Cauchy's Theorem on Abelian Groups

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D&F pp 102

Theorem. If G is a finite Abelian group of order divisible by a prime p, then G contains an element of order p.

Proof. Enough to find an element of order divisible by p; if z is of order $p \cdot n$, z^n would be of order p. Pick $x \in G$, $x \neq 1$. If $p \mid |x|$, we're done. Otherwise $p \mid |G/\langle x \rangle|$, so by induction, $\exists y \in G$ s.t. $|\bar{y}| = p$ in $G/\langle x \rangle$. So $y \notin \langle x \rangle$ yet $y^p \in \langle x \rangle$, So $|y^p| < |y|$, so $p \mid |y|$ $\square$

$y^p = x^\alpha$ $\qquad |y| = pk + r \qquad 0 \leq r < p$

$e = y^{pk+r} = x^{\alpha k} y^r \implies y^r \in \langle x \rangle \implies r = 0,$
as $|\bar{y}| = p$.