AKT-091126, Hour 30-31: Peter Lee on Braided Monoidal Categories, the Pentagon and the Hexagons

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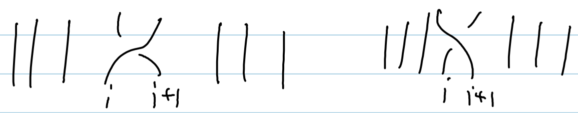
8:51 AM

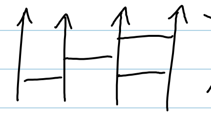
Peter Lee on braided monoidal categories, the pentagon and the hexagons.

The braid group B_n is generated by



$B_{i,i+1}$ $\quad$ $B_{i,i+1}$ $\quad$ / mod R2 & R3

Recall

$A_n = K\langle$  $\rangle / 4T$, $t^{ij} = t^{ji}$, $[t^{ij}, t^{kl}] = 0$ if $|\{i,j,k,l\}| = 4$

$= K\langle t^{ij}\rangle / 4T : [t^{ij}, t^{ik} + t^{jk}] = 0$

Can we find $B = \uparrow\uparrow + \cdot\ \uparrow\!-\!\uparrow$ + higher order satisfying R3?

deg 0: $\quad 1 = 1$

deg 1: $\quad t^{12} + t^{13} + t^{23} = t^{23} + t^{13} + t^{12}$

deg 2: $\quad [t^{12}, t^{13} + t^{23}] + [t^{13}, t^{23}] \overset{?}{=} 0$ false!

Recall: If $\mathfrak{g}$ is a f.d. metrized Lie algebra w/ a f.d. rep. R, we can associate

$- - - - - \longleftrightarrow t \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ or $t^{-1} \in \mathfrak{g} \otimes \mathfrak{g}$

$\longrightarrow \quad \longrightarrow Id \in End(R)$

$- - - \uparrow \quad \longrightarrow r \in \mathfrak{g}^* \otimes R^* \otimes R$

So we can think of elements of A_n as elements in $U(\mathfrak{g})^{\otimes n}$, acting on $\underbrace{R \otimes R \ldots \otimes R}_{n \text{ times}}$

$\longrightarrow$ Look for invariants which take values in tensor power of rep'n of $U(\mathfrak{g})$

If $\mathfrak{g}$ is a Lie algebra, we define a category $Rep(\mathcal{U}(\mathfrak{g}))$ whose objects are $\mathcal{U}(\mathfrak{g})$ modules with morphism being intertwiners.

<u>Note</u> If V_1 & V_2 are $\mathcal{U}(\mathfrak{g})$-modules, then so is $V_1 \otimes V_2$

via
$$g.(v_1 \otimes v_2) = (gv_1) \otimes v_2 + v_1 \otimes (g v_2)$$
$$= (\Delta g)(v_1 \otimes v_2)$$

where $\Delta g = g \otimes 1 + 1 \otimes g$. Δ extends to

$\Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ .....

$\vdots$

<u>Def</u> A bi-algebra A over a field k is a k-algebra together with linear maps $\begin{cases} \Delta : A \to A \otimes A \\ \epsilon : A \to k \end{cases}$

s.t.

1. Δ is co-associative.
2. $(\epsilon \otimes I) \circ \Delta = (I \otimes \epsilon) \circ \Delta = I$
3. Δ is an algebra homomorphism.
4. ϵ is an algebra homomorphism.

<u>Def</u> A monoidal category is a category $\mathcal{C}$ together with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ s.t.

1. $\exists$ natural isomorphisms $\Phi_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$

$$\begin{tikzcd}
& U(V(WZ)) & \\
(UV)(WZ) \arrow[ur] & & U((VW)Z) \arrow[ul] \\
((U \otimes V) \otimes W) \otimes Z \arrow[u, "\Phi_{U\otimes V, W, Z}"] \arrow[rr, "\Phi_{U,V,W}"'] & & (U \otimes (V \otimes W)) \otimes Z \arrow[u]
\end{tikzcd}$$

2. $\exists\ \mathbb{1} \in \mathcal{C}$ with natural iso

$$\rho_u : U \otimes 1 \xrightarrow{\sim} U$$

$$\lambda_u : 1 \otimes U \xrightarrow{\sim} U$$

w/ some coherence...

<u>Def</u> A bi-algebra is "co-commutative" if ....

It is "almost co-commutative" if $\exists\ R \in A \otimes A$ invertible s.t.

$$(\Delta^{op} a) = R(\Delta a) R^{-1}$$

<u>Def</u> A braided monoidal category .....