November 25, hour 31: Justice for tensor products, uniqueness part for modules over a PID

November-23-10
1:17 PM

IT2C2W: $[M \text{ f.g.}/R \text{ PID} \Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle]$

$\Rightarrow$ structure of f.g. Abelian groups, J.C.F.

Reminder. $M \otimes N$, $\dim V \otimes W = (\dim V)(\dim W)$, $\frac{R}{\langle a\rangle} \otimes \frac{R}{\langle b\rangle} = \frac{R}{\langle \gcd(a,b)\rangle}$

HW. Is $\mathbb{Q}[x,y]/x^2+y^2=1$ a UFD?

The universal property of tensor products, then....

Theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring".

Example. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ "Extension of scalars".

In general, given $\phi: R \to S$ a ring morphism, S is an R module & set $M_S := S \otimes_R M$. Then $R^n_S = S^n$.

Theorem. $(M, N) \longmapsto M \otimes N$ is a "bifunctor" } skipped.

Prop. For any domain R there is a unique field $Q(R)$ "The field of fractions"

s.t.

$$\begin{tikzcd} R \arrow[r, "1-1"] \arrow[dr] & Q(R) \arrow[d, dashed, "\exists!"] \\ & F \end{tikzcd}$$

Proof later.

Claim If M is torsion then $M_{Q(R)} = 0$. } not properly stated.

Prop If $M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle$, then

1. $\dim_{Q(R)} M_{Q(R)} = k$
2. $\dim_{R/\langle p\rangle} M_{R/\langle p\rangle} = k + |\{i : p_i \sim p\}|$ done line.
3. $\dim_{R/\langle p\rangle} \operatorname{im}(m \mapsto p^s m) = k + |\{i : p_i \sim p \ \&\ s < s_i\}|$

as

$$\operatorname{im}(m \mapsto p^s m) \cong \begin{cases} p^s R & \text{on } R \\ R/\langle q^t \rangle & \text{on } R/\langle q^t \rangle \quad q \nsim p \\ 0 & \text{on } R/\langle p^t \rangle \quad s \geq t \\ R/\langle p^{t-s} \rangle & \text{on } R/\langle p^t \rangle \quad s < t \end{cases}$$

and

$$\ker(m \mapsto p^s m) \cong \begin{cases} 0 & \text{on } R \\ 0 & \text{on } R/\langle q^t \rangle \quad q \nsim p \\ R/\langle p^t \rangle & \text{on } R/\langle p^t \rangle \quad s \geq t \\ R/\langle p^s \rangle & \text{on } R/\langle p^t \rangle \quad s < t \end{cases}$$

$R/\langle p^s \rangle \hookrightarrow \ker$ by $[r]_{p^s} \mapsto [p^{t-s} r]_{p^t}$

Localization & Fields of fractions.