Modules over PID's
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Following Selick's notes, page 92–.

Theorem 1 Over a PID, a submodule of a free module is free.

Nice observation: R is a PID says more or less the same as "every submodule of R is free".

Example: $\langle 2, x\rangle$ as a module over $\mathbb{Z}[x]$ is not free, yet it is a submodule of $\mathbb{Z}[x]$.

Theorem 2 Over a PID, a finitely generated torsion-free module is free.

Selick's proof seems a bit wrong:

Theorem 2.10.2. *Over a PID, a finitely generated torsion-free module is free.*

Proof. Let R be a PID and let M be a finitely generated torsion-free R-module. Let $R \hookrightarrow K$ be the inclusion of R into its field of fractions, and let

Need "field of fracs", "tensor product".

$$\tilde{M} := K \otimes_R M$$

be the extension of M to a K-vector space.

Let $x_1, \ldots, x_m \in M$ be a generating set for M. The images of $x_1, \ldots, x_m$ generate $\tilde{M}$, so $\exists$ a subset $y_1 \ldots, y_n$ whose images in $\tilde{M}$ form a basis for $\tilde{M}$. Each x_j can be written in $\tilde{M}$ as a K-linear combination of $y_1, \ldots, y_n$, so clearing denominators gives that $b_j x_j$ is an R-linear combination of $y_1, \ldots, y_n$ $\forall j$.

Set $b = b_1 \cdots b_m$, so that bx_j is an R-linear combination of $y_1, \ldots, y_n$ $\forall j$.

$\therefore$ bz is an R-linear combination of $y_1, \ldots y_n$ $\forall z \in M$, since $x_1, \ldots, x_m$ span M. Since M is torsion-free,

$$b : M \mapsto M$$
$$z \mapsto bz$$

is injective. Hence,

$$M \cong M/\ker\phi \cong \operatorname{Im} b = bM$$

However,

$$\bigoplus_{j=1}^{n} y_j \xrightarrow{\phi} bM$$
$$y_j \mapsto y_j$$

we don't know that $y_j \in bM$! We could artificially "inflate" b by multiplying it by 500 ($R=\mathbb{Z}$), and then certainly $y_j \notin bM$

is an isomorphism (onto since bz is a linear combination of $y_1, \ldots, y_n$ $\forall z \in M$, (1-1) since $y_1, \ldots y_n$ are linearly independent in $\tilde{M}$).

$\therefore$ $M \cong bM \cong$ a free R-module. $\square$

The correct completion of the proof is to map $bM \longmapsto R^n$ by mapping $bz \longmapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, where

$bM \mapsto R^{\cdot}$ by mapping $bz \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, where the a_i's are the unique(!) coefficients in R for which $bz = \sum a_i y_i$. This map is injective, hence bM is a submodule of a free module, hence bM is free, hence M is free. $\square$

Corollary 3 If M is a finitely generated module over a PID then $M \cong Tor(M) \oplus R^n$ for some $n \in \mathbb{N}$. } need to kno that n is uniquely determined.

Proof $M \to M/Tor(M)$ splits as $M/tor(M)$ is torsion free an hence free, so $M \cong Tor(M) \oplus M/Tor(M)$. $\square$

Standing Assumption M is a torsion module over a PID R.

Lemma 5 Let $Ann(M) = \langle a \rangle$ and assume $b \in R$ has $(a,b) = 1$. Then multiplication by b, $\mu_b : M \to M$, is an isomorphism. $\square$

Lemma 6. $M \cong M_{p_1} \oplus \dots \oplus M_{p_k}$, where $u = p_1^{t_1} \dots p_k^{t_k}$ is the decomposition of u, the annihilator of M ($Ann(M) = \langle u \rangle$) into primes and where $M_p := \{x \in M : p^l x = 0 \text{ for some } l\}$.

The Main Theorem If M is a finitely generated module over a PID R, then

$$M \cong R^k \oplus \bigoplus R/(p_i^{s_i}) \quad \text{for } k \in \mathbb{N},\ s_i \in \mathbb{N}, \text{ and } p_i \text{ prime in } R.$$

Furthermore, up to a permutation this decomposition

Furthermore, up to a permutation this decomposition is unique.