

Balls & Boxes, December schedule.

HW. HW4 due, HW5 on web by midnight.

IT 2C2W:  $[M \text{ f.g.}/R \text{ PID} \Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i} \rangle]$

$\Rightarrow$  structure of f.g. Abelian groups, J.C.F.

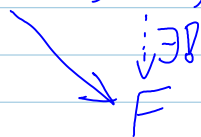
Reminder.  $\frac{R}{\langle a \rangle} \oplus \frac{R}{\langle b \rangle} = \frac{R}{\langle \gcd(a,b) \rangle}$  "on board" in blue!

Theorem.  $(M, N) \mapsto M \otimes N$  is a "bifunctor"

Prop. For any domain  $R$  there is a unique field  $Q(R)$

s.t.  $R \xrightarrow{\iota} Q(R)$

"the field of fractions"



Proof later.

Claim IF  $M$  is torsion then  $M_{Q(R)} = 0$ .

Prop IF  $M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i} \rangle$ , then

1.  $\dim_{Q(R)} M_{Q(R)} = k$

2.  $\dim_{R/\langle p \rangle} M_{R/\langle p \rangle} = k + |\{i : p_i \sim p\}|$

3.  $\dim_{R/\langle p \rangle} \text{im}(M \rightarrow p^s M) = k + |\{i : p_i \sim p \ \& \ s < s_i\}|$

as  $\text{im}(M \rightarrow p^s M) \cong \begin{cases} p^s R & \text{on } R \\ R/\langle q^t \rangle & \text{on } R/\langle q^t \rangle \ q \neq p \\ 0 & \text{on } R/\langle p^t \rangle \ s \geq t \\ R/\langle p^{t-s} \rangle & \text{on } R/\langle p^t \rangle \ s < t \end{cases}$

and  $\text{ker}(M \rightarrow p^s M) \cong \begin{cases} 0 & \text{on } R \\ 0 & \text{on } R/\langle q^t \rangle \ q \neq p \\ R/\langle p^t \rangle & \text{on } R/\langle p^t \rangle \ s \geq t \\ R/\langle p^s \rangle & \text{on } R/\langle p^t \rangle \ s < t \\ R/\langle p^s \rangle \mapsto \text{ker by } [r]_{p^t} \mapsto [p^{t-s}r]_{p^t} \end{cases}$

Localization & Fields of fractions. Let  $R$  be a commutative domain

$\dots$

Def A multiplicative subset  $S$  of  $R \setminus \{0\}$ . (domain contains 1, closed under  $\times$ )

Examples  $R \setminus \{0\}$ ,  $R \setminus P$  ( $P$  prime), Powers of  $a \neq 0$ .

Definition  $S^{-1}R = \left\{ \frac{r}{s} \right\} / \frac{r_1}{s_1} \sim \frac{r_2}{s_2} \text{ if } r_1 s_2 = r_2 s_1$

$$\left[ \begin{array}{l} \frac{r_1}{s_1} \sim \frac{r_2}{s_2}, \frac{r_2}{s_2} \sim \frac{r_3}{s_3} \Rightarrow r_1 s_2 = r_2 s_1, r_2 s_3 = r_3 s_2 \Rightarrow \\ r_1 s_2 s_3 = r_2 s_1 s_3 = s_1 r_3 s_2 \Rightarrow r_1 s_3 = r_3 s_1 \end{array} \right] \quad \begin{array}{l} \frac{r_1}{s_1} + \frac{r_2}{s_2} = \dots \\ \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \dots \end{array}$$

$R \setminus \{0\}$  - "Field of Fractions  $\mathbb{Q}(R)$ "

$R \setminus P$  - "localization at  $P$ "

$\{2^n\}$  - "dyadic rationals".

$R \rightarrow S^{-1}R$   
is injective

Abelian groups & The mult. groups of finite fields

$$A \cong \mathbb{Z}^k \oplus \bigoplus \mathbb{Z}/p_i^{s_i} \cong \mathbb{Z}^k \oplus \mathbb{Z}/a_1 \oplus \mathbb{Z}/a_2 \oplus \dots$$

$a_1 | a_2 | a_3 \dots$

Theorem If  $F$  is finite,  $F^*$  is cyclic.

Proof otherwise,  $x^{a_1} - 1$  has too many roots.

done here

$F[x]$  and the J.C.F.  $T: V \rightarrow V$  makes  $V$  an  $F[x]=R$  module, so  $V \cong R^k \oplus \bigoplus R/\langle p_i \rangle$ . As  $f(T)=0$  for some  $f$ ,  $k=0$ . If  $F$  is alg. closed,  $p_i = x - \lambda_i$

Cayley-Hamilton and practical J.C.F. computations. —