groups & the mult. groups of finite fields

$$A \cong \mathbb{Z}^k \oplus \bigoplus \mathbb{Z}/{p_i}^{s_i} \cong \mathbb{Z}^k \oplus \mathbb{Z}/a_1 \oplus \mathbb{Z}/a_2 \oplus \dots$$

$$a_1 \mid a_2 \mid a_3 \dots$$

Theorem If F is finite, F^* is cyclic.

Proof otherwise, $x^{a_1}-1$ has too many roots.

done here

$F[x]$ and the J.C.F.

$T: V\to V$ makes V an $F[x]=R$ module, so $V\cong R^k\oplus\bigoplus R/\langle p_i^{s_i}\rangle$. As $f(T)=0$ for some f, $k=0$. If F is alg. closed, $p_i = x-\lambda_i$

Cayley-Hamilton and practical J.C.F. computations.