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Zsuzsi Dancso on the Kontsevich integral.

From <http://drorbn.net/images/2/2b/BerkeleyHandout-0811.png>:

THE KONTSEVICH INTEGRAL OF KNOTTED TRIVALENT GRAPHS
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knotted trivalent graphs
trivalent graph: also $\begin{matrix} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{matrix}$ (with arrows) or $\begin{matrix} \circlearrowleft \\ \circlearrowright \end{matrix}$ (with arrows)

knotted $\Gamma \subset \mathbb{R}^3$ up to isotopy
Isotopy: continuous deformation

operations
- orientation switch
- edge delete
- edge unzip
- connected sum

Goal $Z: \text{KTT} \rightarrow \mathbb{C}$
if algebraic, \mathbb{C} is \mathbb{R} , \mathbb{C} is \mathbb{R} defined on \mathbb{R}
= homomorphism

Algebraic
- homomorphism = good
- algebraic knot theory
- \mathbb{R} is finitely generated, \mathbb{C} is not
- All descriptions of Z on knots

The Kontsevich integral for knots [Gus] [Gus] [Gus]

Finite type invariants and the algebra \mathcal{A}
[Bar] $\rightarrow \mathcal{A}$ invariant
what = singular knots
 $H(\mathcal{X}) = H(\mathcal{X}^+) - H(\mathcal{X}^-)$
Link type of type n $\neq 0$ when $2n$ is odd

chord diagram \circlearrowleft only the crossings matter
chord diagram of singular knot
 $A: S^1 \rightarrow \mathbb{R}^2$ connect pairings of double pt
 $\infty \circ \circ \circ \circ$

Γ of type n \rightarrow chord diagram on S^1
chord diagram is a chord, let Γ be any singular knot up to chord diagram D
 $\mathcal{A}(D) = H(\mathcal{A}_\Gamma)$ well defined!

Proposition always
 $H(\mathcal{A}_\Gamma) = 0$ (Framing Independence)
 $H(\mathcal{A}_\Gamma) = H(\mathcal{A}_{\Gamma'}) = H(\mathcal{A}_{\Gamma''}) = 0$
 $H(\mathcal{A}_\Gamma) = H(\mathcal{A}_{\Gamma'}) = H(\mathcal{A}_{\Gamma''}) = 0$ (97)

Def \mathcal{A} = chord diagrams / FI, FT
multiplication: same sum with diff by FT

In type $2 \rightarrow 4$ called **weight system**
Knots integral $Z: \text{KTT} \rightarrow \mathcal{A}$
"universal" finite type invariant: \forall weight system w , Z is w with weight system w $\Rightarrow K \mapsto w(Z(K))$

Z homomorphism $Z(K_1 \# K_2) = Z(K_1) \otimes Z(K_2)$

Definition of Z
 $Z(K) = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \int \prod_{\text{edges}} \omega_{\text{edge}} \prod_{\text{crossings}} \omega_{\text{crossing}}$
 $\omega_{\text{edge}} = \frac{1}{2\pi i} \int_{\mathbb{C} \setminus \{0\}} \frac{e^{-\omega z}}{z} dz$
 $\omega_{\text{crossing}} = \frac{1}{2\pi i} \int_{\mathbb{C} \setminus \{0\}} \frac{e^{-\omega z}}{z} dz$
 $\mathbb{C} \setminus \{0\}$

$\mathbb{C} \setminus \{0\}$ is of type $1/2$ in $(\mathbb{C} \setminus \{0\})$ where Γ decrease along the orientation of \mathbb{C}

Z convergent
power \sum solution \mathbb{C} by FI anyway!

Z almost invariant
symmetric under horizontal difs that leave the cut pos fixed
- wire: S^1 has the Z invariant under "rotating" (rotating "rotates")
- multi Z
Def Z not invariant under changing the # of cuts
Ex $Z(\Gamma) = Z(\Gamma') - Z(\Gamma'')$
Def $Z^*(K) = \frac{Z(K)}{|\text{Aut}(K)|}$ \rightarrow Z^* is invariant
works same same $Z(K) = \int \text{topology}$
 Z^* an isotopy inv of knots (with universal finite type invariants)

Extending Z [not the Maximum-Entropy] **large association** **Problem**
 Γ a chord diagram $\rightarrow Z(\Gamma)$
 $\mathcal{A}(M) = \text{chord diagram in solution } \mathcal{A} \rightarrow \mathcal{A}$
 \forall "wire invariance" $\mathcal{A} \rightarrow \mathcal{A}$
Operations $S_0: \mathcal{A}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$
 $d_0: \mathcal{A}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$
 $X^0: \mathcal{A}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$
 $U_0: \mathcal{A}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$
 $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma) \rightarrow \mathcal{A}(\Gamma)$
making things to chords, well-def

three solution
Use same picture, some formula
- would retain all good properties of Z
Def divergent
 \mathcal{A} don't have FI any more
 \mathcal{A} even FI wouldn't help

Renormalization
- reduce to vertices of X and Y
- shapes X, Y (not allowed: X, Y)
- replace $X \rightarrow Y$ for M $E \rightarrow 0$
- loop when "doubly sum"
also $Y \rightarrow X$ \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow
Call this Z_0
- **convergent**
- but under these difs, rigid motions of cuts and vertices
- commutes w/ ori switch, vertex delete/unzip, $\#$

Def not an isotopy invariant.

Connections
Missing moves:
 $\begin{matrix} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{matrix} \rightarrow \begin{matrix} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{matrix}$
 $\begin{matrix} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{matrix} \rightarrow \begin{matrix} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{matrix}$
"connectors" to find $\mathcal{A} \rightarrow \mathcal{A}$
isagias to help:
 $\begin{matrix} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{matrix} \rightarrow \begin{matrix} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{matrix} = \text{circled}$
 $\begin{matrix} \nearrow \\ \searrow \\ \searrow \\ \nearrow \end{matrix} \rightarrow \begin{matrix} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{matrix} = \text{circled}$
etc. - comes down to circled enough
 circled for just the knot case
 circled making (not independent, but S_0)
also simultaneously
In the end it works! $Z \rightarrow Z_0$
Good properties they except unip. maps to be renormalized
~ Thank you! ~

References
[DK1] D. Bar-Natan. On the Vassiliev knot invariants, Topology 34 (1995) 423-422
[DK2] ——— Algebraic knot theory - a talk for action, http://www.math.toronto.edu/~danczo/papers/AKT-CPA.html
[D] S.V. Chmutov, D. Duzhin. The Kontsevich integral, Acta Applicandae Math., 66 2 (April 2001), 155-190
[K] M. Kontsevich: Vassiliev's knot invariants, Adv. in Soviet Math., 14 2 (1993) 137-150.
[LH] T.-G.T. Le, J. Murakami: Parallel version of the universal Vassiliev-Kontsevich invariant, J. Pure Appl. Algebra, 121 (1997), 271-291
[LHM] T.-G.T. Le, H. Murakami, J. Murakami, T. Ohtsuki: A three-manifold invariant via the Kontsevich integral, Osaka J. math. 36 (1999), 345-345
[M] J. Murakami, T. Ohtsuki: Topological Quantum Field Theory for the Universal Quantum Invariant, Communications in Mathematical Physics, 188 3 (1997), 501-521
[D] Z. D.: On a Kontsevich integral for knotted trivalent graphs, preprint http://www.math.toronto.edu/~danczo (in the days)