AKT-091005, Hours 23-24: uHigh, AKT

November-04-09
4:01 PM

We are here:

	u	v	w
Top	X		
Comb	X		
Low	X		
high			

This remains our subject.

High Algebra:

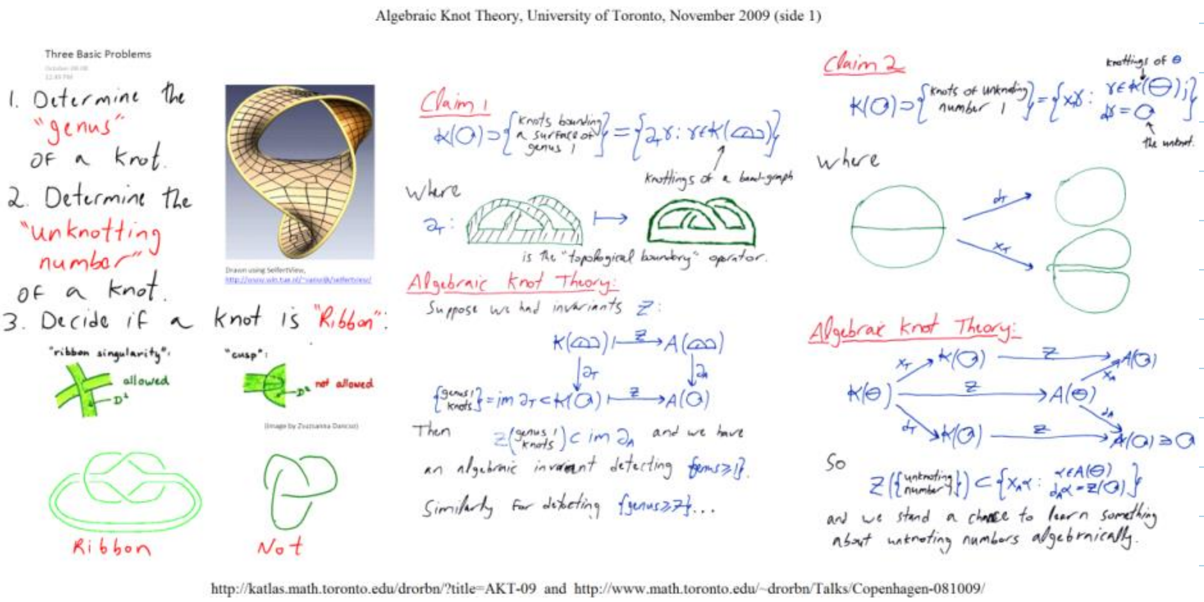
We seek a $\underbrace{\text{homomorphic}}_{\text{huh? (5)}}$ $\underbrace{\text{expansion}}_{\text{huh? (4)}}$ from the $\underbrace{\text{TG-algebra}}_{\text{huh? (3)}}$ $\underbrace{\mathcal{K}^{TG}}_{\text{huh? (1)}}$ to the TG-algebra $\underbrace{\mathcal{A}^{TG}}_{\text{huh? (2)}}$.

Why bother?

1. General optimism in the pursuit of aesthetics.
2. Potential usefullness via "Algebraic Knot Theory".
3. Reduction of the "Fundamental theorem" to a "finite" (high) algebra problem.

in practice, this we do today.

Following these handouts.



Algebraic Knot Theory, University of Toronto, November 2009 (side 1)

Three Basic Problems

1. Determine the "genus" of a knot.
2. Determine the "unknotting number" of a knot.
3. Decide if a knot is "Ribbon":

"ribbon singularity": allowed D^2

"cusp": D^2 not allowed

Drawn using SeifertView, http://www.win.tue.nl/~vanwijk/seifertview/

(image by Zsuzsanna Dancso)

Ribbon Not

Claim 1

$\mathcal{K}(\bigcirc) \supset \left\{\begin{matrix}\text{knots bounding}\\ \text{a surface of}\\ \text{genus 1}\end{matrix}\right\} = \{\partial_T \gamma : \gamma \in \mathcal{K}(\text{band-graph})\}$

knottings of a band-graph

where ∂_T: is the "topological boundary" operator.

Algebraic Knot Theory:

Suppose we had invariants Z:

$$\begin{array}{ccc} \mathcal{K}(\text{band-graph}) & \xrightarrow{Z} & \mathcal{A}(\text{band-graph}) \\ \downarrow \partial_T & & \downarrow \partial_A \\ \left\{\begin{matrix}\text{genus 1}\\ \text{knots}\end{matrix}\right\} = \operatorname{im}\partial_T \subset \mathcal{K}(\bigcirc) & \xrightarrow{Z} & \mathcal{A}(\bigcirc) \end{array}$$

Then $Z\left(\begin{matrix}\text{genus 1}\\ \text{knots}\end{matrix}\right) \subset \operatorname{im}\partial_A$ and we have an algebraic invariant detecting $\{\text{genus} \geq 1\}$.

Similarly for detecting $\{\text{genus} \geq 7\}$...

Claim 2

$\mathcal{K}(\bigcirc) \supset \left\{\begin{matrix}\text{knots of unknotting}\\ \text{number 1}\end{matrix}\right\} = \{x_T\gamma : \gamma \in \mathcal{K}(\Theta);\ d_T\gamma = \bigcirc\}$

knottings of Θ; the unknot.

where $\Theta \xrightarrow{d_T}$, $\Theta \xrightarrow{x_T}$

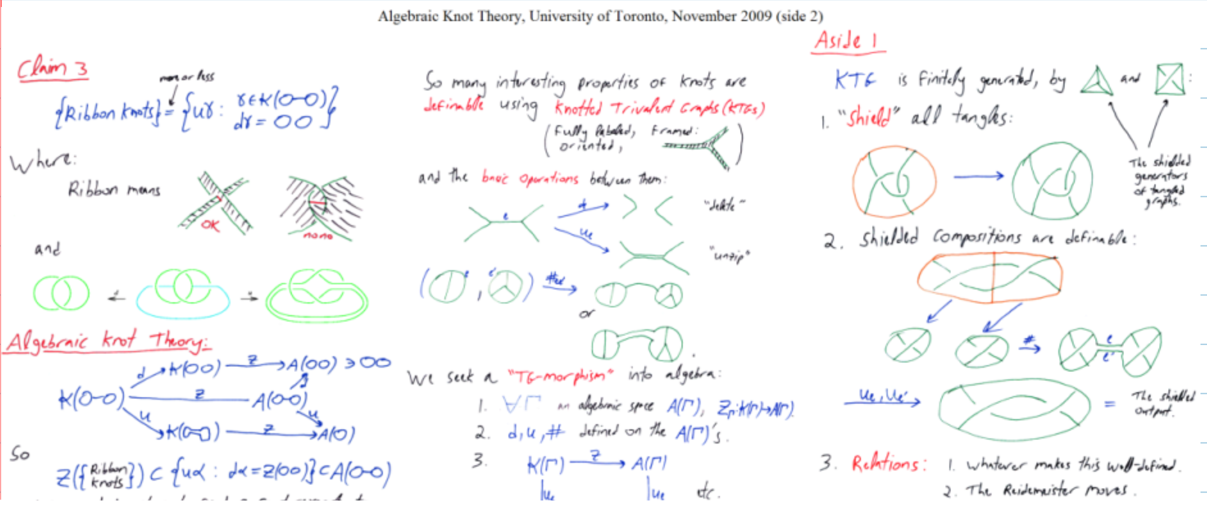
Algebraic Knot Theory:

$$\begin{array}{ccc} & \mathcal{K}(\bigcirc) \xrightarrow{Z} \mathcal{A}(\bigcirc) & \\ \mathcal{K}(\Theta) & \xrightarrow{Z} & \mathcal{A}(\Theta) \\ & \mathcal{K}(\bigcirc) \xrightarrow{Z} \mathcal{A}(\bigcirc) \ni \bigcirc & \end{array}$$

So $Z\left(\left\{\begin{matrix}\text{unknotting}\\ \text{number 1}\end{matrix}\right\}\right) \subset \left\{x_A\alpha : \begin{matrix}\alpha \in \mathcal{A}(\Theta)\\ d_A\alpha = Z(\bigcirc)\end{matrix}\right\}$

and we stand a chance to learn something about unknotting numbers algebraically.

http://katlas.math.toronto.edu/drorbn/?title=AKT-09 and http://www.math.toronto.edu/~drorbn/Talks/Copenhagen-081009/



Algebraic Knot Theory, University of Toronto, November 2009 (side 2)

Claim 3

$\{\text{Ribbon knots}\} \overset{\text{more or less}}{=} \left\{u\gamma : \begin{matrix}\gamma \in \mathcal{K}(\bigcirc\!\!-\!\!\bigcirc)\\ d\gamma = \bigcirc\bigcirc\end{matrix}\right\}$

Where: Ribbon means OK / nono

and

Algebraic Knot Theory:

$$\begin{array}{ccc} & \mathcal{K}(\bigcirc\bigcirc) \xrightarrow{Z} \mathcal{A}(\bigcirc\bigcirc) \ni \bigcirc\bigcirc & \\ \mathcal{K}(\bigcirc\!\!-\!\!\bigcirc) & \xrightarrow{Z} & \mathcal{A}(\bigcirc\!\!-\!\!\bigcirc) \\ & \mathcal{K}(\bigcirc) \xrightarrow{Z} \mathcal{A}(\bigcirc) & \end{array}$$

So $Z\left(\left\{\begin{matrix}\text{Ribbon}\\ \text{knots}\end{matrix}\right\}\right) \subset \{u\alpha : d\alpha = Z(\bigcirc\bigcirc)\} \subset \mathcal{A}(\bigcirc\!\!-\!\!\bigcirc)$

So many interesting properties of knots are definable using Knotted Trivalent Graphs (KTGs) (Fully labeled, oriented, Framed:)

and the basic operations between them: "delete" (d), "unzip" (u), $\#$ ... or ...

We seek a "TG-morphism" into algebra:

1. $\forall\Gamma$ an algebraic space $\mathcal{A}(\Gamma)$, $Z_\Gamma: \mathcal{K}(\Gamma) \to \mathcal{A}(\Gamma)$
2. $d, u, \#$ defined on the $\mathcal{A}(\Gamma)$'s.
3. $\mathcal{K}(\Gamma) \xrightarrow{Z} \mathcal{A}(\Gamma)$, u_e, u_e etc.

Aside 1

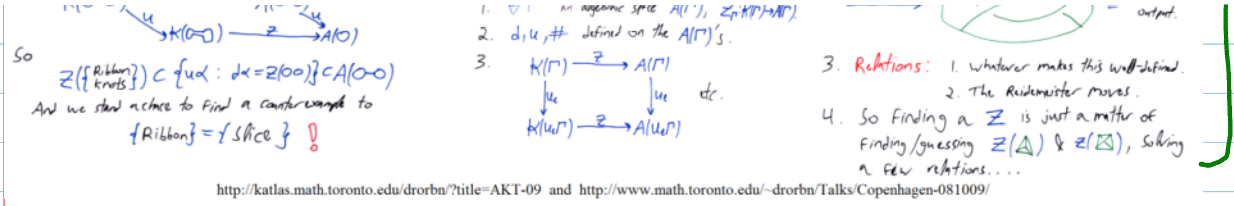
KTG is finitely generated, by and:

1. "Shield" all tangles:

The shielded generators of tangled graphs.

2. Shielded compositions are definable: $\#$; $u_e, u_{e'}$; = The shielded output.

3. Relations: 1. whatever makes this well-defined. 2. The Reidemeister moves.



We've also used http://www.math.toronto.edu/~drorbn/classes/0304/KnotTheory/SeifertAlgorithm.