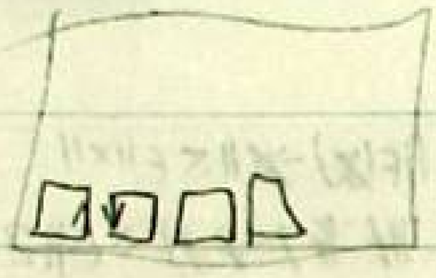
מושגי יסוד בגד"ר, 29 לאוקטובר 2000

1. למצוא סלבוס ורשימת אנשים

2. יעלו הניירות. המשפט הבסיסי של הנושא: סכום שלמקומי



משפט סטוקס: $\int_D dw = \int_{\partial D} w$

3. הספר: ברדון יהיה עם ממתקים בקרוב

4. תלוי בשאלות, יש לקבוע זמנים נוספים

שאלות
1. תבניות אינפי.
2. הגדרת סלבוס והתוצ'
3. משפט הפי הסתומה וזה שאין
4. מרחבים לפי מקולגל, וזה למה זה אומץ.

1. הדיפרנציאל: $f: V \to W$, $f(v+\Delta v) = f(v) + df_v \cdot \Delta v + o(\Delta v)$

2. בקואורדינטות: $f: \mathbb{R}^n_{x_1 \dots x_n} \to \mathbb{R}^m$

$$[df_{\vec{x}}] = \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \\ \partial f_m/\partial x_1 & \cdots & \partial f_m/\partial x_n \end{pmatrix}$$

3. כלל השרשרת: $d(g \circ f)_v = dg_{f(v)} \circ df_v$

4. משפט הפ' הסתומה: תהי $f: V \to V$ דיפרנציאבילית ברציפות בסביבה פתוחה של $a \in V$ ונניח df_a הפיך אזי יש סביבה A של a וסביבה B של $b=f(a)$ כך ש $f|_A: A \to B$ יש הפיך דיפרנציאבילי $f^{-1}: B \to A$ ומתקיים

$$(df^{-1})_z = (df_{f^{-1}(z)})^{-1} \quad z \in B$$

הוכחה בה"כ, $b=a=0$ וכן $df_a = I$

עקרון הקירוב הליני: לכל x,y קרובים קרוב מספיק בסביבת 0,

$$\|f(y)-f(x)-(y-x)\| \leq \epsilon \|y-x\|$$

הוכחה: $g=f-I$; נפרוש g קטנה.

המשך/...

מרצה: יוסף בגדי, 29 לנובמבר 2000, המשך

שלבי ההוכחה:

א. קיום f^{-1}: פתרון $f(x)=z$, יחידות הפתרון.

ב. רציפות f^{-1}

ג. דיפרנציאביליות ב-0 של f^{-1} → $\|x\| \le \|x-f(x)\|$, $\|z\| \le \|z-f(z)\|$

ד. דיפרנציאביליות הרחק מ-0, חישוב הדיפרנציאל.

<u>משפט הפ' הסתומה</u> $f:\mathbb{R}^n_x \times \mathbb{R}^m_y \longrightarrow \mathbb{R}^m$ גזירה ברציפות

בסביבת $(a,b)\in\mathbb{R}^n\times\mathbb{R}^m$ ומקיימת $f(a,b)=0$

נניח גם ש- $(d_yf)_{(a,b)}$ הפיך. אזי קיימות

סביבות $a\in A$, $b\in B$ כך שלכל $x\in A$ קיים

$g(x)\in B$ יחיד כך ש- $f(x,g(x))=0$. הפ' g

דיפרנציאבילית ומתקיים

$$dg_x = -\left(d_yf_{(x,g(x))}\right)^{-1}\circ(d_xf)_{(x,g(x))}$$

<u>הוכחה</u> $\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x\\f(x,y)\end{pmatrix}$; ההופכי מהצורה $\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x\\k(x,y)\end{pmatrix}$

Math 1300 Geom & Top, Tuesday Sep 11 2007 - week 1

* Introduction * Differentials & the chain rule. * Today's office hour only until 1PM.

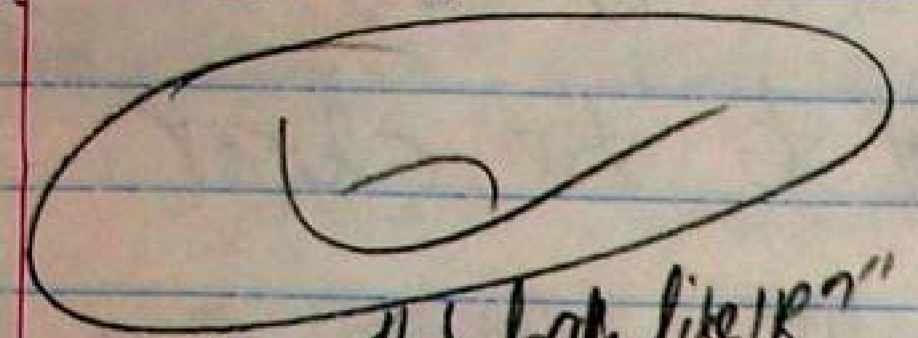
* The inverse & implicit function Theorems.

First technicality: I'm DROR BAR-NATAN

First hour

on board.

goal: understand



"smooth & locally like $\mathbb{R}^n$"

1. locally
2. The fundamental thm of calculus.
3. Globally.

Some teaching / learning philosophy

<u>Def</u> Let $f: V \to W$. f is diffable at $v \in V$ if $\exists\, df_v : V \to W$ linear

s.t. $f(v+\Delta v) = f(v) + df_v \cdot \Delta v + \underbrace{o(\Delta v)}_{\text{define}}$

<u>Thm</u> 1. If such df_v exists, it is unique.

2. If f is linear, $df_v = f$.

3. $d(f+g)_v = df_v + dg_v$

4. For $f: \mathbb{R}^n_{x_1 \dots x_n} \longrightarrow \mathbb{R}^m$,

$$df_x = \begin{pmatrix} \partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & & \partial f_m/\partial x_n \end{pmatrix}$$

5 The chain rule:

$$d(g \circ f)_v = dg_{f(v)} \cdot df_v$$

proofs.

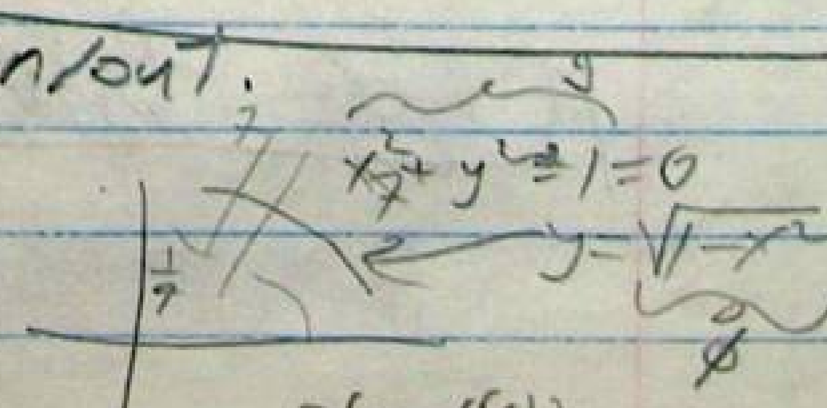
1. Better: search inside yourself.

2. otherwise: See any m.v calculus text.

show proof & example: compute $(x^x)'$

Second hour. * go over "About" handout.

The Implicit Function Theorem



$g(x, \phi(x)) = 0$

\#eqn's = \#unknowns.

Let $g: \mathbb{R}^n_x \times \mathbb{R}^m_y \to \mathbb{R}^m$ be a C^1 function near $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$, assume $g(\xi, \eta) = 0$ & $(\partial_y g)_{(\xi,\eta)}$ is invertible.

Then there exists nbd A of ξ & B of η s.t. there is a unique function $\phi: A \to B$ with $\phi(\xi) = \eta$ & $g(x, \phi(x)) = 0$

Furthermore, 1. ϕ is diffable & $d\phi_x = -(\partial_y g)^{-1}_{(x,\phi(x))} \cdot \partial_x g_{(x,\phi(x))}$ ... cont.

2. If g is C^p, so is ϕ.

Simplifying assumptions:

$(\xi, \eta) = (0,0)$, $(\partial_y g)_{(0,0)} = I$

Idea For a given x, if y_1 is close to satisfying $g(x,y) = 0$,

then $y_2 = y - g(x,y)$ ought to be closer...

So set $f(x,y) = y - g(x,y)$ & look for ϕ s.t.

$\phi(x) = f(x, \phi(x))$

Claim The operator $\phi \mapsto T\phi := f(x, \phi(x))$

is a "contraction" on $F = \{\phi : A \to B, \ \phi(0)=0, \text{ cont}\}$

for some nbd's A of $0 \in \mathbb{R}^n$ & B of $0 \in \mathbb{R}^m$

Def contraction, the metric on F.

Thm The Banach contraction principle

Pf of claim

The fine print

Math 1300 Geom & Top, Thu Sep 13 2007 - Week 1

on board.

The Implicit Function Theorem

Let $g: \mathbb{R}^n_x \times \mathbb{R}^m_y \to \mathbb{R}^m$ be a C^1 function near $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$, assume that $g(x_0, y_0) = 0$ & $(\partial_y g)_{(x_0, y_0)}$ is an invertible $m \times m$ matrix

Then there exist a nbd A of x_0 & B of y_0 s.t. there is a unique function $\phi: A \to B$ with $\phi(x_0) = y_0$ & $g(x, \phi(x)) \equiv 0$. Furthermore,

1. ϕ is diff-able & $d\phi_x = -(\partial_y g)^{-1}_{(x, \phi(x))} \partial_x g_{(x, \phi(x))}$
2. If g is C^l, so is ϕ

Simplifying assumptions: $(x_0, y_0) = (0, 0)$, $(\partial_y g)_{(0,0)} = I$ (Can you do that formally??)

Idea: For a given x, if y_1 is close to satisfying $g(x, y) = 0$, then $y_2 = y_1 - g(x, y)$ ought to be closer.

So set $f(x, y) = y - g(x, y)$ & ~~look~~ look for ϕ s.t.

$\phi(x) = f(x, \phi(x))$.

Claim The operator $\phi \mapsto T\phi := f(x, \phi(x))$ is a "contraction" on $\Phi = \{\phi: A \to B, \ \phi(0) = 0, \ \phi \text{ cont.}\}$ for some nbds A of $0 \in \mathbb{R}^n$ (open) & B of $0 \in \mathbb{R}^m$ (closed).

Def Contraction, the metric on Φ.

Thm The Banach contraction principle.

Proof of claim (use MVT on the side); Φ is complete; Pf of Banach/more.

Further claims

1. ϕ is diff-able

at 0:

$$0 = g(x,\phi(x)) = \overset{g(0,0)+}{(\partial_x g)\cdot x} + (\partial_y g)\cdot \phi(x) + o(x)$$

$$\Rightarrow \phi(x) = -(\partial_y g)^{-1}(\partial_x g)x \cancel{+} - (\partial_y g)^{-1}(o(x))$$

(and we also found the differential!)

2. If g is C^1, so is ϕ:

$$(d\phi)_x = -(\partial_y g)^{-1}_{(x,\phi(x))}\partial_x g_{(x,\phi(x))}$$

3. The inverse function Theorem

4. The equivalence of the two Theorems.

The MVT: on $\mathbb{R}$ ~~~ on $\mathbb{R}^n$

for some $t\in[x,h]$

$$\frac{f(x+h)-f(x)}{h} = f'(t)$$

$$= h\cdot f'$$

$$f(b)-f(a) = (b-a)\cdot(\nabla f)(tb+(1-t)a)$$

Save on corner

מושגי יסוד בגבג"ו, 31 לאוקטובר 2000

chart = מפה.

1. סיום משפט הפ' ההפוכה והפ' הסתומה.

<u>הגדרה</u> יריעה דיפרנציאבילית ממימד-n או n-יריעה M^n: מרחב האוסדורף עם בסיס בן מניה עם אוסף של "מפות":

1. מפה היא הומאומורפיזם $\phi: \underset{\text{פתוחה}}{U} \to \underset{\text{פתוחה}}{\tilde{U}}$, $U \subset M$, $\tilde{U} \subset \mathbb{R}^n$ פתוחות.
2. כל $x \in M^n$ נמצא בתחום של איזו מפה.
3. אם ϕ, ψ מפות, אזי $\phi\psi^{-1}: \psi(U \cap V) \to \phi(U \cap V)$ היא C^∞.
   $\phi: U \to \tilde{U}$, $\psi: V \to \tilde{V}$
4. האוסף הוא מקסימלי ביחס ל-1-3.

"אטלס" על M^n הוא אוסף מפות המקיים רק את 1-3.
קל לראות שאטלס מגדיר מבנה יחיד של n-יריעה.

<u>הגדרה</u> "מבנה פונקציונלי" על מרחב X הוא $F_X: \{\text{פתוחות ב-}X\} \to (\text{אלגבראות})$ כך ש-

(1) $F_X(U)$ היא תת אלגברה של אלגברת הפ' הממשיות הרציפות על U.

(2) $F_X(U)$ מכילה את הקבועים

(3) $V \subset U$, $f \in F_X(U) \Rightarrow f|_V \in F_X(V)$

(4) $U = \bigcup U_\alpha$, $\forall \alpha\ f|_{U_\alpha} \in F_X(U_\alpha) \Rightarrow f \in F_X(U)$

<u>הערה</u> מספיק לבדוק את (4) עבור U_α-ות "קטנות"

<u>דוגמאות</u> פ' רציפות, C^k, C^∞, C^ω

<u>הגדרה</u> מורפיזם $(X, F_X) \xrightarrow{\phi} (Y, F_Y)$: מיפוי $\phi: X \to Y$ רציף

$$(f \mapsto f \circ \phi): F_Y(U) \to F_X(\phi^{-1}(U))$$

איזומורפיזם: ϕ^{-1} גם הוא מורפיזם.

<u>הגדרה</u> יריעה n ממדית: מרחב האוסדורף עם בסיס בן מניה ומבנה פונקציונלי איזומורפי מקומית ל-$\mathbb{R}^n$.

מושגי יסוד בגד"ר, 5 לנובמבר 2000

1. ספרים הגיעו 2. לא קבלתי ל emails 3. למה משמש ה' אסמן?

4. ידיעות בשיעור הקודם.

5. הוכחות שקולות 2 ההגדרות.

6. הגדרה: מ"ר F.M "חלקה".

7. ידיעה וורונולית וידיעות עם שפה.

דוגמאות: X מבנה פונקציונלי

1. $f: X \to Y$ "המבנה הפונקציונלי המושרה על Y"

2. $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ א. הצגות מכות
ב. דיון במבנה פונקציונלי.

3. S^2 א. הצגת הגלוש סטראוגרפית. (מפה)
ב. כמנה של נספי $\mathbb{R}^3$.

4. $\mathbb{RP}^2$

5. המבנה הפונקציונלי המושרה על תת-קבוצה.

6. מכפלה.

מושגי יסוד בגד"ר 7 לנובמבר 2000: ס חלוקת התמורות?
1. המשך כשיעור הקודם.

Math 1300 Geom & Top, Tue Sep 18 2007 - Week 2.

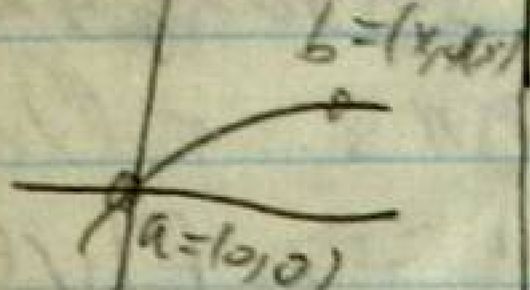
The MVT

on $\mathbb{R}$: For some $t\in[a,b]$, $\frac{f(b)-f(a)}{b-a} = f'(t)$

on $\mathbb{R}^n$: For some t on the straight line between a & $b \in \mathbb{R}^n$, $f(b)-f(a) = (b-a)\cdot\nabla f(t)$.

__Claim__ If $g(x,\phi(x))=0$ (g as before), then $\phi(x)=O(x)$.

__Pf__ By MVT_ϕ on g_i:



$$0 = x\cdot\nabla_x g_i^{(t_i)} + \phi(x)\cdot\nabla_y g_i(t_i)$$

or

$$A\phi(x) = -Bx, \text{ where}$$

$$A=\begin{pmatrix}\nabla_y g_1(t_1)\\ \vdots \\ \nabla_y g_m(t_m)\end{pmatrix} \text{ is very near } \partial_y g(0)$$

$$B=\begin{pmatrix}\nabla_x g_1(t_1)\\ \vdots \\ \nabla_x g_m(t_m)\end{pmatrix} \quad \text{''} \quad \partial_x g(0).$$

__Claim__ If g is C^∞, so is ϕ.

__Pf__ $g(x,\phi(x))=0. \Rightarrow d\phi_x = -(\partial_y g)^{-1}_{(x,\phi(x))}(\partial_x g)_{(x,\phi(x))}$

__Thm__ __The inverse function theorem__: If $f:\mathbb{R}^m \to \mathbb{R}^m$ & $(df)_0$ is invertible, then f is invertible near 0.

__Proof__ solve $x=f(y)$

__Claim__ Inverse implies explicit:

$$\begin{pmatrix}x\\y\end{pmatrix} \xmapsto{g} \begin{pmatrix}x\\ f(x,y)\end{pmatrix}$$

__Def 1__ An n-manifold (n-dim manifold) M^n: A Hausdorff second-countable space with a collection of "charts" s.t.

"Atlas" on M^n:
1. A chart is a homeomorphism $\phi: U_\phi \to U'_\phi$ where $U_\phi \subset M^n$ & $U'_\phi \subset \mathbb{R}^n$ are open.
2. The domain of the charts cover M^n: $M = \bigcup_\phi U_\phi$
3. For any charts ϕ & ψ $\phi\psi^{-1}: \psi(U_\phi \cap U_\psi) \to \phi(U_\phi \cap U_\psi)$ is C^∞.

4. The collection is maximal rel 1, 2, 3.

__Claim__ Every Atlas on M^n can be enlarged uniquely to a maximal one.

... /cont.

"a func. structured space".

Conti.

<u>Def</u> A "functional structure" on a T.S. X is an assignment $U \mapsto F_X(U)$ subalg. on the opens of X s.t.

1. $F_X(U)$ is a subalgebra of the alg. of conti. fncns.
2. $F_X(U)$ contains all constants.
3. $V \subset U$, $f \in F_X(U) \Rightarrow f|_V \in F_X(V)$
4. $U = \bigcup U_\alpha$, $f|_{U_\alpha} \in F_X(U_\alpha)\ \forall \alpha \Rightarrow f \in F_X(U)$.

(Enough to check/impose 4 on "small U_α".)

<u>Examples</u> Conti., C^∞, analytic, C^p,

<u>Claim</u> A subspace of a F.S.S. is a F.S.S.

<u>Def</u> A morphism $\phi : (X, F_X) \to (Y, F_Y)$, an isomorphism.

$$\mathcal{O} \xrightarrow{\phi} \mathcal{O} \xrightarrow{f}$$

<u>Def 2</u> an n-manifold is a second-countable Hausdorff F.S.S. which is locally isomorphic as F.S.S to $\mathbb{R}^n$.

<u>Thm</u> Def 1 & 2 are equiv.

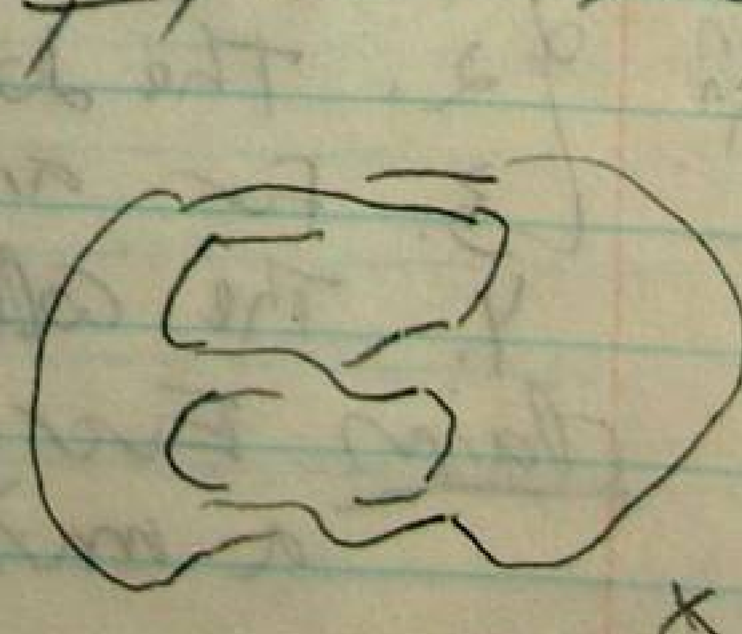
<u>Examples</u> A. S^2 in 3 ways: 1. Direct atlas
2. Induced by $p : \mathbb{R}^3 \setminus \{0\} \to S^2$
3. Induced by $i : S^2 \hookrightarrow \mathbb{R}^3$

B. $\mathbb{RP}^2$

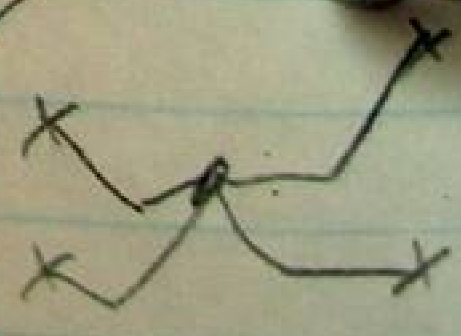
C. $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ (generalization for $n > 2$)

$S^1 = T^1$

Products; $T^n = (S^1)^n$.

<u>Asides</u> 1. 

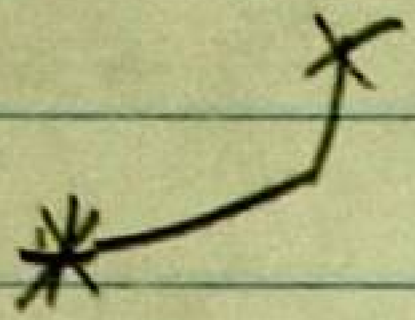
2. Glue disk along bdry

3. 

Tangent vectors, push forwards, differentials.

Math 1300 Geom & Top, Thu Sep 20 2007 – week 2.

Continue with functional structures as on Sep 18, 2007

start with 

on board: def 1.

מבוא יסודי לגאומטריה דיפרנציאלית, 20 לנובמבר 2000

Extra 1: Feb 4,6
Extra 2: Feb 11,13

עניני מנהלה: שעות קבלה, בנין ותרגילים, שיעורים נוספים. חלוקת דף כתובות

1. גישות לכן גבוהה:
״הדבקה חלקה של עותקי $\mathbb{R}^n$״
״מרחב ועליו משפחה חוץ F כי ׳חלקות׳ השקולות מקומית לחלקות על $\mathbb{R}^n$״.

וקטורים משיקים:

הגדרה $\gamma: \mathbb{R} \to M$, $\gamma(0)=p$, $f: M \to \mathbb{R}$: $D_\gamma f := \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$

הגדרה האופרטור D_γ נקרא ״וקטור משיק בנק׳ p״. $T_p(M) := \{D_\gamma\}_{\text{כל }\gamma}$

הגדרה נבט של פונקציה חלקה בנק׳ p.

תכונות D_γ מוגדר על נבטים ומקיים
(1) $D(af+bg) = aDf + bDg$
(2) $D(fg) = f(p)Dg + g(p)Df$

טענה כל אופרטור D המקיים את התכונות הנ״ל הוא וקטור משיק (ובמהלך ההוכחה נבין יותר טוב מהם וקטורים משיקים)

הלמה של Hadamard: אם f חלקה בסביבת $0 \in \mathbb{R}^n$, אזי $f(x) = f(0) + \sum x_i g_i(x)$ כאשר $g_i(x)$ חלקות.

הוכחת הטענה בעזרת הלמה (ראינו כבר את $D_\gamma f$ עבור $\gamma(t)=tx$ ואת DF באופן כללי.) (תרגיל)

מסקנה: $T_p(M)$ מרחב וקטורי n־ממדי.

הוכחת הלמה.

הגדרה $d\phi = \phi_*: T_p(M) \to T_{\phi(p)}(N)$ ע״י $\phi_*(D_\gamma) = D_{\phi\circ\gamma}$

תרגיל: 1. ϕ_* מוגדרת ומקיימת $(\phi_* D)(g) = D(g\circ\phi)$
2. ϕ_* ליניארית
3. $\phi_*\psi_* = (\phi\circ\psi)_*$

מושגי יסוד בגיאומטריה דיפרנציאלית, 12 לנובמבר 2000

1. יום ראשון ⇐ שפרינגר ; 24 ; סידור ה... מבולל.

2. וקטור משיק: א. מחלקת שקילות של מסילות דרך p, שקילות.
ב. גזירה של נקודה ב p.

3. מכפלה קרטזית ע"י כל אחת מהגדרות.

3.5 הערה: קואורדינטות מקומיות, $\frac{\partial}{\partial x_i}$

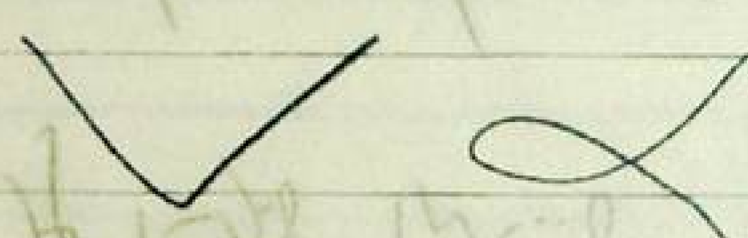
4. תכונות: Φ_* ליניארית; $(\Phi\circ\Psi)_* = \Phi_*\Psi_*$

5. איך מגדירים את Φ_* באופן אינווריאנטי?

$\Phi_* \sim \bar{\Phi}_*$

$$\begin{array}{ccc} U\subset M & \xrightarrow{\Phi} & W\subset N \\ \downarrow & & \downarrow \\ U'\subset\mathbb{R}^m & \xrightarrow{\bar{\Phi}} & V'\subset\mathbb{R}^n \end{array}$$

6. אימרסיה; דוגמאות



7. משפט: כל אימרסיה נראית מקומית כמו $\mathbb{R}^n \to \mathbb{R}^m$.
כלומר, אם $\Theta: \mathbb{R}^n \to M^m$ ו- Θ_* חח"ע בנקודה p, אזי קיימות מפות ϕ,ψ ב p ש

$$\begin{array}{ccc} N^n \supset U & \underset{p\to 0}{\xrightarrow{\phi}} & U'\subset\mathbb{R}^n \\ \Theta\downarrow & & \downarrow \sigma \quad x\mapsto (x,0)\in\mathbb{R}^n\times\mathbb{R}^{m-n} \\ M^m\supset V & \underset{\Theta(p)\to 0}{\xrightarrow{\psi}} & V'\subset\mathbb{R}^m \end{array}$$

הוכחה

$$\begin{array}{ccccccc} N^n\supset U_p & \xrightarrow{\phi_1} & U_1 & \xrightarrow{\text{הזזה}} & U_2 & = & U_3 \quad x \\ \Theta\downarrow & & \downarrow\Theta_1 & & \downarrow\Theta_2 & & \downarrow\Theta_3 \quad \downarrow \\ M^m\supset V & \xrightarrow{\psi_1} & V_1 & \xrightarrow{\text{הזזה ופונקציה ליניארית}} & V_2 & \xleftrightarrow{\Theta_3(x)} & V_3 \quad \binom{x}{0} \end{array}$$

$\zeta\binom{x}{y} = \Theta_2(x) + \binom{0}{y}$

$\Theta_{2*} T_0\mathbb{R}^n = \left\{\binom{x}{0}\right\} \subset T_0\mathbb{R}^m$

* class photo
* Math in Moscow.
* HW pingott: 1 is out, 3 is in.

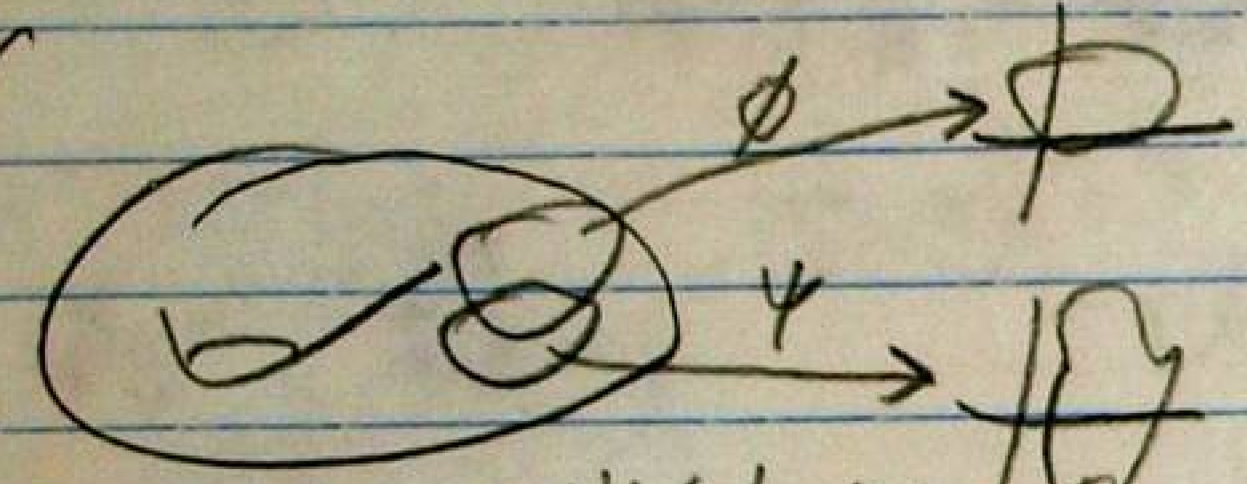
Math 1300 Geom & Top, ~~Thu~~ Tue Sep 25 2007 - Week 3

on board: two defs of manifolds equiv, need to show

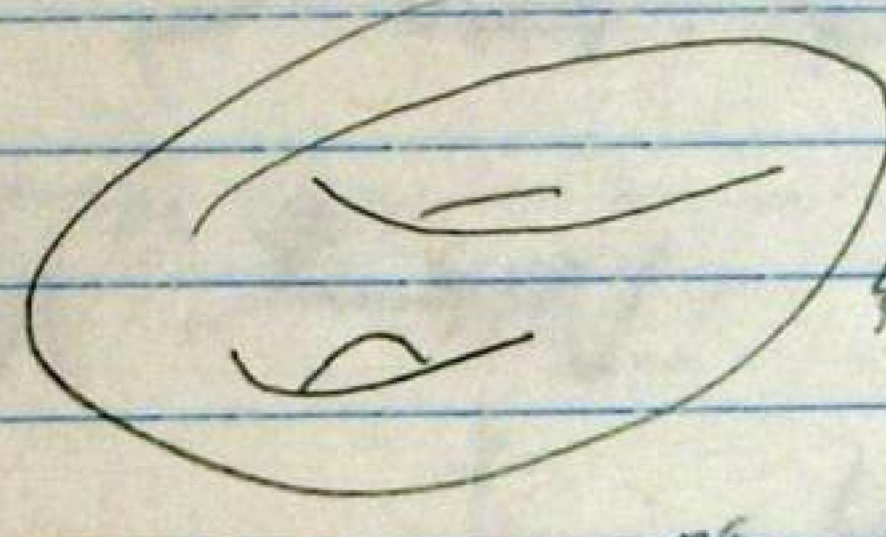
* Induced structures, examples, products
* smooth functions in two ways
* tangent vectors in two ways & their equivalence.
* pushforwards / differentials.

---

on board.



$\psi\phi^{-1}$ is smooth



Functional structure locally iso. to $\mathbb{R}^n$.

$\Longrightarrow$

F is "smooth" $\Leftrightarrow$ $F\circ\phi^{-1}$ is

$\Longleftarrow$

(First discuss induced functional structures)

Atlas: {local isos $\phi: U \to V$}

More examples: $S^2 = \mathbb{R}^3/\mathbb{R}^\times$, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, $\mathbb{C}P^n = \mathbb{C}^{n+1}/\mathbb{C}^\times$

Def smooth $\phi: M \to N$ in two ways; equivalent

Thm 1 The two defs are equivalent.

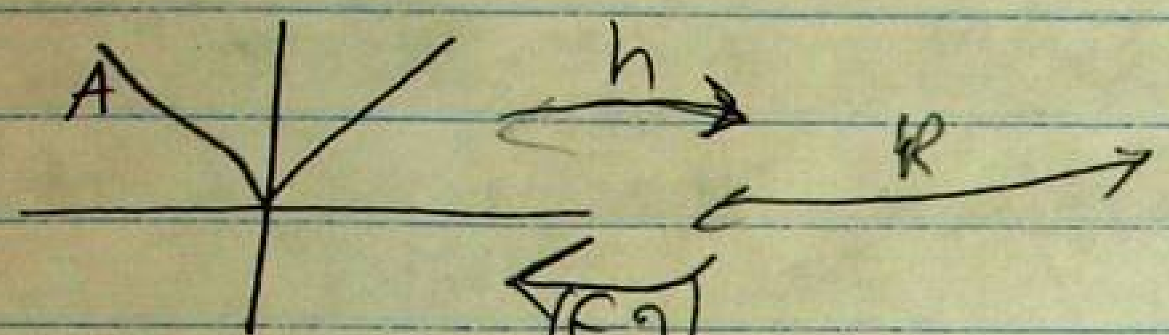
Thm 2 smooth manifolds form a category

Tangent vectors in two ways & their equiv.

Pushforwards & differentials.

Math 1300 Geom&Top, Thu Sep 27 2007, Week 3.

* Photo at 12:55
* Wiki words
  1. use history/recent changes.
  2. Uploads & new pages must be linked. (or I will remove them)
* PY page 71



is $A \cong \mathbb{R}$ as F.S.s?

complete either one.

1. (Karshon) $h \circ (f,g) = Id$
   $\Rightarrow h_x f' + h_y g' = 1$, but $f = \pm g$ and both occur
2. (Dror) a. when a car approaches a bow in the road, it must stop.
   b. When you stop, nothing moves.

3. Find two lin. indep dir. derivatives on A at (0,0)

Tangent vcts at $p \in M$

$$\left\{ \frac{\gamma : \mathbb{R} \to M \quad \gamma(0)=p}{\gamma_1 \sim \gamma_2 \text{ if } \gamma_1'(0) = \gamma_2'(0) \text{ in some chart}} \right\} \xleftrightarrow{\text{Thm}} \left\{ D : \{\text{smooth functions near } p\} \to \mathbb{R} \;\; \begin{matrix} 1. \text{ linear} \\ 2. \text{ Leibnitz} \end{matrix} \right\}$$

{key pt on $\mathbb{R}^n$: $f(p) = f(0) + \sum x_i g_i(p)$ — i.e. A function vanishing at 0 is a "linear comb" of the x_i's

$\Rightarrow$ Any D is determined by n constants (Dx_i)

$\Rightarrow$ The iso. holds.

$\Rightarrow$ $T_0\mathbb{R}^n = \{\text{all t.v. to } \mathbb{R}^n \text{ at } 0\}$ is an n-dim V.S.

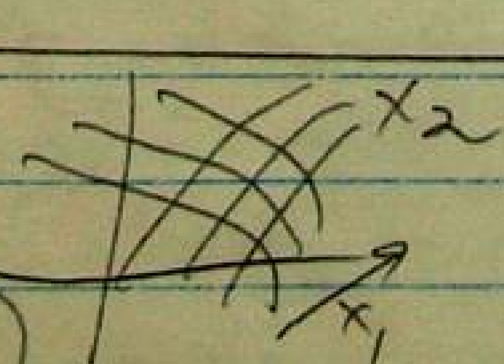
$\Rightarrow$ $T_pM^n = \{\text{all t.v. to } M \text{ at } p\}$ is an n-dim V.S.}.

Pf of Hadamard's lemma:

$$p = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad f(p) - f(0) = \int_0^1 \frac{d}{dt} f(tp)\,dt = \int_0^1 \sum x_i \frac{\partial}{\partial x_i} f(tp)\,dt$$

$$= \sum x_i \underbrace{\int_0^1 \left(\frac{\partial}{\partial x_i} f\right)(tp)\,dt}_{g_i(p)} \qquad \Rightarrow g_i(0) = \left(\frac{\partial f}{\partial x_i}\right)(0).$$

In local coordinates:



$$DF = \sum (Dx_i) \frac{\partial}{\partial x_i} F$$

$$D = \sum (Dx_i) \frac{\partial}{\partial x_i}$$

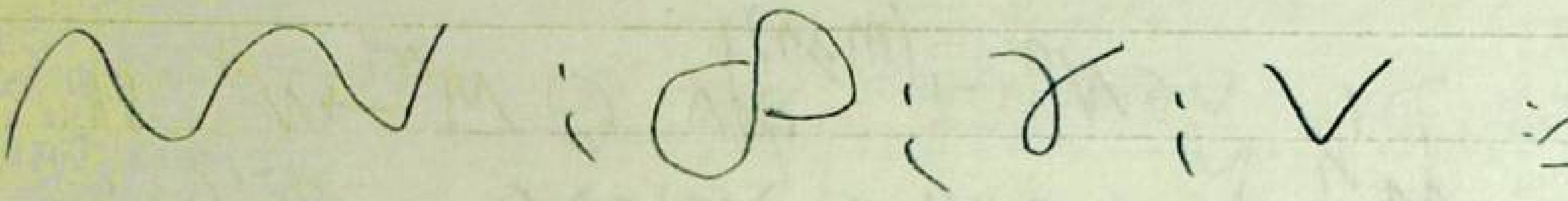
Pushforwards & the differential.

מושגי יסוד בגאומטריה דיפרנציאלית 19 לנובמבר 2000

0. תזכורת: תהי $\phi: M \to N$ אז $\phi_*: TM \to TN$; עם זיהויים נכונים ϕ_* הוא $d\phi$; אם ϕ לינארית אזי $\phi_* = \phi$.

1. הגדרות: אימרסיה: ϕ_* חח"ע
תת-יריעה: אימרסיה חח"ע
שיכון: תת-יריעה שהטופולוגיה עליה היא הטופולוגיה המושרה

2. דוגמאות:



3. המשפט על המבנה המקומי של אימרסיה, כבשיעור הקודם

4. מסקנה: המבנה הפונקציונלי על תת-יריעה משוכנת הוא המבנה הפונקציונלי המושרה.

5. הגדרה: סבמרסיה. משפט: כל סבמרסיה נראית מקומית כמו ההטלה $\mathbb{R}^m \to \mathbb{R}^n$. במדויק, אם $\theta_*: T_p(M^m) \to T_{\theta(p)}(N^n)$ הוא על בנקודה $p \in M^m$, אזי קיימות מפות ϕ, ψ כך ש-

$$\begin{array}{ccc} M^m \supset U & \xrightarrow[p \to 0]{\phi} & V' \subset \mathbb{R}^m \\ \downarrow \theta & & \downarrow \theta \\ N^n \supset V & \xrightarrow[\theta(p) \to 0]{\psi} & V'' \subset \mathbb{R}^n \end{array}$$

$(x,y) \in \mathbb{R}^n \times \mathbb{R}^k$ $(m=n+k)$
$\downarrow$
x

הוכחה.

$$\begin{array}{ccccccc} M^m \supset U & \xrightarrow{\phi_1} & V_1 & \xrightarrow{\text{הזזה}} & V_2 & \xrightarrow{\xi(x,y) = (\theta_2(x,y),y)} & V_3 \\ \downarrow \theta & & \downarrow \theta_1 & & \downarrow \theta_2 & & \downarrow \text{הטלה} \\ N^n \supset V^\theta & \xrightarrow{\psi} & V_1 & \xrightarrow{\text{הזזה}} & V_2 & = & V_4 \end{array}$$

נותר להראות ש- $\xi_* = \xi$ היא הפיכה. $\theta_{2*}: \mathbb{R}^m \to \mathbb{R}^n$ היא ההטלה.

6. הגדרה נק' רגולרית - הדרגה של ϕ_* מקסימלית
ערך רגולרי - ...

משפט סרד: כמעט כל הערכים הם רגולריים

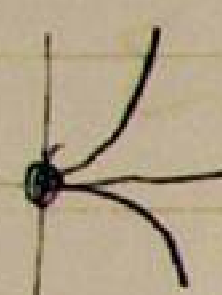
משפט אם $\phi: M^m \to N^n$ חלקה ו- $y \in N^n$ ערך רגולרי, אזי $\phi^{-1}(y)$ תת-יריעה משוכנת של M^m ממימד $m-n$.

מושגי יסוד בגאומטריה דיפרנציאלית, 21 לנובמבר 2000

<u>הגדרה</u> נק' רגולרית: דרגת ϕ_* מקסימלית.
ערך רגולרי:

<u>משפט סרד</u>: כמעט כל הערכים הם רגולריים.

<u>משפט</u>: אם $\phi: M^m \to N^n$ חלקה $(m \geq n)$ ו-$y \in N^n$ ערך רגולרי, אזי $\phi^{-1}(y)$ תת-יריעה משוכנת של M^m ממימד $m-n$.

<u>דוגמה 1</u> $x^2+y^2=1$ דוגמה 2. $x^3=y^2$ 

<u>הערה</u> למשפט יש הפוך אבל - מקומית, כל שיכון $K \hookrightarrow M$ הוא תמונה הפוכה של ערך רגולרי של העתקה חלקה.

<u>הערה</u> אם $K \hookrightarrow M$ שיכון אזי $T_pK \hookrightarrow T_pM$. יתר על כן, אם $\theta: M \to N$ ו-y ערך רגולרי ו-$K=\theta^{-1}(y)$, אזי

$$T_pK = \ker \theta_*$$

<u>הגדרה</u> חיתוך טרנסברסלי של יריעות $N_1 \pitchfork N_2$ $N_i \subset M$

<u>משפט</u> אם $N_1^{n_1} \pitchfork N_2^{n_2}$ בתוך M^m, אזי $N_1 \cap N_2$ תת-יריעה משוכנת של M^m וממימד n_1+n_2-m. מקומית, $N_1 = \mathbb{R}^{n_1} \times \{0\}$ ו-$N_2 = \{0\} \times \mathbb{R}^{n_2}$

<u>דוגמה</u>

*photo
*Questionnaire.
*HW1

Math 130 Geom & Top, Tue Oct 2 2007, Week 4

Push Forward/Pullbacks

$X \longrightarrow Y$

pts, paths, funct. str., Tangent V.s. — fncns,

$(\phi_* D)f = D\phi^* f$ $\quad$ $\phi_*[\gamma] := [\phi_*\gamma]$; $D_\gamma f = \frac{d}{dt}(f\circ\gamma)|_{t=0}$

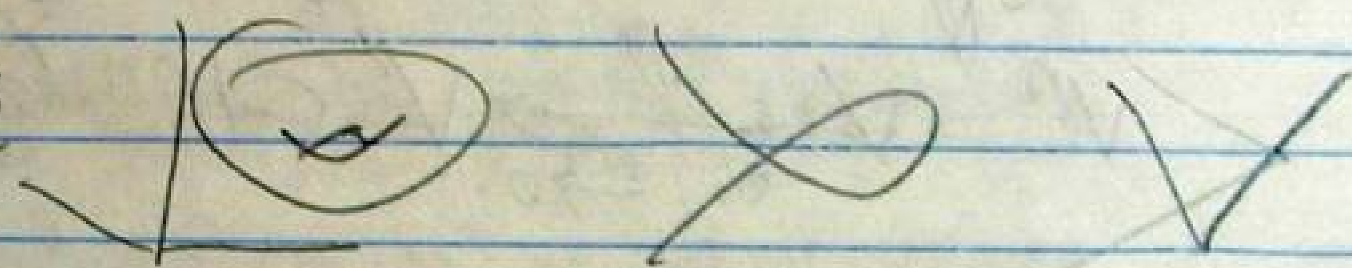
$(D_{\phi\circ\gamma}) \overset{?}{=} (\phi_* D_\gamma)$ $\quad$ $\frac{d}{dt}f(\phi(\gamma(t))) = \frac{d}{dt}(f\circ\phi)(\gamma(t))$

Properties: 1. Linearity

2. Functoriality: $\phi_* \circ \psi_* = (\phi\psi)_*$] on $\mathbb{R}^n$, $\phi_* = d\phi$ & this is the chain rule!

Definition Immersion: $\phi: N^n \to M^m$ s.t. ϕ_* is 1-1.

Examples

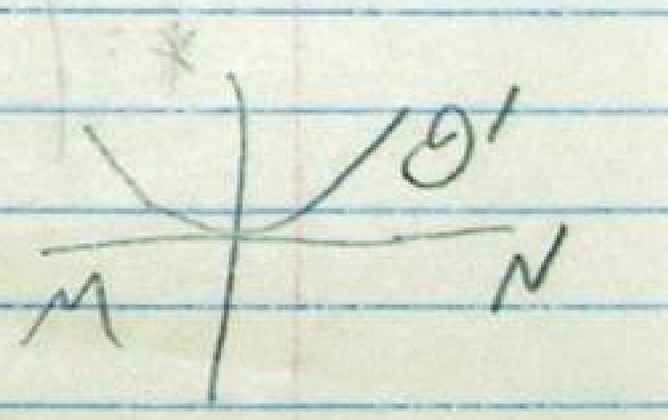


$\lambda(x) = e^{-1/x^2}$ for $x>0$, 0 for $x \le 0$

$x \mapsto (\text{sign}\,x \cdot \lambda(x), \lambda(x))$

Thm Locally, every immersion looks like $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$ $(n \le m)$. Precisely, if $\phi: N^n \to M^m$ & ϕ_* is 1-1 @ p, then $\exists$ charts ϕ, ψ s.t.

$$\begin{array}{ccc} N^n \supset U & \xrightarrow[p\to 0]{\phi} & U' \subset \mathbb{R}^n \\ \downarrow \theta & & \theta' \downarrow \quad \begin{matrix} x \\ \downarrow \\ (x,0) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \end{matrix} \\ M^m \supset V & \xrightarrow[\theta(p)\to 0]{\psi} & V' \subset \mathbb{R}^m \end{array}$$



To do.

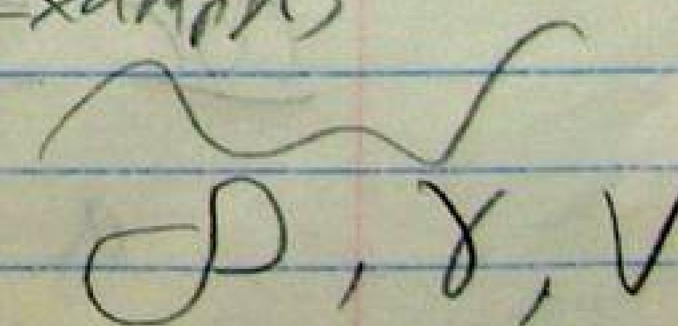
Pf

$$\begin{array}{ccccc} N^n \supset U_0 & \xrightarrow[p\to 0]{\phi} & U_1 & \longrightarrow & U_2 = U' \\ \downarrow \theta & & \downarrow \theta_1 & & \\ M^m \supset V_0 & \xrightarrow[\substack{\theta(p)\to 0 \\ \theta_* T_p N \to \mathbb{R}^n}]{\psi} & V_1 & \longrightarrow & V_2 = V' \end{array}$$

Def Submanifold: 1-1 immersion

Embedding: submanifold whose top is the induced top.

Examples



Cor: The functional structure on an embedded submanifold is the induced structure.

~~Review Proof~~

Pf

... cont.

Cont:

The story for submersions.

Def. Submerssion.

Thm A submerssion looks locally like the projection $\mathbb{R}^m \to \mathbb{R}^n$. Precisely, if $\theta_*: T_p(M^m) \to T_{\theta(p)} N^n$ is onto at $p \in M^m$, then $\exists$ charts ϕ, ψ s.t.

$$\begin{array}{ccccc} M^m \supset U & \xrightarrow[p \mapsto 0]{\phi} & U' \subset \mathbb{R}^m & & (x,y) \in \mathbb{R}^n \times \mathbb{R}^k \quad m = n+k \\ \downarrow \theta & & \downarrow \theta' & & \downarrow \\ N^n \supset V & \xrightarrow[\theta(p) \to 0]{\psi} & V' \subset \mathbb{R}^n & & x \end{array}$$

Pf

$$\begin{array}{ccccc} M^m \supset U & \xrightarrow[p \mapsto 0]{\phi_1} & V_1 & \xrightarrow[(\theta_1(x,y),\, y)]{\xi(x,y) =} & U_2 \\ \downarrow \theta & & \downarrow \theta_1 = \psi_1 \circ \theta \circ \phi_1^{-1} & & \downarrow \text{Proj.} \\ N^n \supset V & \xrightarrow[\theta(p) \to 0]{\psi_1} & V_1 & \longrightarrow & V_2 \end{array}$$

$* = \left(\ ',\ ', \ 0 \right)$

Def. A regular point p: $\operatorname{rank} \theta_p$ is maximal.
Regular value: y s.t. every $p \in \theta^{-1}(y)$ is a reg pt. } Sard's thm: There are plenty.

Cor If $\theta: M^m \to N^n$ is smooth & $y \in N^n$ is a reg. val, then $\theta^{-1}(y)$ is an embedded submanifold of M^m of dim $m-n$.

Math 1300 Geom & Top, Thu Oct 4 2007, hour 12

TE1 on Thu Nov 8 at 6PM ? (on Tue Y arrives at 7pm)

Mon Nov 5 –"– 2°

HW2 / Questionnaire

Outside info

<u>Theme</u> Locally things look like their differentials.

Immersions:

$d\theta_p$ is 1-1

$$\begin{tikzcd} N^n \supset U \arrow[r, "\phi"] \arrow[d, "\theta"] & U' \subset \mathbb{R}^n \arrow[d, hook, "\iota"] \\ M^m \supset V \arrow[r, "\psi"] & V' \subset \mathbb{R}^m \end{tikzcd}$$

Submersion:

$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \Big| 0 \Big) = d\theta_p$ is onto

$$\begin{tikzcd} M^m \supset U \arrow[r, "\phi"] \arrow[d] & U_1 \subset \mathbb{R}^m \arrow[d, "\pi = \left(\begin{smallmatrix}1&&\\&1&\\&&1\end{smallmatrix} \middle| 0\right)"] \\ N^n \supset V \arrow[r, "\psi"] & V_1 \subset \mathbb{R}^n \end{tikzcd}$$

$(x,y) \in \mathbb{R}^n \times \mathbb{R}^{m-n}$
$\downarrow$
x

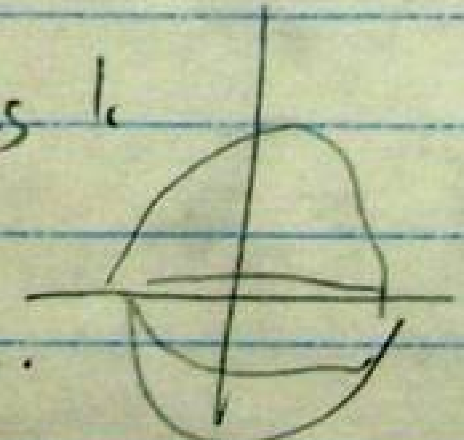
<u>Pf</u> as last time

<u>Def</u> Reg/critical pts $\theta: M^m \to N^n$

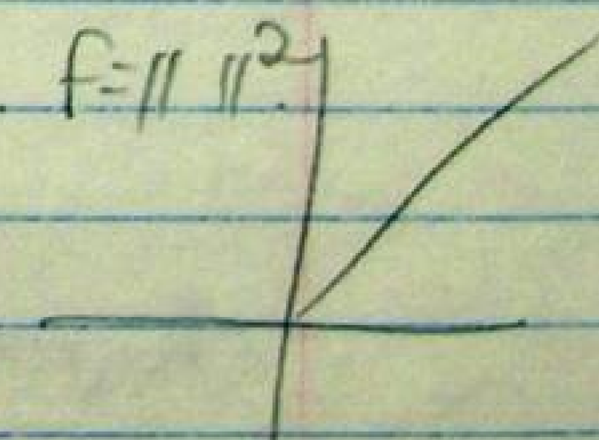
Reg: rank $d\theta_p$ is as large as it can be n.

reg/critical value.

examples 1. squash a sphere.



2. $f = \| \|^2$



Sards theorem.

<u>Cor.</u> If $\theta: M^m \to N^n$ is smooth and $y \in N^n$ is a reg. val., then $\theta^{-1}(y)$ is an embedded submanifold of M^m of dim $m-n$.

Math 1300 Geom & Top, Tue Oct 9 2007, hours 13-14.

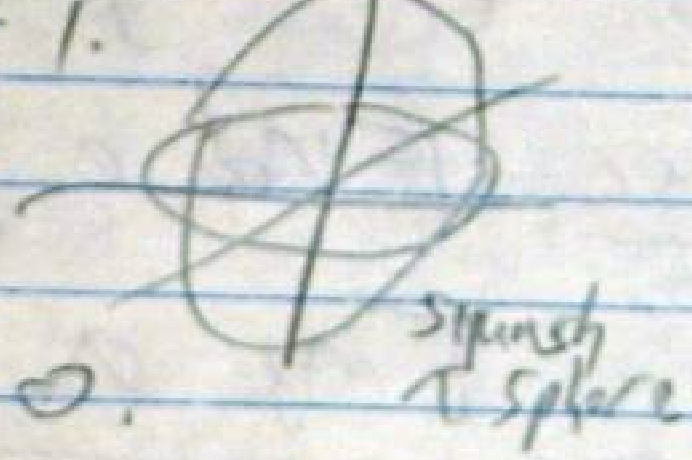
Reminder: An immersion locally looks like $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$, $x \mapsto (x,0)$

A submersion —//— $\mathbb{R}^m \to \mathbb{R}^n$, $(x,y) \mapsto x$

Def Reg/Crit pts of $\theta: M^m \to N^n$ (reg: rank $d\theta_p = n$)

reg/crit value

Examples 1.



Sard's theorem The set of critical values is of meas. 0.

Cor If $\theta: M^m \to N^n$ is smooth and $y \in N^n$ is a reg. val, then $\theta^{-1}(y)$ is an embedded submanifold of M^m of dim $= m-n$.

(moral: locally a submanifold is both an image and a kernel)

Def N_1, N_2 embedded submanifolds of M.
N_1 & N_2 "transverse" at $p \in N_1 \cap N_2$ if $T_p(N_1) + T_p(N_2) = T_p(M)$.
N_1 & N_2 are "transversal", $N_1 \pitchfork N_2$, if they are transverse at every $p \in N_1 \cap N_2$.

Thm If $N_1^{n_1} \pitchfork N_2^{n_2}$ in M^m, then $N_1 \cap N_2$ is a submanifold of M^m of $\dim(N_1 \cap N_2) = n_1 + n_2 - m$; locally, $N_1 = \mathbb{R}^{n_1} \times \{0\}$ and $N_2 = \{0\} \times \mathbb{R}^{n_2}$

Example as on other side.

Pf. 1. Submanifolds: $N_i \cap U = \phi_i^{-1}(0)$ $\phi_i : U^m \to \mathbb{R}^{m-n_i}$

rank $d(\phi_1 \times \phi_2)$ is maximal

2. $N_1 \cap N_2 = \Psi(\mathbb{R}^{n_1+n_2-m})$; where $\Psi: V \to \mathbb{R}^{n_1+n_2-m}$; consider $\theta: V \to \mathbb{R}^m$

$\theta(x) := (\phi_2(x), \Psi(x), \phi_1(x))$

.../cont.

Sard's theorem

Ⓐ A smooth function $f: \mathbb{R} \to \mathbb{R}$ w/ critical values <u>a</u> cantor set.

<u>Q1</u> Can you get <u>every</u> Cantor set?

<u>Q2</u> Can you get <u>the</u> cantor set?

Prove "baby Sard" for $f: \mathbb{R} \to \mathbb{R}$

<u>Exercise</u> Find a C^1 $f_2 : \mathbb{R}^2 \to \mathbb{R}$ w/ crit values $= \mathbb{R}$.

Find a C^2 $f_3 : \mathbb{R}^3 \to \mathbb{R}$ w/ —"—

Pf of Sard's theorem.

<u>Example for transversality</u>

$$M = \mathbb{C}^3 \setminus \{0\} \quad (6d)$$

$$N_1 = V = [z_1^3 + z_2^2 + z_3^2 = 0] \quad 4d$$

$$N_2 = S^5 = [|z_1|^2 + |z_2|^2 + |z_3|^2 = 1] \quad 5d.$$

$N_1 \pitchfork N_2$ is the 3d $L(3,1)$.

$N_1 \pitchfork N_2$ because of $\gamma_t = (t^2 z_1, t^3 z_2, t^3 z_3)$

0708-1300/Class notes for Thursday, October 11

From Drorbn

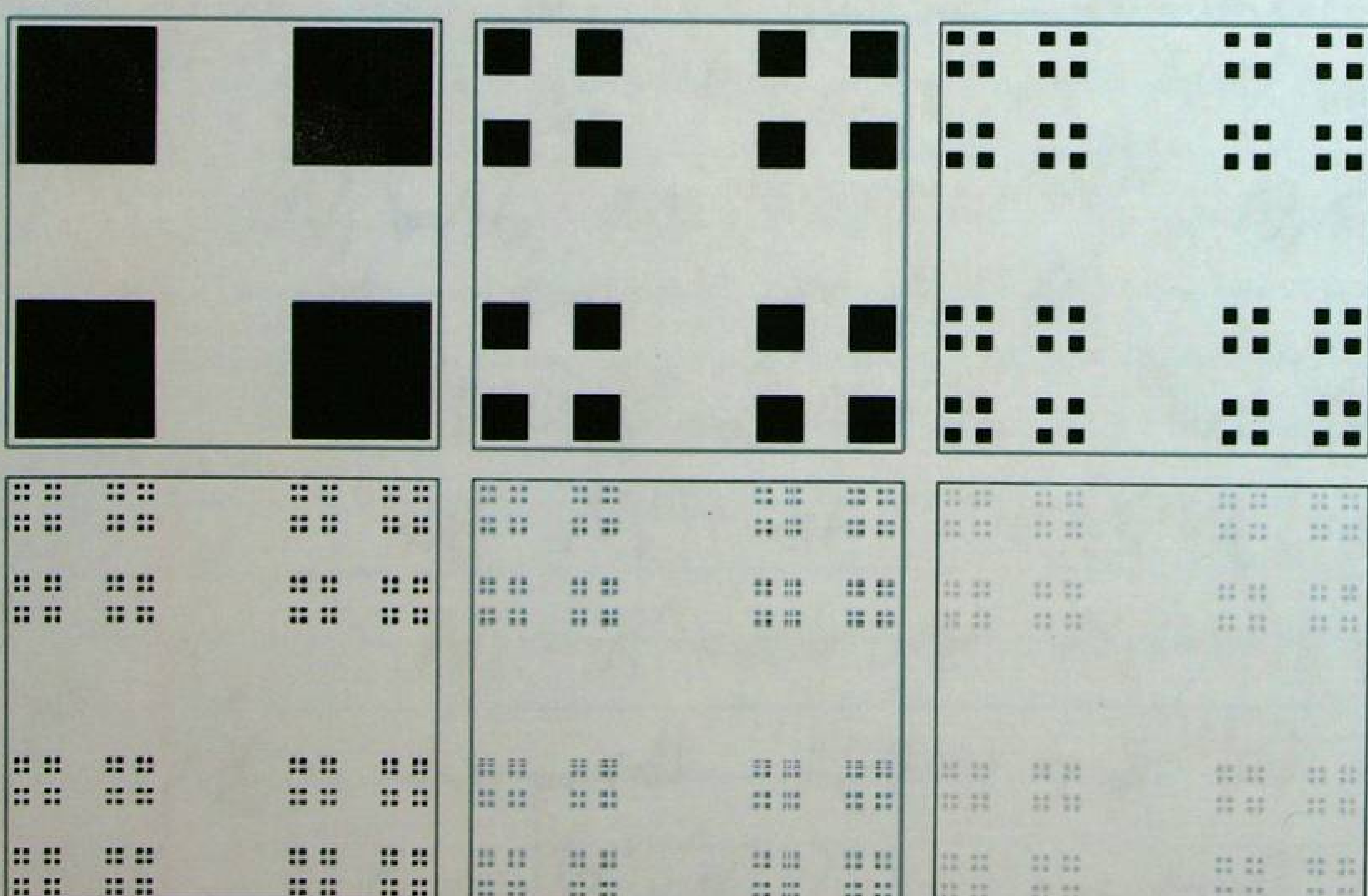
Today's Agenda: Intro to Sard

- The Cantor set C.
- Sets of measure zero.
- C is of measure zero.
- But $C+C$ is $[0,2]$ (in two ways).
- Thick Cantor sets.
- "Measure zero" is a smooth invariant.
- Measure 0 makes sense on manifolds.
- Baby Sard for C^1 functions $f:\mathbb{R}\to\mathbb{R}$:
  - Such an f with a Cantor set of singular points and values.
  - And yet a proof.
- A counterexample for Sard for C^1 functions $f:\mathbb{R}^2\to\mathbb{R}$.

The Cantor Aerogel

```
In[1]:= GraphicsGrid[Partition[Table[
    ArrayPlot[Table[
      If[MemberQ[IntegerDigits[x, 3]~Union~IntegerDigits[y, 3], 1], 0, 1],
      {x, 0, 3^n-1}, {y, 0, 3^n-1}
     ]],
    {n, 1, 6}
   ], 3]]
```

Out[1]=



Strange as it may seem, the faded and barely visible Cantor aerogel in the square at the bottom right of the image above is still thick enough to block all diagonal light rays.

Proof:

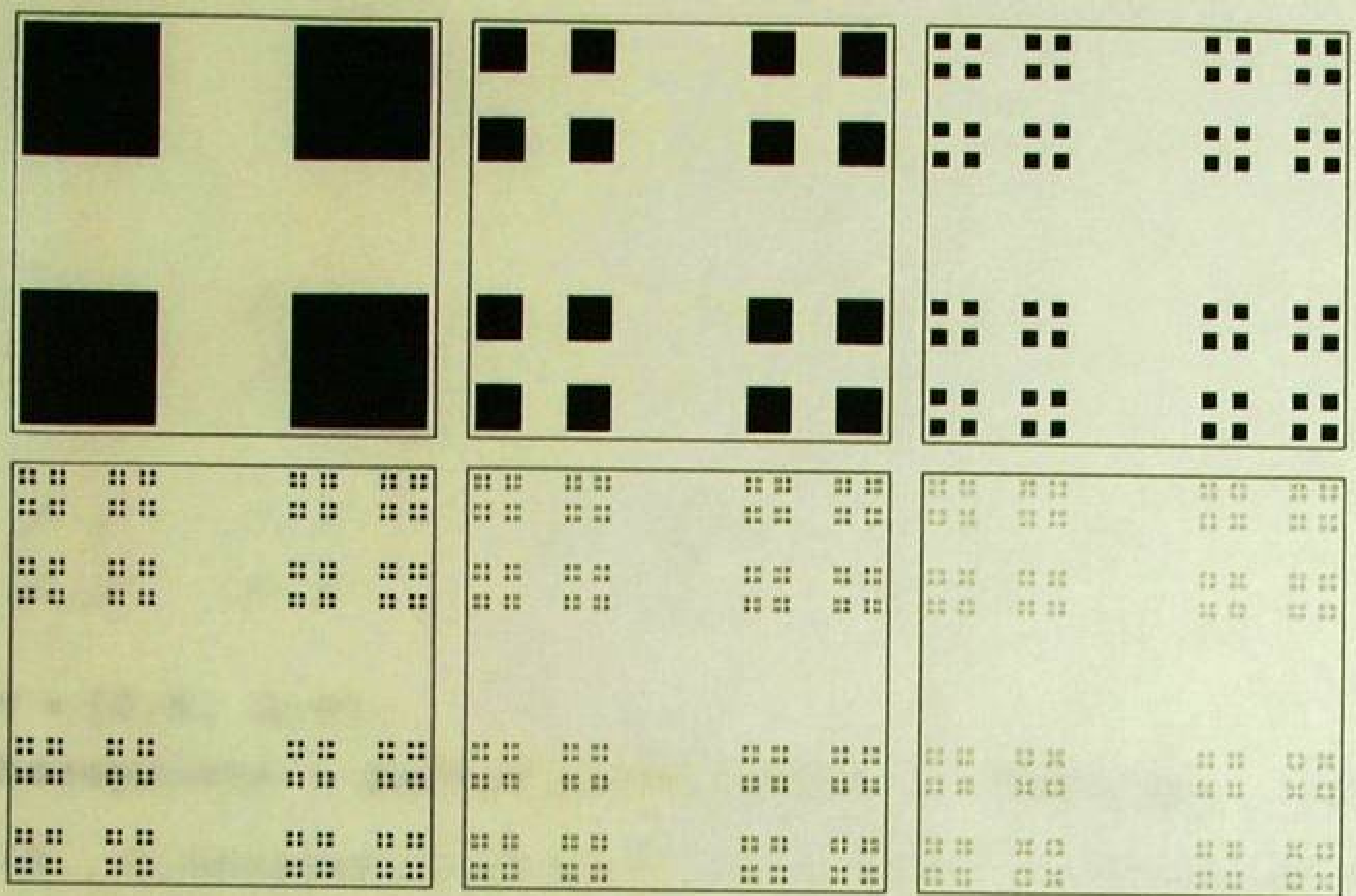
TEI: Thu Nov 8, 6PM, SS 1084.

Math 1300 Geom & Top, Thu Oct 11 2007, hour 15.

* An extra tiny on transversality.
* Intro. to Sard.

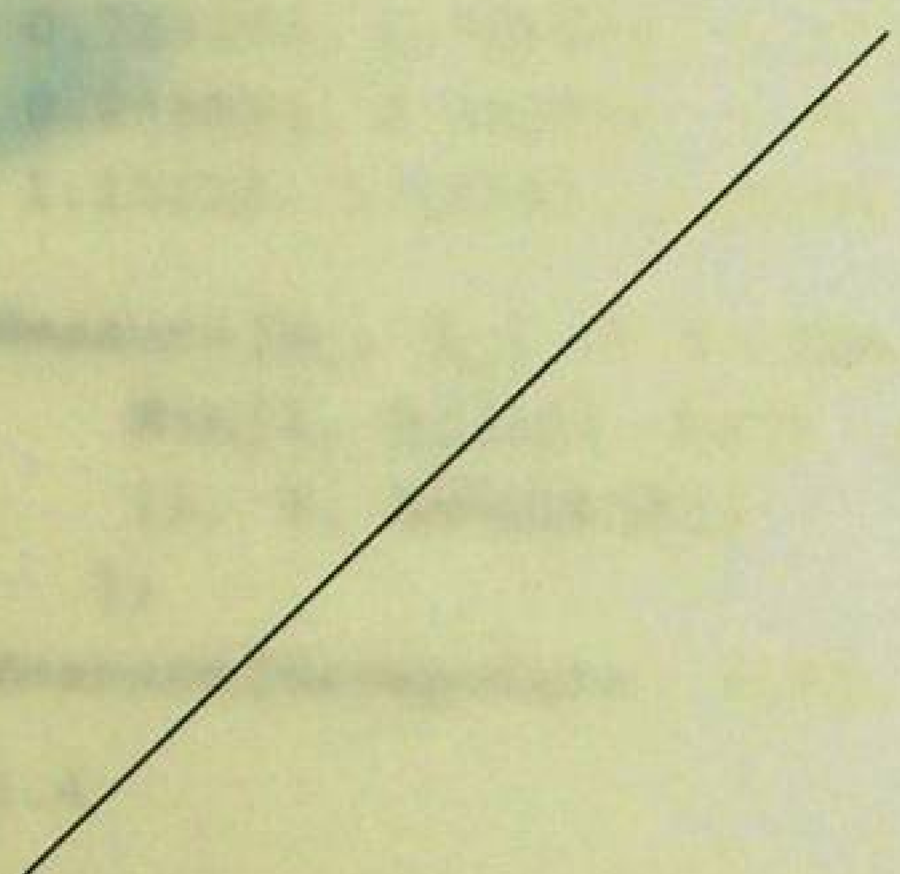
In[1]:=
```
GraphicsGrid[Partition[Table[
   ArrayPlot[Table[
     If[MemberQ[IntegerDigits[x, 3] ~Union~ IntegerDigits[y, 3], 1], 0, 1],
     {x, 0, 3^n - 1}, {y, 0, 3^n - 1}
    ]],
   {n, 1, 6}
  ], 3]]
```

Out[1]=



In[2]:=
```
CC[0] = Line[{{0, 0}, {1, 1}}];
Graphics[CC[0]]
```

Out[3]=

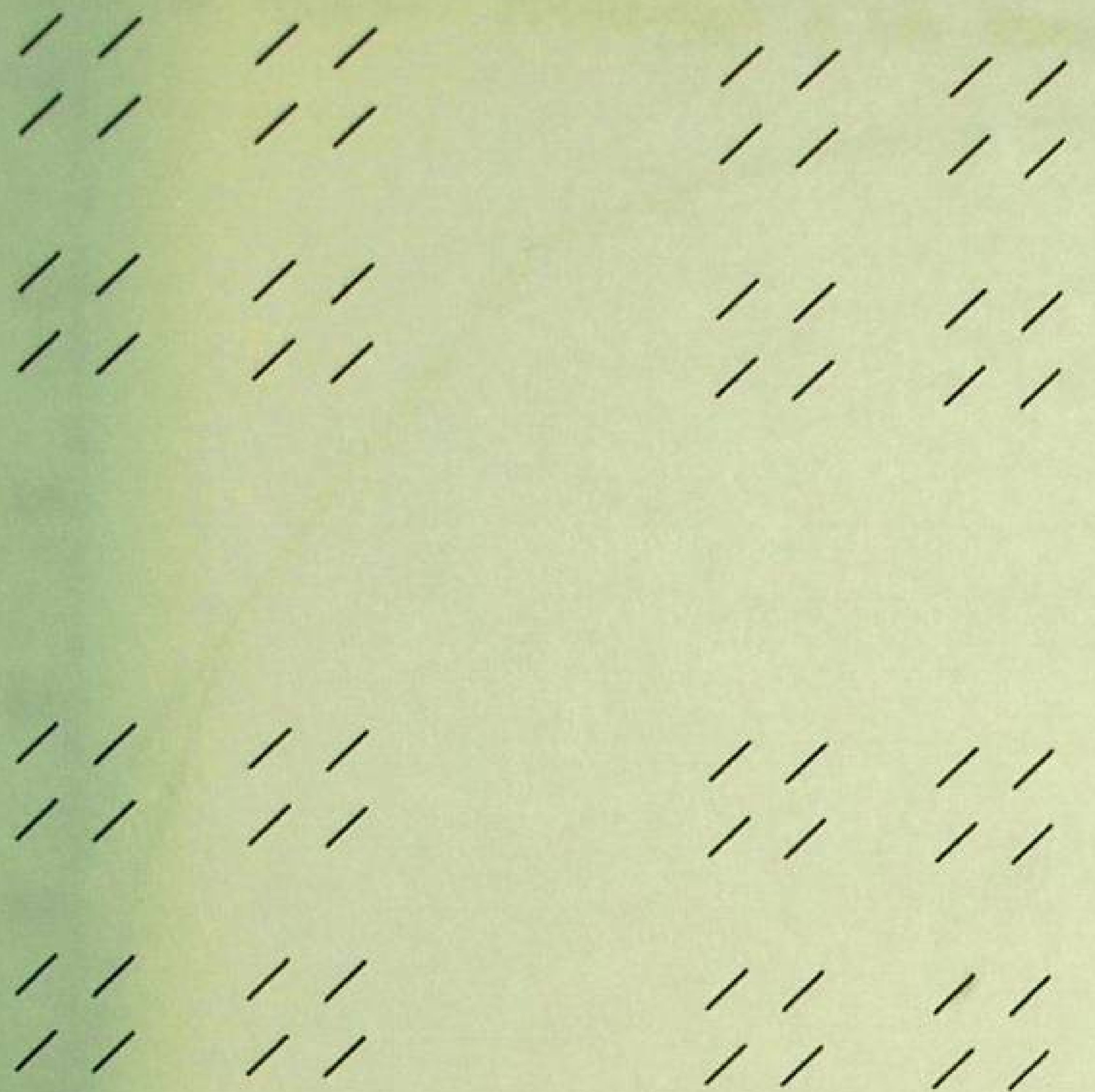


In[4]:=
```
Evolve[Line[{a_, b_}]] := Module[{v, n},
   v = b - a; n = {-v[[2]], v[[1]]};
   {
    Line[{a, a + v / 3}], Line[{a + 2 v / 3, b}],
    Line[{a + v / 3 + n / 3, a + 2 v / 3 + n / 3}],
    Line[{a + v / 3 - n / 3, a + 2 v / 3 - n / 3
     }]
   }];
CC[n_] := CC[n] = CC[n - 1] /. l_Line :> Evolve[l];
CC[1]
```

Out[6]= $\left\{\text{Line}\left[\left\{\{0,\ 0\},\ \left\{\frac{1}{3},\ \frac{1}{3}\right\}\right\}\right],\ \text{Line}\left[\left\{\left\{\frac{2}{3},\ \frac{2}{3}\right\},\ \{1,\ 1\}\right\}\right],\ \text{Line}\left[\left\{\left\{0,\ \frac{2}{3}\right\},\ \left\{\frac{1}{3},\ 1\right\}\right\}\right],\ \text{Line}\left[\left\{\left\{\frac{2}{3},\ 0\right\},\ \left\{1,\ \frac{1}{3}\right\}\right\}\right]\right\}$

In[7]:= `Graphics[CC[3]]`

Out[7]=



In[8]:=
```
v = {0.6, 0.8};
basepoints = Sort[Flatten[CC[3] /. Line[{a_, _}] :> a.v]]
```

Out[9]= {0., 0.0444444, 0.0592593, 0.103704, 0.133333, 0.177778, 0.177778, 0.192593, 0.222222, 0.237037, 0.237037, 0.281481, 0.311111, 0.355556, 0.37037, 0.4, 0.414815, 0.444444, 0.459259, 0.503704, 0.533333, 0.533333, 0.577778, 0.577778, 0.577778, 0.592593, 0.592593, 0.622222, 0.637037, 0.637037, 0.637037, 0.666667, 0.681481, 0.711111, 0.711111, 0.711111, 0.725926, 0.755556, 0.755556, 0.77037, 0.77037, 0.77037, 0.814815, 0.814815, 0.844444, 0.888889, 0.903704, 0.933333, 0.948148, 0.977778, 0.992593, 1.03704, 1.06667, 1.11111, 1.11111, 1.12593, 1.15556, 1.17037, 1.17037, 1.21481, 1.24444, 1.28889, 1.3037, 1.34815}

In[10]:=
```
Measure[b_, l_] := l + Sum[
    Min[l, b[[i]] - b[[i - 1]]],
    {i, 2, Length[b]}
   ];
Measure[basepoints, v.{1, 1} / 3^3]
```

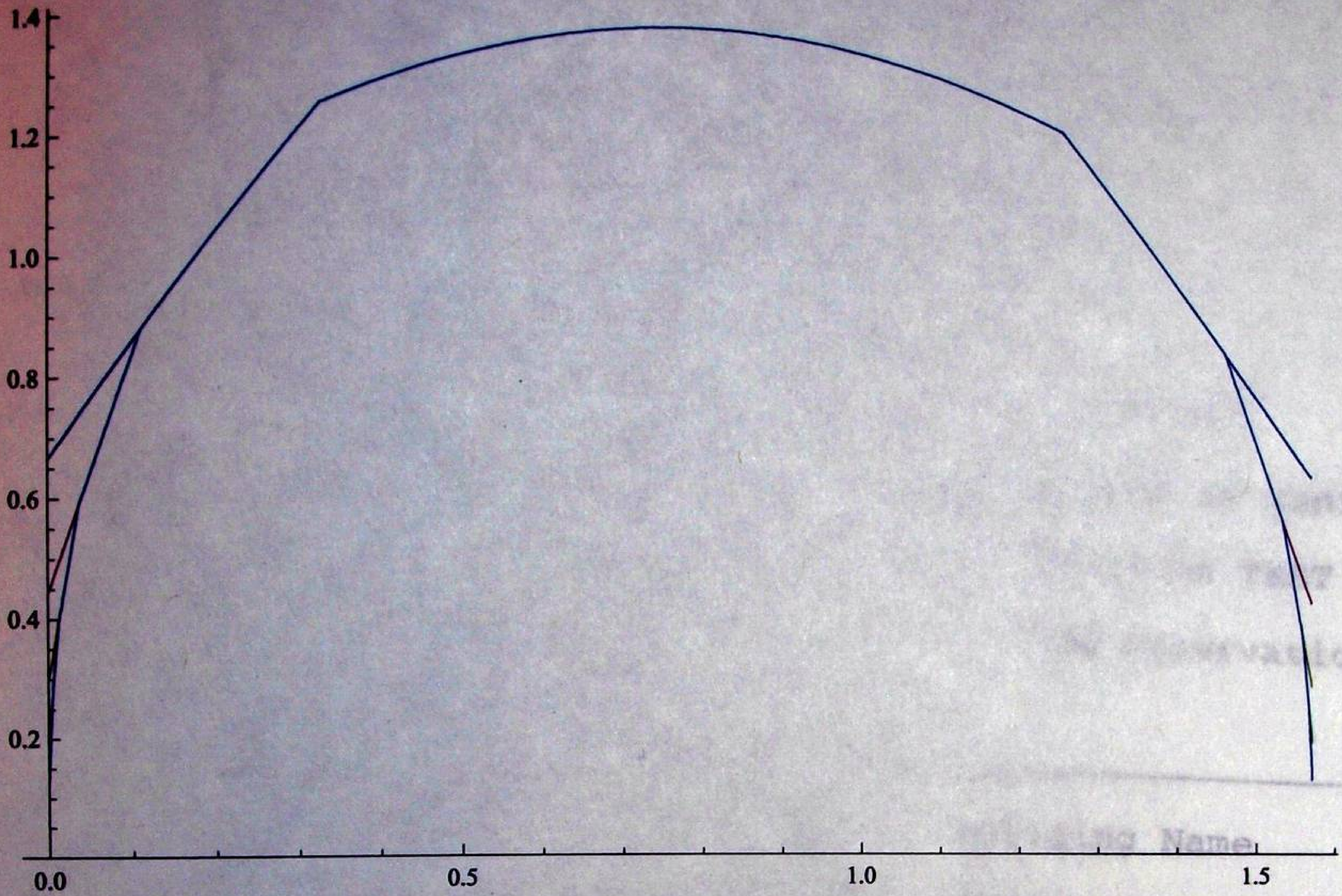
Out[11]= 1.4

In[12]:=
```
CCShadow[n_, t_] := Module[{v, basepoints},
   v = {Cos[t], Sin[t]};
   basepoints = Sort[Flatten[CC[n] /. Line[{a_, _}] :> a.v]];
   Measure[basepoints, v.{1, 1} / 3^n]
  ];
{CCShadow[3, 0], CCShadow[3, Pi / 4]}
```

Out[13]= $\left\{\frac{8}{27}, \sqrt{2}\right\}$

```
In[14]:= Plot[{CCShadow[1, t], CCShadow[2, t], CCShadow[3, t], CCShadow[4, t], CCShadow[5, t]},
  {t, 0, Pi / 2}, PlotRange → {0, Sqrt[2]}]
```

Out[14]=



תרגול יסוד בגיאומטריה דיפרנציאלית, 26 לנובמבר 2000

0. תזכורת על המחלק

<u>משפט</u> אם $\phi: M^m \to N^n$ חלקה ו- $y \in N^n$ ערך רגולרי ($\exists x \in f^{-1}(y)$ $\mathrm{rank}\, d\phi_x = n$) אז $\phi^{-1}(y)$ תת-יריעה משוכנת של M^m שמימדה $m-n$.

<u>הוכחה</u> לכל $x \in \phi^{-1}(y)$ קיימות $M \supset U_x \xrightarrow[x \mapsto 0]{\phi_x} U'_x \subset \mathbb{R}^m$ ו- $N \supset V_x \xrightarrow[y \mapsto 0]{\psi_x} V'_x \subset \mathbb{R}^n$ עם $\phi: U_x \to V_x$, $\pi: U'_x \to V'_x$

$U_x \cap \phi^{-1}(y) = \phi_x^{-1}(\mathbb{R}^k \cap U'_x)$

$\pi: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$

$k = m-n$

ולכן $\eta_x = \phi_x|_{U_x \cap \phi^{-1}(y)} : U_x \cap \phi^{-1}(y) \to \mathbb{R}^k \cap U'_x$ הומאומורפיזם.

נקח את $\{\eta_x\}_{x \in \phi^{-1}(y)}$ כאטלס עבור $\phi^{-1}(y)$. צריך להראות שלכל x_1, x_2, $\eta_{x_1} \circ \eta_{x_2}^{-1}$ חלקה...

המשך בשיעור הקרוב!

<u>דוגמא:</u> $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus 0 \mid z_1^3 + z_2^2 + z_3^2 = 0\}$ יריעה 4 מימדית.

$S = S^5 = [\,|z_1|^2 + |z_2|^2 + |z_3|^2 = 1\,]$ יריעה 5 מימדית.

אם $S \pitchfork V$, אז $S \cap V$ יריעה 3 מימדית (שזה דומה לטענה דבר שכבר ראינו)

<u>טענה:</u> אכן $S \pitchfork V$.

<u>הוכחה</u> $(z_1, z_2, z_3) \mapsto (2z_1, 3z_2, 3z_3)$

<u>משפט</u> אם M יריעה חלקה ו- $\{U_\alpha\}$ כיסוי פתוח, אז קיים פירוק יחידה חלק $\{f_\beta\}$ הכפוף ל- $\{U_\alpha\}$

מושגי יסוד בגאומטריה דיפרנציאלית - משפט וויטני - 3/12/00

"כל יריעה חלקה M^n ניתנת לשיכון כתת-יריעה סגורה של $\mathbb{R}^{2n+1}$"

<u>שלב I</u>: קומפקטית $\Leftarrow$ $\mathbb{R}^N$ רעיון: נכסה את מקומינו על פני היריעה בעזרת מספר סופי של מפות, ומחליפים אותה במקום של האטלס לפונות בעזרת "פונקציות מציינות".

<u>שלב II</u>: קומפקטית $\Leftarrow$ $\mathbb{R}^{2n+1}$ רעיון: שימוש במשפט סרד כדי להטיל ולרדת במימד מרחב המטרה. צריך לשמור על חח"ע (2n+1 ממדים) ולהימנע מ- TM (2n ממדים).

<u>שלב III</u>: כללית $\Leftarrow$ $\mathbb{R}^N$ רעיון: מוצאים "פונקציות נאותות" ומשתמשים בה כדי לוודא שהחלקים המרוחקים של היריעה יהיו משוכנים הרחק.

<u>שלב IV</u>: כללית $\Leftarrow$ $\mathbb{R}^{2n+1}$ טכני.

<u>תזכורת</u>: 1. אם U_α כיסוי פתוח של M^n אזי יש כיסוי פתוח V_α עם $\overline{V_\alpha} \subset U_\alpha$ $\forall \alpha$

2. גרסה חלקה למשפט טיצה (Tietze) אם $K \subset M$ סגורה ו- $f: K \to \mathbb{R}$ חלקה, אזי ל- f יש הרחבה חלקה $f: M \to \mathbb{R}$

3. הגדרת היריעה TM

4. משפט סרד

5. קיום פונקציה חלקה נאותה; "פונקציות נאותות": כלי: שהתמונה ההפוכה של קומפקטית היא קומפקטית.

6. פונקציה נאותה על מרחב קומפקטי היא סגורה.

<u>משפט</u> יהי $\{U_\alpha\}$ כיסוי פתוח של M^n. אזי יש פירוק יחידה חלק $\{f_\beta\}$ הכפוף ל- $\{U_\alpha\}$. דהיינו: 1. יש עידון סופי מקומית $\{V_\beta\}$ של $\{U_\alpha\}$ ופ. חלקות $f_\beta: M \to [0,1]$ עם $\operatorname{supp} f_\beta \subset V_\beta$ $\forall \beta$

2. $\sum f_\beta(x) = 1$ לכל $x \in M$.

מושגי יסוד בגאומטריה דיפרנציאלית 28 לנובמבר 2000

1. תלות ד"פ על טרנסוורסליות

2. וויטני עבור קומפקטית לתוך $\mathbb{R}^{N}$

3. וויטני עבור קומפקטית לתוך $\mathbb{R}^{2n+1}$

4. וויטני עבור כללית לתוך $\mathbb{R}^{4n+3}$:

5. "כל יריעה ח"מ ניתנת לשיכון כתת-יריעה סגורה של $\mathbb{R}^{2n+1}$"

<u>חזרות</u> (חלקן יוחזרו)

1. אם U_α כיסוי פתוח של ח"מ M אזי יש כיסוי פתוח V_α עם $\bar{V}_\alpha \subset U_\alpha$ $\forall\alpha$
(נכון לכל מרחב פאראקומפקטי, וכל מרחב האוסדורף קומפקטי מקומי בα_2 הוא כזה)

2. גרסה חלקה למשפט טיצה (Tietze)
אם $K \subset M$ סגורה ו $f: K \to \mathbb{R}$ חלקה, אזי ל f הרחבה חלקה $\bar{f}: M \to \mathbb{R}$

3. עוד על TM

4. משפט סרד

5. קיום פונקציה נאותה "פונקציית מיצוי".

6. פ' נאותה על מרחבים קומפקטיים מקומית היא סגורה.

<u>משפט</u> יהי $\{U_\alpha\}$ כיסוי פתוח של יריעה חלקה ח"מ. אז יש פירוק יחידה חלק $\{f_\beta\}$ הכפוף ל$\{U_\alpha\}$, דהיינו

1. יש עידון $\{V_\beta\}$ (סופי מקומית) של $\{U_\alpha\}$ ופונקציות חלקות $f_\beta: M \to [0,1]$ עם $\operatorname{supp} f_\beta \subset V_\beta$ $\forall\beta$

2. $\sum_\beta f_\beta(x) = 1$ לכל $x \in M$.

<u>טענה</u> נשלים את חוב 2 ותוך 5.

Math 1300 Geom & Top, Tue Oct 16 2007, hours 16–17

1. Projections of $\mathbb{C}\times\mathbb{C}$.
2. $F(a\, G_1) = $ the G_1
3. Proof of Sard:

On board:
$f: M^m \to N^n$ smooth:
$\Rightarrow$ crit values of f are of measure 0.

* Enough to argue locally.

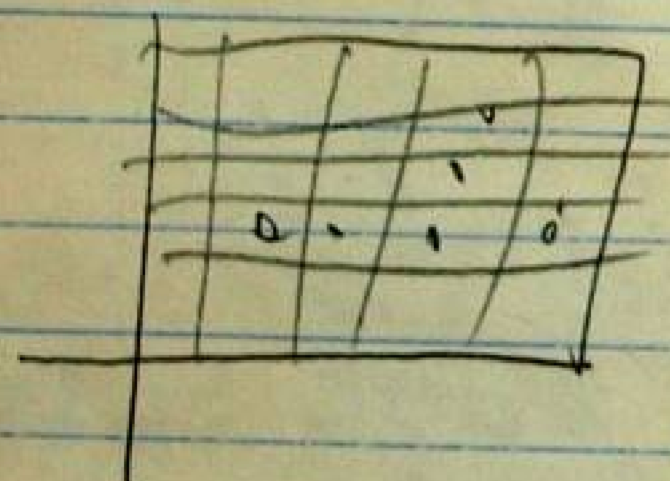
Set $D_k = \{p : \text{all partial derivatives of } f \text{ of order} \leq k \text{ vanish}\}$

and $D_0 = \{p : df_p \text{ is not onto}\} = \text{crit pts}(f)$

Then $D_0 \supset D_1 \ldots \supset D_m$

WLOG $f: \mathbb{R}^m \to \mathbb{R}^n$

Claim $f(D_m)$ has measure 0.



Recall Taylor:
$$g(x) = \sum_{j=0}^{m} g^{(j)}(x_0)\frac{(x-x_0)^j}{j!} + \frac{g^{(m+1)}(t)}{(m+1)!}\cdot(x-x_0)^{j+1}$$

Claim $f(D_k - D_{k+1})$ has measure 0 for $1 \leq k \leq m$

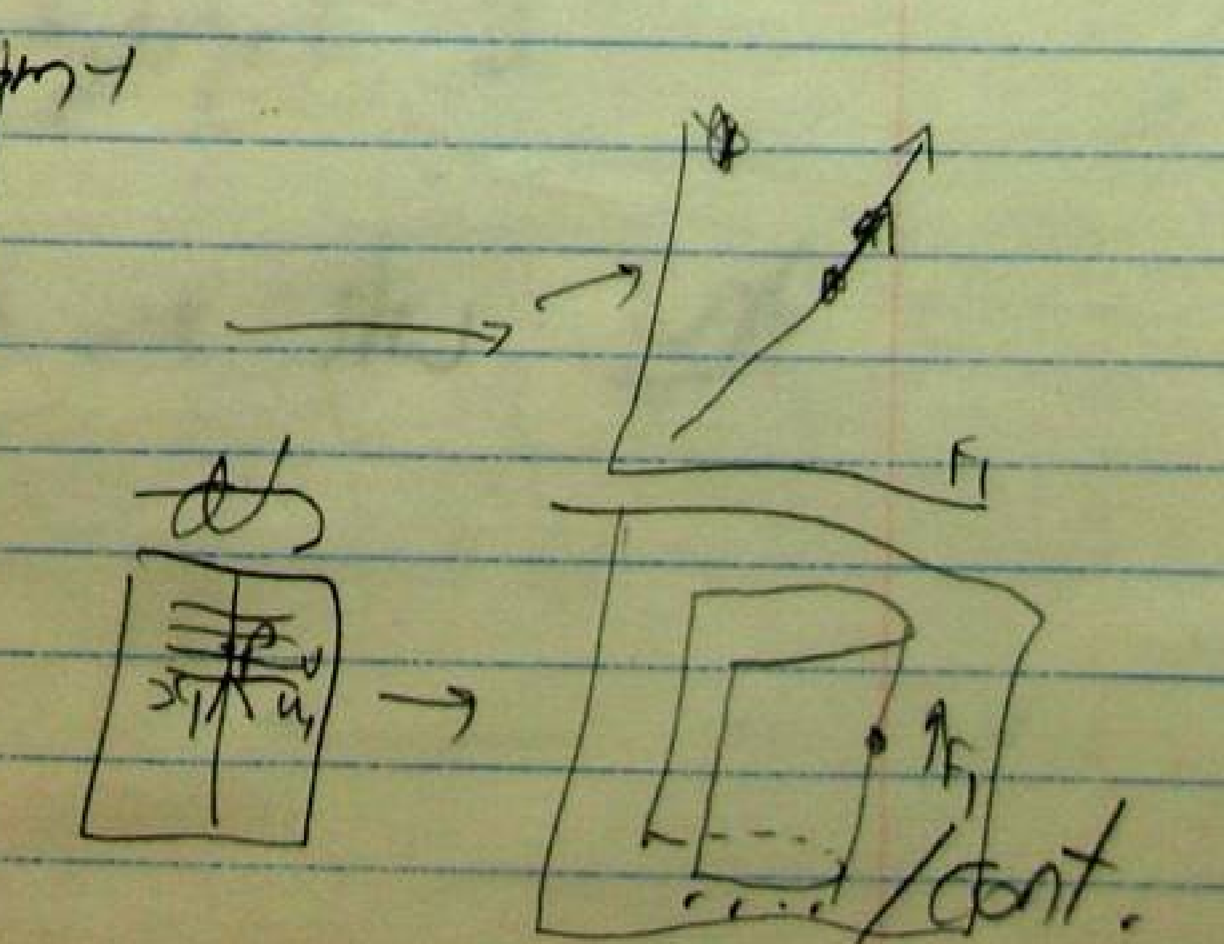
Pf ... Let g be a kth derivative of f with $dg \neq 0$; then $D_k \subset g^{-1}(0)$ which is of lower dimension.

Claim $f(D_0 - D_1)$ has measure 0.

Suppose $\frac{\partial}{\partial x_1} f_1 \neq 0$. Let $U_t = f_1^{-1}(t) = \mathbb{R}^{m-1}$

then $\text{Crit}(f) \subset \bigcup_t \text{Crit } f_{\text{rest}}|_{U_t}$

Also: Pf of mini-Fubini:



/cont.

Cont.

The Whitney embedding thm

Every manifold M^m can be embedded in $\mathbb{R}^{2m+1}$

Pf 1. M compact, in some $\mathbb{R}^N$

2. N can be cut to $2m+1$

3. M can be general.

Math 1300 Geom&Top, Thu Oct 18 2007, hour 18.

Handout on baby Fubini.

Today: The Whitney embedding Theorem:

Every manifold M^m can be embedded in $\mathbb{R}^{2m+1}$

Proof: 1. M Compact $\hookrightarrow \mathbb{R}^N$, N big. } fade in Handout using "partitions of unity"

2. N can be cut to 2m+1 } Sard!

3. M can be general. } "the zebra trick"

---

1. Given a finite atlas $\phi_i : U_i \to \mathbb{R}^m$

$p \mapsto (\qquad \phi_1(p), \phi_2(p), \quad \dots \quad \phi_k(p))$

"the globe embeds in the product of all page in the atlas"

Need a "partition of unity" λ_i:

Def'n A (smooth) partition of unity subordinate to an open cover;

Thm from topology: Manifolds are "paracompact":

1. Every open cover has a locally finite refinement.

2. If U_α is locally finite, it can be "shrunk" to V_α s.t. $\forall \alpha\ \overline{V_\alpha} \subset U_\alpha$.

Thm If $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ is an atlas on M, there's a smooth partition of unity λ_β subordinate of the cover U_α of M.

... /cont.

Cont.

pf of Att 2 of Whitney.

Pages 203-204 of Guillemin-Pollack's "Differential Topology."

For the proof of Sard's theorem, we will need the measure zero form of Fubini's theorem. Suppose that $n = k + l$, and write $\mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^l$. For each $c \in \mathbf{R}^k$, let V_c be the "vertical slice" $\{c\} \times \mathbf{R}^l$. We shall say that a subset of V_c has measure zero in V_c if, when we identify V_c with $\mathbf{R}^l$ in the obvious way, the subset has measure zero in $\mathbf{R}^l$.

Fubini Theorem (for measure zero). Let A be a closed subset of $\mathbf{R}^n$ such that $A \cap V_c$ has measure zero in V_c for all $c \in \mathbf{R}^k$. Then A has measure zero in $\mathbf{R}^n$.

Proof. Because A may be written as a countable union of compacts, we may, in fact, assume A compact. Also, by induction on k, it suffices to prove the theorem for $k = 1$ and $l = n - 1$. We will divide the proof into several lemmas.

Lemma 1. Let $S_1, \ldots, S_N$ be a covering of the closed interval $[a, b]$ in $\mathbf{R}^1$. Then there exists another cover $S'_1, \ldots, S'_M$ such that each S'_j is contained in some S_i, and

$$\sum_{j=1}^{M} \text{length}\,(S'_j) < 2(b - a).$$

Proof. Left to the reader.

Given a set $I \subset \mathbf{R}$, let $V_I = I \times \mathbf{R}^{n-1}$ in $\mathbf{R}^n$.

Lemma 2. Let A be a compact subset of $\mathbf{R}^n$. Suppose that $A \cap V_c$ is contained in an open set U of V_c. Then for any suitably small interval I about c in $\mathbf{R}$, $A \cap V_I$ is contained in $I \times U$.

Proof. If not, there would exist a sequence of points (x_j, c_j) in A such that $c_j \to c$ and $x_j \notin U$. Replace this sequence by a convergent one to get a contradiction. Q.E.D.

Proof of Fubini. Since A is compact, we may choose an interval $I = [a, b]$ such that $A \subset V_I$. For each $c \in I$, choose a covering of $A \cap V_c$ by $(n - 1)$ dimensional rectangular solids $S_1(c), \ldots, S_{N_c}(c)$ having a total volume less than ϵ. Choose an interval $J(c)$ in $\mathbf{R}$ so that the rectangular solids $J(c) \times S_i(c)$ cover $A \cap V_J$ (Lemma 2). The $J(c)$'s cover the line interval $[a, b]$, so we can use Lemma 1 to replace them with a finite collection of subintervals J'_j with total length less than $2(b - a)$. Each J'_j is contained in some interval $J(c_j)$, so the solids $J'_j \times S_i(c_j)$ cover A; moreover, they have total volume less than $2\epsilon(b - a)$. Q.E.D.

0708-1300/Homework Assignment 3

From Drorbn

Contents

Reading

Read sections 8-10 of chapter II of Bredon's book three times:

- First time as if you were reading a novel - quickly and without too much attention to detail, just to learn what the main keywords and concepts and goals are.
- Second time like you were studying for an exam on the subject - slowly and not skipping anything, verifying every little detail.
- And then a third time, again at a quicker pace, to remind yourself of the bigger picture all those little details are there to paint.

Also, read section 12 of chapter I of Bredon's book, but you can be a little less careful here.

Doing

Solve the following problems from Bredon's book, but submit only the solutions of the problems marked with an "S":

problems	on page(s)
S1, S2, 3, S4, S5	88
S1, 2, 3, S4, 5	89

Note that these problems largely concern with material that we will not cover in class.

Due Date

This assignment is due in class on Thursday November 1, 2007.

Just for Fun

- Trace the proof of the Whitney embedding theorem to find an embedding of the two dimensional real projective plane, $\mathbb{RP}^2 = S^2/(p = -p)$, inside $\mathbb{R}^5$. Do not do anything explicitly; just convince yourself that indeed you can find a small atlas (how small?), use it to embed $\mathbb{RP}^2$ in some large $\mathbb{R}^N$ (how large?), and figure out how many times you will need to use Sard's theorem before you're down to the target, $\mathbb{R}^5$.
- Now see if you can come up with some cleverer way of viewing $\mathbb{RP}^2$, that will allow you to explicitly embed it in $\mathbb{R}^5$.

Summary of Proposal for Public Release, 2007

(To be submitted to the NSERC)

Why are mathematicians fascinated by the whole numbers? Certainly not because of the beauty inherent in staring at numbers such as 9,465,438. Neither it is due to the difficulty in figuring out that 9,465,438 is 2x3x1,577,573. The true reason is that the whole numbers are surprisingly deep, and the study of whole numbers, also known as "number theory", forced us to better understand, and indeed develop, many other useful and beautiful techniques, concepts and ideas. Number theory just seems to be related to everything.

Likewise, though on a smaller scale, many knot theorists such as myself care little about shoelaces, yet care a lot about the unexpected ways by which the study of knotted shoelaces is intricately and deeply related to such a priori remote subjects as 3-dimensional manifolds, hyperbolic geometry, quantum field theory, differential geometry, Lie theory and representation theory, quantum algebra, combinatorics, homological algebra and sophisticated algorithmics.

My research for this project will concentrate on the further elaboration of these unexpected links, using both analytical and computational tools. My primary goal will be to complete our understanding of the relationship between algebra and the so-called "Kontsevich integral of knotted graphs"; I expect this will benefit knot theory via the tools and techniques of "algebraic knot theory", and I expect this will benefit algebra by providing a unified framework for the study of all quantum groups.

I tend to write expositions and give expository talks, draw pictures and write computer programs. Thus much of my work in this project will end up finding its way to my already-comprehensive web site, at http://www.math.toronto.edu/~drorbn/.

Summary of Proposal, 2007

(Submitted to the NSERC)

In the space provided below, state the objectives of the proposed research program and summarize the scientific approach, highlighting the novelty and expected significance of the work to a field or fields in the natural sciences and engineering. [...] Your summary must not exceed forty-five lines on the printed copy.

Over the next five years I plan to pursue following three projects.

* **Algebraic Knot Theory.** For many years now the Kontsevich integral Z (a universal finite type invariant of knots and links) is appreciated for its strength. It is stronger than all known "quantum invariants" taken together. But only recently I understood that that might not be where the real power of this invariant lies: Z beautifully extends to an invariant of knotted trivalent graphs which is well behaved under certain natural operations defined on graphs - edge deletion, "unzipping" and connected sums. Several well known and hard-to-detect properties of knots are "definable" using these operations, including the knot genus, unknotting numbers and the property of being a ribbon knot. Thus, at least in principle, each of these properties can be translated to a simple algebraic "equation" involving Z of the knot being studied. But turning this principle into results is a five-year project. We still have to identify and study appropriate quotients of the target space of Z which make Z computable to all orders. Every classical knot polynomial (Alexander's, Jones', etc.) defines such a quotient, and there are many other quotients to choose from, but even the simplest of these quotients, corresponding to a canonical extension of the Alexander polynomial to graphs, is poorly understood. See http://www.math.toronto.edu/~drorbn/Talks/Aarhus-0706.
* **Knot Theoretic Algebra.** My paper "On Associators and the Grothendieck-Teichmuller Group" indicates strongly that the right context for understanding Drinfel'd's theory of formal associators is a certain category of braid groups and operations mapping such braids groups to each other. There is strong evidence that the theory of quantum groups (specifically, quantum universal enveloping algebra and/or quasi-triangular Hopf algebras) should be related in a similar manner to virtual knots and braids and operations among them. Indeed, one of the starting points of the theory of quantum groups is the quantization of Lie bialgebras, and the "universal" diagrammatic theory underlying Lie bialgebras is the same as the diagrammatic theory that underlies finite type invariants of virtual knots/braids. (Compare with the well-understood relationship between knots, chord diagrams and Lie algebras). Over the grant period I plan to fully understand a universal theory of quantum groups as a natural object within the context of virtual knot theory. See http://www.math.toronto.edu/~drorbn/Talks/Kyoto-0705/ and ---/Tianjin-0707/.
* **Computations in Knot Theory and the Knot Atlas.** I Plan to continue contributing to the computational package KnotTheory` and to the Knot Atlas. Both projects were founded by me but by now have received contributions by many others (especially S. Morrison). See http://katlas.org.

What about Khovanov homology? I made significant contributions to the highly fashionable subject of Khovanov homology (in fact, while Khovanov is definitely the father of the field, I share the credit for making it fashionable...). Yet at the moment I don't feel mature enough to study this topic any further. I'd rather "categorify" knot invariants only after I properly understand the "algebra" on which they ought to be defined (in the sense of my first project above). And how can I even start categorifying other aspects of the theory of quantum groups, when in my opinion this theory in itself is so poorly understood (at least in the sense of my second project)? With luck, at the end of this grant period I will be ready to return to Khovanov homology and categorification in general.

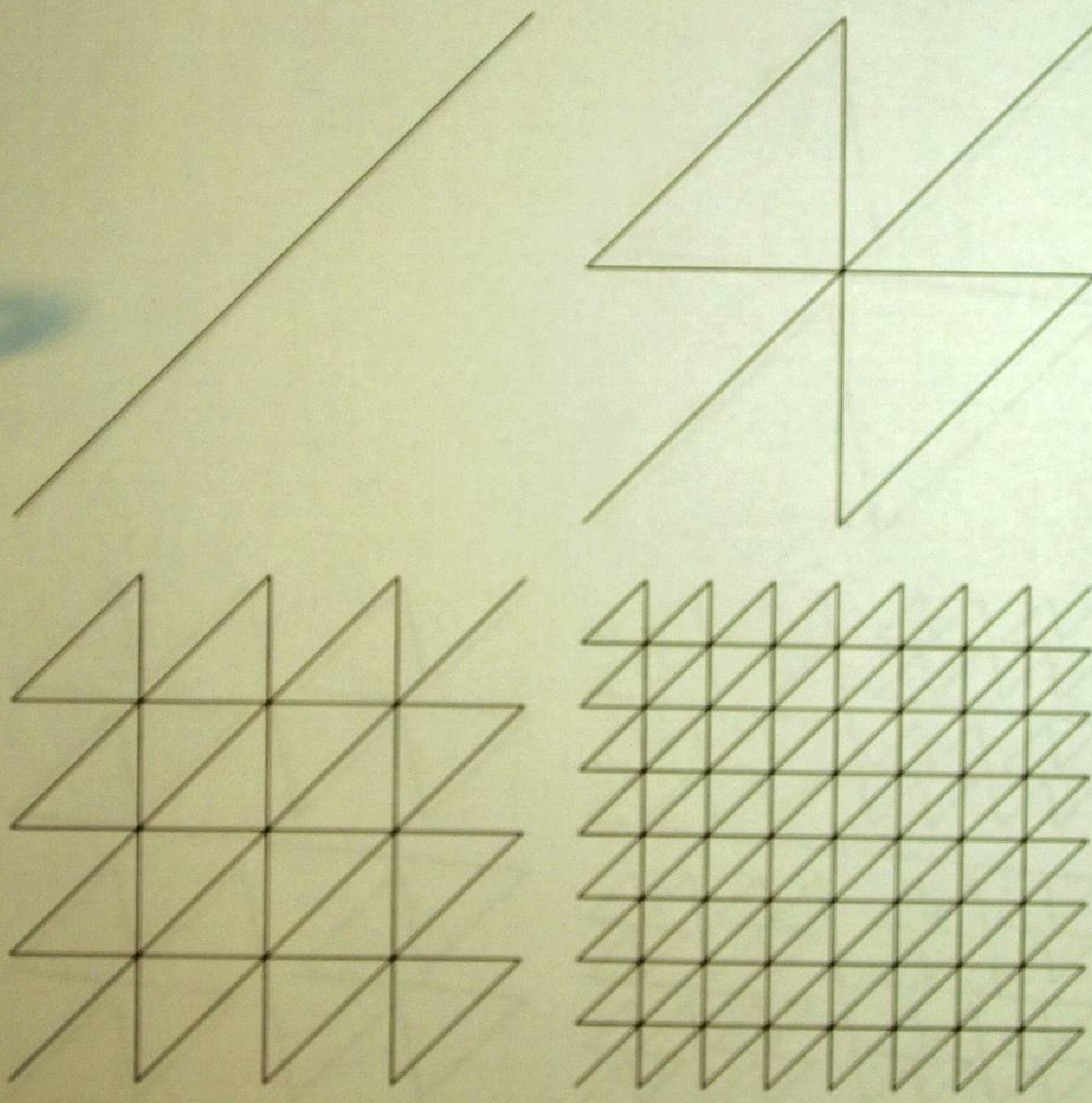
In science, though, the predicted is always less interesting than the unpredictable. With luck, at least some of my work in the next five years will be on topics I haven't yet heard of.

Dror Bar-Natan: Classes: 2007-08: Geom & Top: 071023

A ~~the~~ Peano Curve

```
In[1]:= Peano[0] = DLine[{{0, 0}, {1, 1}}];
        Peano[n_] := Peano[n - 1] /. DLine[{{x0_, y0_}, {x1_, y1_}}] :> {
            dx = x1 - x0; dy = y1 - y0;
            DLine[{{x0, y0}, {x0 + 1/2 dx, y0 + 1/2 dy}}],
            Line[{{x0 + 1/2 dx, y0 + 1/2 dy}, {x0 + 1/2 dx, y0}}],
            DLine[{{x0 + 1/2 dx, y0}, {x0 + dx, y0 + 1/2 dy}}],
            Line[{{x0 + dx, y0 + 1/2 dy}, {x0, y0 + 1/2 dy}}],
            DLine[{{x0, y0 + 1/2 dy}, {x0 + 1/2 dx, y0 + dy}}],
            Line[{{x0 + 1/2 dx, y0 + dy}, {x0 + 1/2 dx, y0 + 1/2 dy}}],
            DLine[{ {x0 + 1/2 dx, y0 + 1/2 dy}, {x0 + dx, y0 + dy}}]
          };
        GraphicsGrid[Partition[
          Table[Graphics[Peano[n] /. DLine -> Line], {n, 0, 3}],
          2]]
```

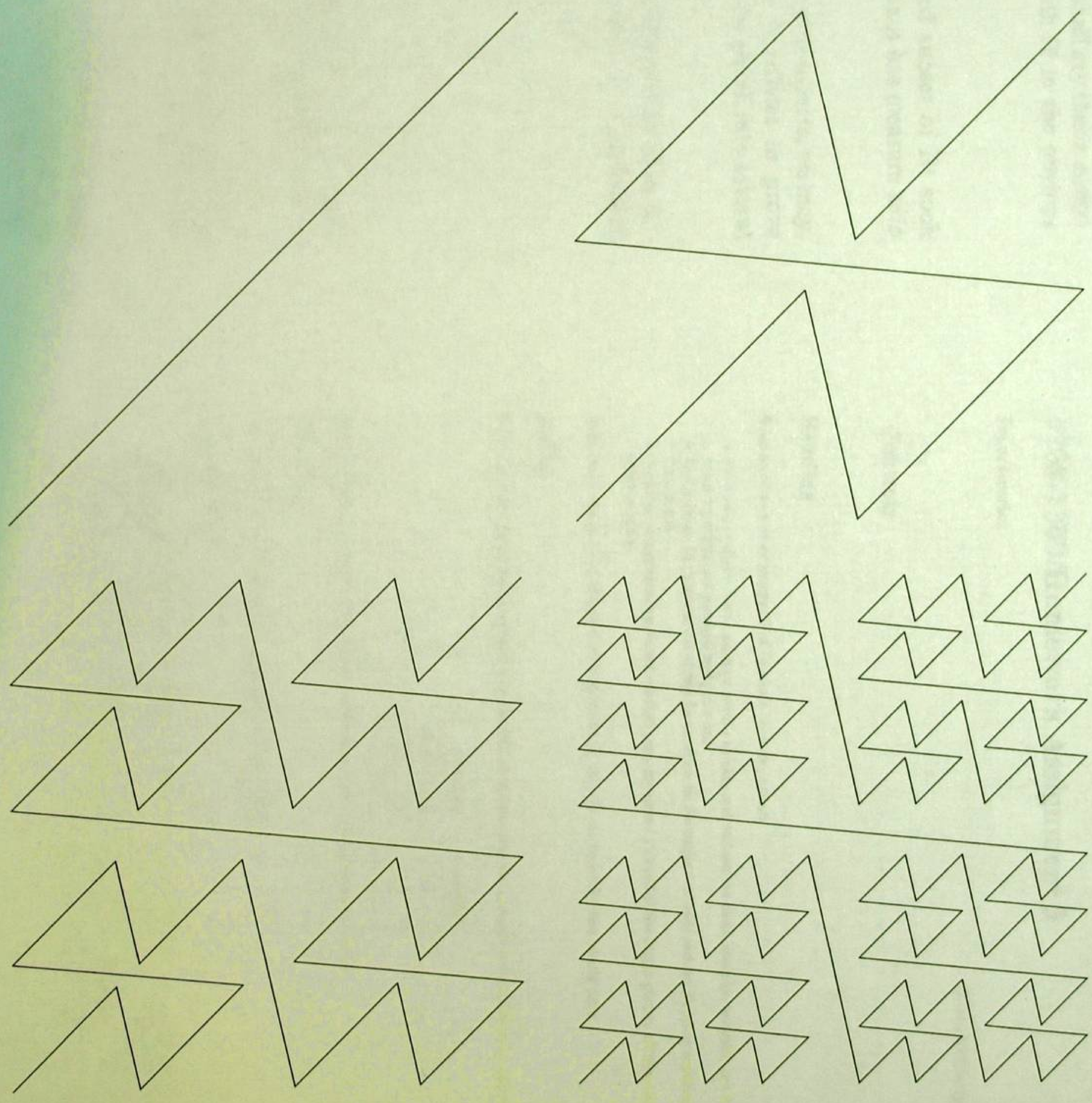
Out[3]=



* Talk today: 12-1, on Rubik's Cube.
* Office hours today: 1-2.

In[4]:=
```
AlmostPeano[0] = DLine[{{0, 0}, {1, 1}}];
AlmostPeano[n_] := AlmostPeano[n - 1] /. DLine[{{x0_, y0_}, {x1_, y1_}}] :> {
    dx = x1 - x0; dy = y1 - y0;
    DLine[{{x0, y0}, {x0 + 0.45 dx, y0 + 0.45 dy}}],
    Line[{{x0 + 0.45 dx, y0 + 0.45 dy}, {x0 + 0.55 dx, y0}}],
    DLine[{{x0 + 0.55 dx, y0}, {x0 + dx, y0 + 0.45 dy}}],
    Line[{{x0 + dx, y0 + 0.45 dy}, {x0, y0 + 0.55 dy}}],
    DLine[{{x0, y0 + 0.55 dy}, {x0 + 0.45 dx, y0 + dy}}],
    Line[{{x0 + 0.45 dx, y0 + dy}, {x0 + 0.55 dx, y0 + 0.55 dy}}],
    DLine[{ {x0 + 0.55 dx, y0 + 0.55 dy}, {x0 + dx, y0 + dy}}]
   };
GraphicsGrid[Partition[
  Table[Graphics[AlmostPeano[n] /. DLine → Line], {n, 0, 3}],
  2]]
```

Out[6]=



Office hour today: ~~B~~ 1-2
Talk: 12-1.

Math 1300 Geom&Top, Tue Oct 23, 2007, hours 19-20

Comments/omissions: 1. The Peano curve (duty, Authority, discipline, compactness)

2. Redefine Locally finite.

Thm The Whitney embedding thm

1. Compact $M^m \hookrightarrow \mathbb{R}^N$
2. ~~Arbitrary~~ Compact $M^m \hookrightarrow \mathbb{R}^{2m+1}$
3. Arbitrary $M^m \hookrightarrow \mathbb{R}^{4m+3}$
4. Arbitrary $M^m \hookrightarrow \mathbb{R}^{2m+1}$

If time — smooth ~~Brauw~~ Brouwer

Math 1300 Geom&Top, Thu Oct 25 2007, hour 21

Exam specs: next Thursday.

Thm 2 classification of ~~2~~ surfaces
what about ~~x~~ rough

Thm 1 classification of 1-manifolds.

Back to whitney.

Smooth ~~Brauw~~ Brouwer & applications.

hours 22-23

0708-1300/Class notes for Tuesday, October 30

From Drorbn

0708-1300/Navigation Panel [Show]

Contents

Today's Agenda

Debts

A bit more about proper functions on locally compact spaces.

Smooth Retracts and Smooth Brouwer

Theorem. There does not exist a smooth retract $r : D^{n+1} \to S^n$.

Corollary. (The Brouwer Fixed Point Theorem) Every smooth $f : D^n \to D^n$ has a fixed point.

Suggestion for a good deed. Tell Dror if he likes the Brouwer fixed point theorem, for he is honestly unsure. But first hear some drorpaganda on what he likes and what he doesn't quite.

Corollary. The sphere S^n is not smoothly contractible.

Challenge. Remove the word "smooth" everywhere above.

Smooth Approximation

Theorem. Let A be a closed subset of a smooth manifold M, let $f : M \to \mathbb{R}$ be a *continuous* function whose restriction $f|_A$ to A is smooth, and let ϵ be your favourite small number. Then there exists a *smooth* $g : M \to \mathbb{R}$ so that $f|_A = g|_A$ and $||f - g|| < \epsilon$. Furthermore, f and g are homotopic via an ϵ-small homotopy.

Theorem. The same, with the target space replaced by an arbitrary compact metrized manifold N.

Tubular Neighborhoods

Theorem. Every compact smooth submanifold M^m of $\mathbb{R}^n$ has a "tubular neighborhood".

Entertainment

A student told me about this clip (http://youtube.com/watch?v=UTby_e4-Rhg) on YouTube (lyrics (http://www.math.northwestern.edu/~matt/kleinfour/lyrics/finite.html)). Enjoy!

There is this one too but it is in Spanish. Romance of the Derivative and the Arctangent (http://matematicas.uis.edu.co/~marsan/ROMANCE%20DE%20LA%20DERIVADA%20Y%20EL%20ARCOCOSENC

Math 1300, Term Exam I, Nov 2007.

"compute" (Explicit)
"Think" (Very on theory)
"Reproduce"
"sketch".

* The inverse function Theorem/implicit
* Differentials & the chain rule
* manifolds / functional structures.
* Tangent vectors, differentials
* Immersions, submersions, transversality
* Sard.
* Whitney
* Brouwer
* Tubular nbds & smooth approx.

4 Q's worth
30 pts each,
A+ = 100

Not on: Cantor & Peano, general topology, partitions of unity.

<u>Goals</u>: 1. Make you study.
2. Give you feedback
3. Give me feedback
4. Give you a grade.

Math 1300 Geom&Top, ~~[illegible]~~ Thu Nov 1 2007, hour 24

* Proper functions.

* complete tubular nbd.

* the sphere is not contractible.

$M^?$: compact submanifold of $\mathbb{R}^n$

$N(M) := \{(x, v) : x \in M, v \in T_x\mathbb{R}^n, v \perp T_x M\}$

$\Phi : N(M) \longrightarrow \mathbb{R}^n$ by $(x, v) \longmapsto x + v$

$N(M, \epsilon) := \{(x, v) \in N(M) : \|v\| < \epsilon\}$

$B(M, \epsilon) := \{y \in \mathbb{R}^n : d(y, M) < \epsilon\}$

<u>Claim</u> $\Phi|_{N(M,\epsilon)} : N(M, \epsilon) \longrightarrow B(M, \epsilon)$ is a diffeomorphism for small enough ϵ

(hence smooth approximation works for compact-manifold-valued functions)

0708-1300/Class notes for Thursday, November 1

From Drorbn

Today's Agenda

- HW4 and TE1.
- Continue with Tuesday's agenda:
  - Debt on proper functions.
  - Prove that "the sphere is not contractible".
  - Complete the proof of the "tubular neighborhood theorem".

0708-1300/Navigation Panel [Hide]

Add your name / see who's in!

#	Week of...	Links
		Fall Semester
1	Sep 10	About, Tue, Thu
2	Sep 17	Tue, HW1, Thu
3	Sep 24	Tue, Photo, Thu
4	Oct 1	Questionnaire, Tue, HW2, Thu
5	Oct 8	Thanksgiving, Tue, Thu
6	Oct 15	Tue, HW3, Thu
7	Oct 22	Tue, Thu
8	Oct 29	Tue, HW4, **Thu**
9	Nov 5	TE1 on Thu
10	Nov 12	HW5
11	Nov 19	
12	Nov 26	HW6
13	Dec 3	
		Spring Semester
14	Jan 7	HW7
15	Jan 14	
16	Jan 21	HW8
17	Jan 28	
18	Feb 4	HW9
19	Feb 11	TE2; Feb 17: last chance to drop class
R	Feb 18	
20	Feb 25	HW10
21	Mar 3	
22	Mar 10	HW11
23	Mar 17	
24	Mar 24	HW12
25	Mar 31	
26	Apr 7	

Errata to Bredon's Book

Proper Implies Closed

Theorem. A proper function $f:X\to Y$ from a topological space X to a locally compact (Hausdorff) topological space Y is closed.

Proof. Let B be closed in X, we need to show that $f(B)$ is closed in Y. Since closedness is a local property, it is enough to show that every point $y\in Y$ has a neighbourhood U such that $f(B)\cap U$ is closed in U. Fix $y\in Y$, and by local compactness, choose a neighbourhood U of y whose close $\bar{U}$ is compact. Then

$$f(B)\cap U = f(B\cap f^{-1}(U))\cap U\subset f(B\cap f^{-1}(\bar{U}))\cap U\subset f(B)\cap U,$$

so that $f(B)\cap U = f(B\cap f^{-1}(\bar{U}))\cap U$. But $\bar{U}$ is compact by choice, so $f^{-1}(\bar{U})$ is compact as f is proper, so $B\cap f^{-1}(\bar{U})$ is compact as B is closed, so $f(B\cap f^{-1}(\bar{U}))$ is compact (and hence closed) as a continuous image of a compact set, so $f(B)\cap U$ is the intersection $f(B\cap f^{-1}(\bar{U}))\cap U$ of a closed set with U, hence it is closed in U.

0708-1300/Homework Assignment 4

From Drorbn

0708-1300/Navigation Panel [Show]

Reading

Read section 11 of chapter II and sections 1-3 of chapter V of Bredon's book three times:

- First time as if you were reading a novel - quickly and without too much attention to detail, just to learn what the main keywords and concepts and goals are.
- Second time like you were studying for an exam on the subject - slowly and not skipping anything, verifying every little detail.
- And then a third time, again at a quicker pace, to remind yourself of the bigger picture all those little details are there to paint.

Doing

Solve the following problems from Bredon's book, but submit only the solutions of the problems marked with an "S":

problems	on page(s)
S1, S2	100-101
S1, S2, 3	264

Also, solve and submit the following question:

Question 6.

1. Show that if $n \neq m$ then $\mathbf{R}^n$ is not diffeomorphic (homeomorphic via a smooth map with a smooth inverse) to $\mathbf{R}^m$.
2. Show that if $n \neq m$ then $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^m$.

Note that a priori the second part of this question is an order of magnitude harder than the first, though with the techniques we already have, it is not too bad at all.

Due Date

This assignment is due in class on Thursday November 15, 2007.

Just for Fun

Find a *geometric* interpretation to the formula

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

(Of course, you have to first obtain a *geometric* understanding of $[X,Y]$, and this in itself is significant and worthwhile).

0708-1300/Term Exam 1

From Drorbn

Term Exam 1 will take place on Thursday November 8, 2007, at 6PM, at Sydney Smith 1084.

Dror's Internal Notes

Math 1300, Term Exam I, Nov 2007.

"compute" (Explicit) → * The inverse function Theorem/implicit
"Think" (Very short theory) → * Differentials & The chain rule
"Reproduce" * manifolds / functional structures.
"sketch". * Tangent vectors, differentials
* Immersions, submersions, transversality
4 Q's worth * Sard.
30 pts each, * Whitney ~~Easy~~
A+ = 100 * Brouwer
* Tubular nbds & smooth approx.

Not on Cantor & Peano, general topology, partitions of unity.

Goals. 1. Make you study.
2. Give you feedback
3. Give me feedback.
4. Give you a grade.

Dror's notes above / Student's notes below

Some Additional Reading

There are some lectures notes (http://ocw.mit.edu/OcwWeb/Mathematics/18-965Fall-2004/LectureNotes/index.htm) of the MIT Open Course Ware (http://ocw.mit.edu/OcwWeb/Mathematics/18-965Fall-2004/CourseHome/index.htm) . This can be an additional reading for us. There are some exercises (http://ocw.mit.edu/OcwWeb/Mathematics/18-965Fall-2004/Assignments/index.htm) too.

More lectures notes (http://www.maths.tcd.ie/~zaitsev/ln.pdf) from the University of Dublin. This one has exercises.

From Wien (http://www.mat.univie.ac.at/~michor/dgbook.pdf) with exercises too.

$$\int_{D} d\omega = \int_{\partial D} \omega$$

מושגי יסוד בגאומטריה דיפרנציאלית, 31 לדצמבר 2000

1. נושאים מהליך.

2. משפט סטוקס $\int_M d\omega = \int_{\partial M} \omega$

3. $A^p(V)$; המכפלה החיצונית, ליניאריות, אסוציאטיביות, סופר-קומוטטיביות.

4. טענה אם $\omega_1, \dots, \omega_n \in V^*$ בסיס, אזי $\{\omega_{i_1} \wedge \dots \wedge \omega_{i_p} : i_1 < \dots < i_p\}$ בסיס ל-$A^p(V)$, ולכן $\dim A^p(V) = \binom{n}{p}$

תבניות דיפרנציאליות $\Omega^p(M)$

טענה קיים אופרטור לינארי יחיד $d : \Omega^p \to \Omega^{p+1}$ המקיים

1. d מרחיב את $(df)(X) = Xf$

2. $d^2 = 0$

3. $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$

(דוגמאות) d ב-$\mathbb{R}^3$

מושגי יסוד בגיאומטריה דיפרנציאלית, 16 לינואר 200[illegible]

1. הגדרות: A^p, $\wedge$, מימדים, Ω^p, דוגמה.

2. משפט: קיים אופרטור לינארי יחיד $d:\Omega^p\to\Omega^{p+1}$ המקיים:

1. d מרחיב את $(df)(X)=Xf$

2. $d^2=0$

3. $d(\omega\wedge\eta)=(d\omega)\wedge\eta+(-1)^{\deg\omega}\omega\wedge d\eta$

3. $\mathbb{R}^3$

4. הגדרה של אינטגרציה, משיכה לאחור.

משפטי יסוד בגאומטריה דיפרנציאלית, 19 לנובמבר 2001

1. נסח קיום ויחידות של d, הערה. $d(w\wedge\eta)=$

2. $\mathbb{R}^3$

3. הכללה של אינטגרציה וכו..

4. משיכה לאחור והתכונות: 1. $(\theta\circ\psi)^*=\psi^*\circ\theta^*$

   2. $\wedge$

   3. $\theta^*(df)=d(\theta^*f)$

   4. $\theta^*(dw)=d\theta^*w$

5. אינטגרציה של בעלת תומך קומפקטי בתוך $U\subset\mathbb{R}^n$

6. אינטגרציה בכלל.

מושגי יסוד בגאומטריה דיפרנציאלית 21 לינואר 2001

1. $\int w_2 = \int \vec{F} \cdot \vec{n} \, d\sigma$ $\qquad$ $\int w_1 = \int \vec{F} \cdot \vec{T} \, ds$

$\int w_3 = \int F \, dV$ $\qquad$ $\int w_0 = \sum F$

+ המשפט הגאוס-בונה.

2 משפט סטוקס.

Do not turn this page until instructed.

Math 1300 Geometry and Topology

Term Test

University of Toronto, November 8, 2007

Solve the 4 problems on the other side of this page.

Each problem is worth 30 points.

You have two hours to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Good Luck!

Solve the following 4 problems. Each problem is worth 30 points. You have two hours. **Neatness counts! Language counts!**

Problem 1 "Compute". Let $\phi : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$ be given by $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$, let $f : \mathbb{R}^2_{u,v} \to \mathbb{R}$ be given by $f(u,v) = u^2 + v^2$, and let $\xi \in T_{(0,1)}\mathbb{R}^2_{x,y}$ be $\xi = \partial/\partial x$. Compute the following quantities (with at least some justification):

1. $\phi_*\xi$.
2. ϕ^*f.
3. $(\phi_*\xi)f$.
4. $\xi(\phi^*f)$.

Problem 2 "Reproduce". The tangent space $T_0\mathbb{R}^n$ to $\mathbb{R}^n$ at 0 can be defined in the following two ways:

1. $T_0^1\mathbb{R}^n$ is the set of all smooth curves $\gamma : \mathbb{R} \to \mathbb{R}^n$ satisfying $\gamma(0) = 0$, modulo the equivalence relation $\sim$, where $\gamma_1 \sim \gamma_2$ iff $\dot\gamma_1(0) = \dot\gamma_2(0)$, where in general, $\dot\gamma$ denotes the derivative of $\gamma(t)$ with respect to t.
2. $T_0^2\mathbb{R}^n$ is the set of all linear functionals D on the vector space of smooth functions on $\mathbb{R}^n$, which also satisfy Leibnitz' rule, $D(fg) = (Df)g(0) + f(0)(Dg)$.

Prove that these two definitions are equivalent (i.e., that there is a natural bijection between $T_0^1\mathbb{R}^n$ and $T_0^2\mathbb{R}^n$). If you use a non-trivial lemma from calculus, state it precisely but you don't need to prove it.

Problem 3 "Think". Let $f : M \to M$ be a smooth function from a compact manifold M to itself. Prove that there is a point $y \in M$ so that $f^{-1}(y)$ is finite. (In fact, there are many such points).

Problem 4 "Sketch". Sketch to the best of your understanding the proof of the Whitney embedding theorem, paying close attention to what is important and little attention to what is not. Here, more than anywhere else, **neatness and language count!**

Good Luck!

Math 1300 Geom & Top, Tue Nov 6 2007, hours 25-26

On board

$$\int_M d\omega = \int_{\partial M} \omega$$

Then follow Dec 31, 2000.

Math 1300 Geom & Top, Thu Nov 8 2007, hour 27

* Brief correction re. retracts.
* Contractability of S^∞ following Franklin.
* Q&A.

Math 1300 Geom & Top, Tue Nov 13 2007, hours 28-29.

On board:

today's goal. done last time

maybe today $\int_M d\omega = \int_{\partial M} \omega$

done, tested

next week.

No office hour today!

$\omega \in \Omega^k(M)$:

1. Eats k t.v. w/ same base point, spits a number.
2. multilinear & A.S.
3. $(\omega_p, \lambda_q) \mapsto \omega\wedge\lambda \in \Omega^{p+q}$
4. $\wedge$ is bilinear, assoc., super comm
5. If $\omega_1 \dots \omega_n$ a basis of V^*, $\omega_{i_1}\wedge\dots\wedge\omega_{i_k}$ a basis of A^k.

Continue as on Dec 31, 2000.

claim
If ω vanishes on U, so does $d\omega$.
pf choose λ w/ $\operatorname{supp}\lambda \subset U$
$0 = d(\lambda\omega) = d\lambda\wedge\omega + \lambda\wedge d\omega$

Prop $\exists!$ linear $d: \Omega^p \to \Omega^{p+1}$ s.t.

1. $(df)(X) = Xf$
2. $d^2 = 0$
3. $d(\omega\wedge\lambda) = (d\omega)\wedge\lambda + (-1)^{\deg\omega}\,\omega\wedge d\lambda$ ⟸ Super philosophy.

Proof. Uniqueness in $\mathbb{R}^n$, existence in $\mathbb{R}^n$, uniqueness on M
Existence on M.

The case of $\mathbb{R}^3$; $\boxed{\text{bring ball}}$

If time: Integration w/ compact support in $\mathbb{R}^n$.

On board: Apologies: 1. Missed classes 2. TEI delays, 3. HW4 confusions 4. HW5 delay.

1 hour skipped for sickness, should have been ↑31-32.

Math 1300 Geom & Top, Tue Nov 20, 2007, hours 30-31

On board

$$\int_M d\omega = \int_{\partial M} \omega \qquad \exists!\ \overset{\text{linear}}{d} : \Omega^* \to \Omega^{*+1} \text{ st.}$$

1. $(df)(X) = Xf$ 2. $d^2 = 0$

3. $d(\omega\wedge\lambda) = (d\omega)\wedge\lambda + (-1)^{\deg \omega}\omega\wedge d\lambda$

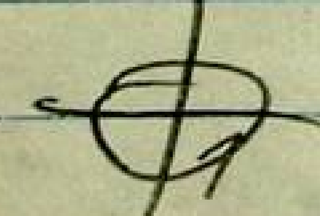
$$\Rightarrow d\left(\sum f_I dx^I\right) = \sum (df_I)\wedge dx^I = \sum_{j,I} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx^I$$

$\Rightarrow$ uniqueness in $\mathbb{R}^n$, if existence

Do.
0. Geometric interp.
1. Existence of d in $\mathbb{R}^n$
2. ~~the~~ uniqueness on M
3. Existence on M.
4. Integration on $\mathbb{R}^n$ / change of variables.

integration on M / orientation, partitions of unity, well definedness

Define orientation using
1. bases of TM
2. sections of $\Omega^n(M)$
3. transition functions

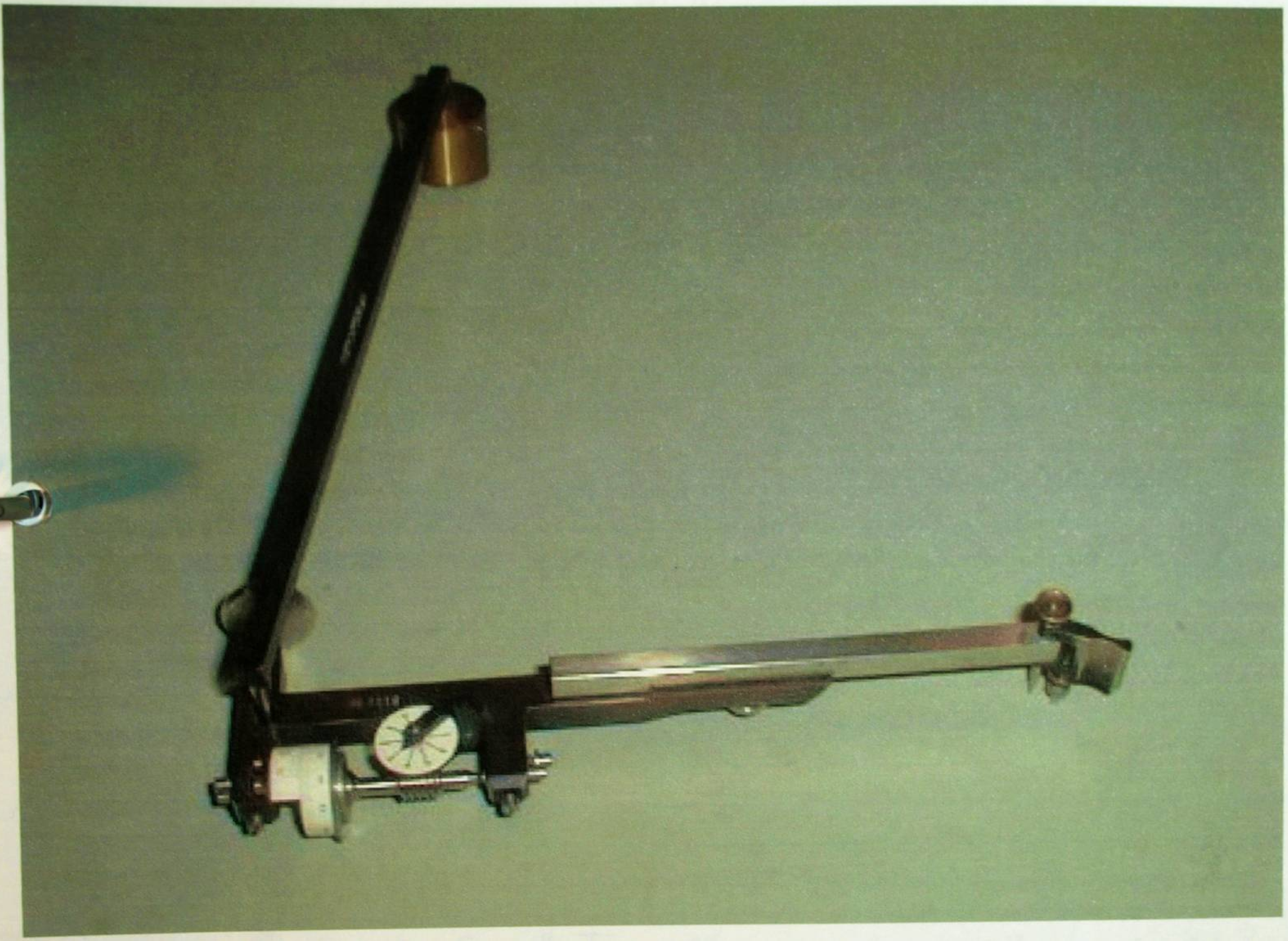
Example  $xdy - ydx$

5. Boundaries
6. Stokes'.

Math 1300 Geom & Top, Thu Nov 22 2007, hour 32

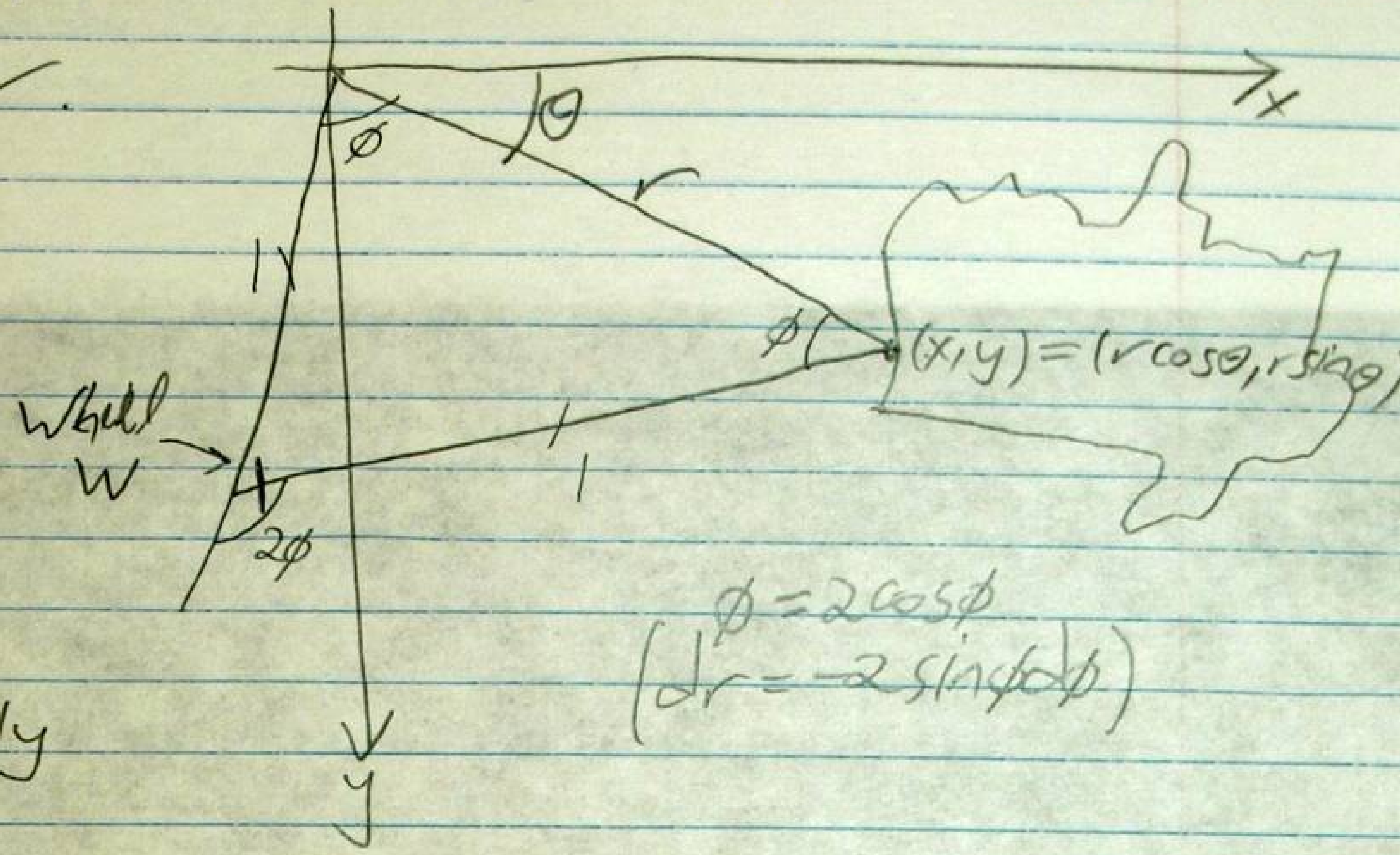
cont. from pen line.

http://whistleralley.com/planimeter/plan1.jpg



Math 1300 Geom&Top, Tue Nov 27 2007, hours 33-34.

* The planimeter.



$$r = 2\cos\phi \qquad (dr = -2\sin\phi\, d\phi)$$

$$W = \cos 2\phi \; d(\theta+\phi)$$

$$dW = -2\sin 2\phi\, d\phi \wedge d\theta$$
$$= -4\sin\phi\cos\phi\, d\phi\wedge d\theta$$
$$= 2\cos\phi\, dr\wedge d\theta$$
$$= r\, dr\wedge d\theta = dx\wedge dy$$

Proof of Stokes'

Back to 3D

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3$$

$w_0 = f$ $\quad w_1 = f_1\,dx_1 + \dots$ $\quad w_2 = g_1\,dx_2\wedge dx_3 + cyc$ $\quad w_3 = g\,dx_1\wedge dx_2\wedge dx_3$

$$f \xrightarrow{grad} F = \begin{pmatrix} f_1 \\ \vdots \\ f_3 \end{pmatrix} \xrightarrow{curl} G = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \xrightarrow{div} g$$

$$\int_{\{P_i\}} w_0 = \sum f(P_i) \qquad \int_\gamma w_1 = \int \vec{F}\cdot\vec{T}\,ds \qquad \int_\Sigma w_2 = \int_A \vec{G}\cdot\vec{n}\,d\sigma \qquad \int_D w_3 = \int g\,dV$$

$$\int_\gamma \operatorname{grad} f\cdot \vec{T}\,ds \qquad \int_{\partial\Sigma} \vec{G}\cdot\vec{T}\,ds = \int_\Sigma \operatorname{curl}\vec{G}\cdot\vec{n}\,d\sigma \qquad \int_{\partial D} \vec{G}\cdot\vec{n}\,d\sigma = \int_D \operatorname{div}\vec{G}\,dV$$

---

de Rham, functoriality, homotopy, Poincaré.

Math 1300 Geometry and Topology, Nov 29 2007, hour 35

Maxwell eqns as on handout & on Nov 21, 1996.

מכניקה קלסית קוונטית למתמטיקאים, 12 לנובמבר 1996

3 שעורים: מקסוול/תיפ, הסתברות קוונטית, קוונטיזציה; 3×מילואים, 3×תרגיל + השלמות.

עוד על תבניות: 1. תזכורת Stokes

2. הלמה של פואנקרה. $I = P\circ d + d\circ P$ (מעל ~~$\mathbb{R}^n$~~)

3. תרגיל: הוכח $\mathcal{L}_X = d\circ i_X + i_X\circ d$

4. אינטגרציה בחלקים $\int \omega\wedge d\sigma = -(-1)^{\deg\omega}\int d\omega\wedge\sigma$ על תחום נטול שפה

5. האופרטור $*$: $\omega\wedge(*\omega) = \|\omega\|^2 dx^1\wedge\ldots\wedge dx^n$

מקיים $\omega\wedge*\sigma = \sigma\wedge*\omega$ עבור σ,ω מאותה דרגה.

משוואות מקסוול:

A: על $\mathbb{R}^4_{t,x,y,z}$ 1-תבנית

$$S(A) = \int_{\mathbb{R}^4} \tfrac{1}{2}\|dA\|^2 + \underset{\uparrow\ \text{3-תבנית}}{J\wedge A}$$

הערה: מכאן שאם תנאי יהיה מותר, ומשוואה לא יהיה יחיד.

אוילר לגרנז': $d*dA = J \implies$ א. תנאי $dJ = 0$

ב. כדאי לסמן $F = dA$.

$\implies dF = 0\ ;\ d*F = J$ אלה הן משוואות מקסוול!

נכתוב $F = E_x dx\wedge dt + E_y dy\wedge dt + E_z dz\wedge dt + B_x dy\wedge dz + \ldots$

$J = \rho\, dx\wedge dy\wedge dz - j_x\, dy\wedge dz\wedge dt - \ldots$

$dJ = 0 \implies$ חוק שימור המטען.

$$dF = \begin{cases} dx\wedge dy\wedge dz & \partial_x B_x + \partial_y B_y + \partial_z B_z \\ dy\wedge dz\wedge dt & \partial_y E_z - \partial_z E_y + \partial_t B_x \\ \vdots & \end{cases}$$

$\operatorname{div} B = 0$. אין מונופולים מגנטיים.

$\operatorname{curl} E = -\frac{\partial B}{\partial t}$ כבא"ע וכו' אלקטרומגנטי

$*F = -B_x dx\wedge dt - B_y dy\wedge dt \ldots - E_x dy\wedge dz + \ldots$

$$d*F = J \implies \begin{cases} -\operatorname{div} E = \rho & \text{זורי אלקטרוסטטיקה!} \\ \operatorname{curl} B = -\underbrace{\frac{\partial E}{\partial t}}_{\text{תרומת מקסוול, קריטי!}} + j & \text{כמו מגנוט דרקין} \end{cases}$$

אבוי! יצאה משוואה אלקטרודינמית הגיעו לנו. שכחנו את איינשטיין.

$$S = \int_{a_1}^{a_2} mc\, ds + eA = mc\int_{t_1}^{t_2}\sqrt{1-v^2/c^2}\,dt + e\left(A_t + A_x\dot{x} + A_y\dot{y} + A_z\dot{z}\right)dt$$

חלקיק בשדה E-M: $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

$ds = c\cdot dt\sqrt{1 - v^2/c^2}$, $v = \frac{dq}{dt}$

$\left(E = Mc^2\ ;\ E = p\dot{q} - F = \frac{mc^2}{\sqrt{1-v^2/c^2}}\ ;\ p = \frac{\partial F}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1-v^2/c^2}}\right.$:הערה, $\left. e=0\right)$!

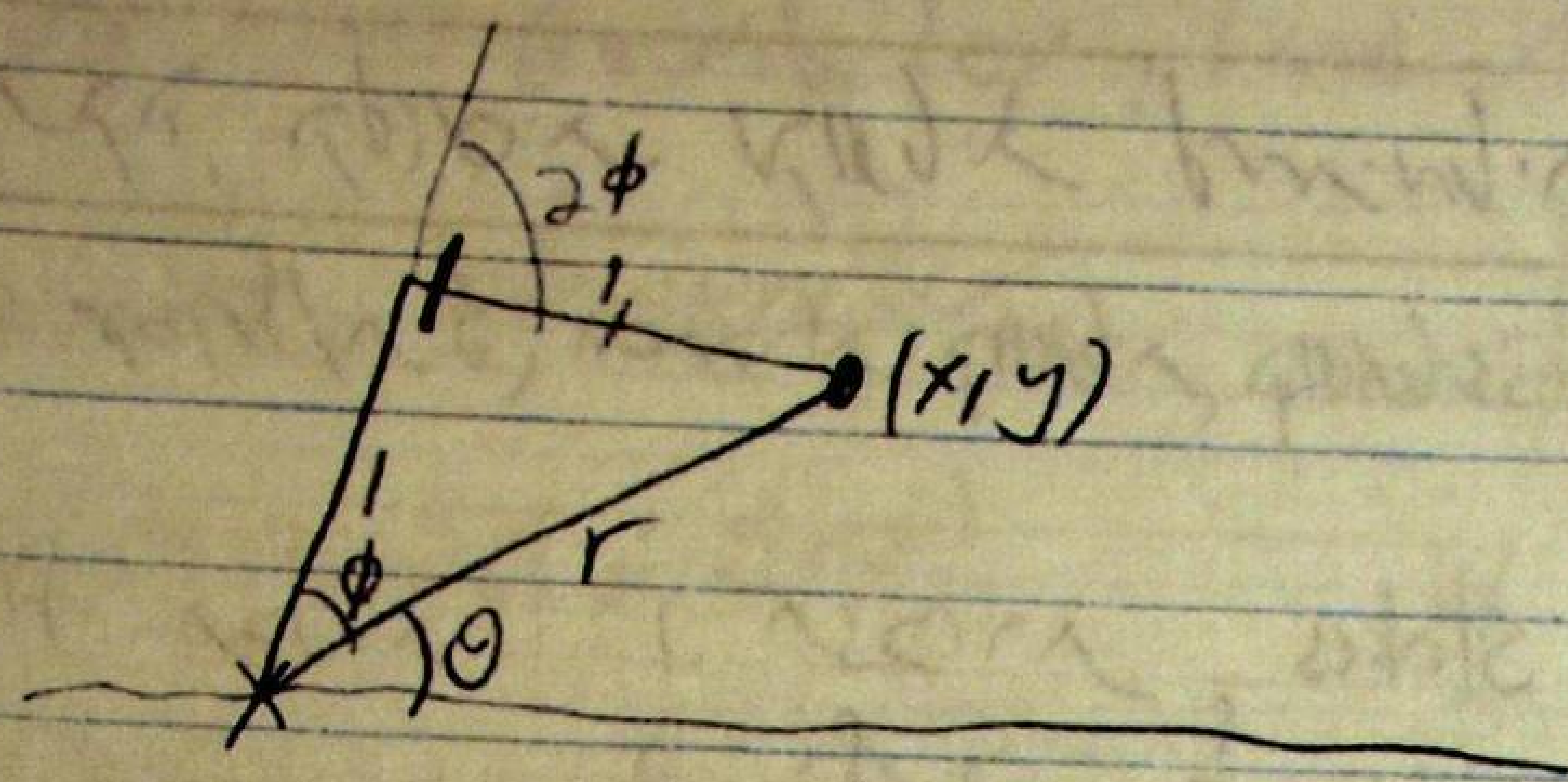
משוואות E-L:

$\partial_x A_t + (\partial_x A_x)\dot{x} + \partial_x A_y\dot{y} + (\partial_x A_z)\dot{z} = \left(\frac{\ldots}{\sqrt{1-v^2/c^2}}\right)'$

MooKeuk

מושגי יסוד בגאומטריה דיפרנציאלית, 23 לינואר 2001

הפלנימטר:



$$r = 2\cos\phi$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$W = \cos 2\phi \, d(\theta+\phi)$$

$$dw = -2 \sin 2\phi \, d\phi \wedge (d\theta + d\phi) = -2\sin 2\phi \, d\phi \wedge d\theta$$

$$= -4 \sin\phi\cos\phi (d\phi \wedge d\theta) =$$

$$\boxed{dr = -2\sin\phi \, d\phi}$$

$$= 2\cos\phi \, dr \wedge d\theta = r \, dr \wedge d\theta = dx \wedge dy$$

קוהומולוגיית דה-רהם; הדוגמה של T^2.

הלמה של פואנקרה ורעיון ההוכחה.

Table 18-1 Classical Physics

Maxwell's equations

I. $\nabla\cdot\boldsymbol{E}=\dfrac{\rho}{\epsilon_0}$	(Flux of $\boldsymbol{E}$ through a closed surface) $=$ (Charge inside)$/\epsilon_0$
II. $\nabla\times\boldsymbol{E}=-\dfrac{\partial\boldsymbol{B}}{\partial t}$	(Line integral of $\boldsymbol{E}$ around a loop) $=-\dfrac{d}{dt}$(Flux of $\boldsymbol{B}$ through the loop)
III. $\nabla\cdot\boldsymbol{B}=0$	(Flux of $\boldsymbol{B}$ through a closed surface) $=0$
IV. $c^2\nabla\times\boldsymbol{B}=\dfrac{\boldsymbol{j}}{\epsilon_0}+\dfrac{\partial\boldsymbol{E}}{\partial t}$	c^2(Integral of $\boldsymbol{B}$ around a loop) $=$ (Current through the loop)$/\epsilon_0$ $+\dfrac{\partial}{\partial t}$(Flux of $\boldsymbol{E}$ through the loop)

[**Conservation of charge**

$\nabla\cdot\boldsymbol{j}=-\dfrac{\partial\rho}{\partial t}$ (Flux of current through a closed surface) $=-\dfrac{\partial}{\partial t}$(Charge inside)]

Force law

$$\boldsymbol{F}=q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})$$

Law of motion

$$\frac{d}{dt}(\boldsymbol{p})=\boldsymbol{F},\quad\text{where}\quad \boldsymbol{p}=\frac{m\boldsymbol{v}}{\sqrt{1-v^2/c^2}}$$

(Newton's law, with Einstein's modification)

Gravitation

$$\boldsymbol{F}=-G\frac{m_1m_2}{r^2}\boldsymbol{e}_r$$

A BIT ON MAXWELL'S EQUATIONS

Prerequisites:

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\omega = ||\omega||^2 dx_1 \cdots dx_n$ and $\omega \wedge \star\eta = \eta \wedge \star\omega$ whenever ω and η are of the same degree.
- Integration by parts: $\int \omega \wedge d\eta = -(-1)^{\deg \omega} \int (d\omega) \wedge \eta$ on domains that have no boundary.

The Action Principle: The *Vector Field* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2}||dA||^2 dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current.*

The Euler-Lagrange Equations is this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\dfrac{\partial \rho}{\partial t} + \operatorname{div} j = 0$	"conservation of charge"
	$\operatorname{div} B = 0$	"no magnetic monopoles"
$dF = 0 \Longrightarrow$	and	
	$\operatorname{curl} E = -\dfrac{\partial B}{\partial t}$	that's how generators work!
	$\operatorname{div} E = -\rho$	"electrostatics"
$d * F = J \Longrightarrow$	and	
	$\operatorname{curl} B = -\dfrac{\partial E}{\partial t} + j$	that's how electromagnets work!

Exercise: Use the Lorentz metric to fix the sign error!

0708-1300/Homework Assignment 6

From Drorbn

Contents

0708-1300/Navigation Panel [Hide]

Add your name / see who's in!

#	Week of...	Links
		Fall Semester
1	Sep 10	About, Tue, Thu
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3	Sep 24	Tue, Photo, Thu
4	Oct 1	Questionnaire, Tue, HW2, Thu
5	Oct 8	Thanksgiving, Tue, Thu
6	Oct 15	Tue, HW3, Thu
7	Oct 22	Tue, Thu
8	Oct 29	Tue, HW4, Thu, Hilbert sphere
9	Nov 5	Tue,Thu, TE1
10	Nov 12	Tue, ~~Thu~~
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12	Nov 26	Tue, Thu
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		Errata to Bredon's Book

Reading

At your leisure, read your class notes over the break, and especially at some point right before classes resume next semester. Here are a few questions you can ask yourself while reading:

- Do you understand pullbacks of differential forms?
- Do you think you could in practice integrate any differential form on any manifold (at least when the formulas involved are not too messy)?
- Do you understand orientations and boundaries and how they interact?
- Why is Stokes' theorem true? Both in terms of the local meaning of d, and in terms of a formal proof.
- Do you understand the two and three dimensional cases of Stokes' theorem?
- Do you understand the Hodge star operator $\star$?
- How did we get $d\star dA = J$ from the least action principle?
- Do you understand how Poincare's lemma entered the derivation of Maxwell's equations?
- Do you understand the operator P? (How was it used, formally derived, and what is the intuitive picture behind it?)
- What was H_{dR} and how did it relate to pullbacks and homotopy.

Doing

Solve the following problems and submit your solutions of problems 1, 3 and 4. This is a very challenging collection of problems; I expect most of you to do problem 2 with no difficulty (it is a repeat of an older problem), problem 1 with some effort, and I hope each of you will be able to do at least one further problem. It will be great if some of you will do all problems!

Problem 1. If M is a compact orientable n-manifold with no boundary, show that $H^n_{dR}(M)\neq 0$.

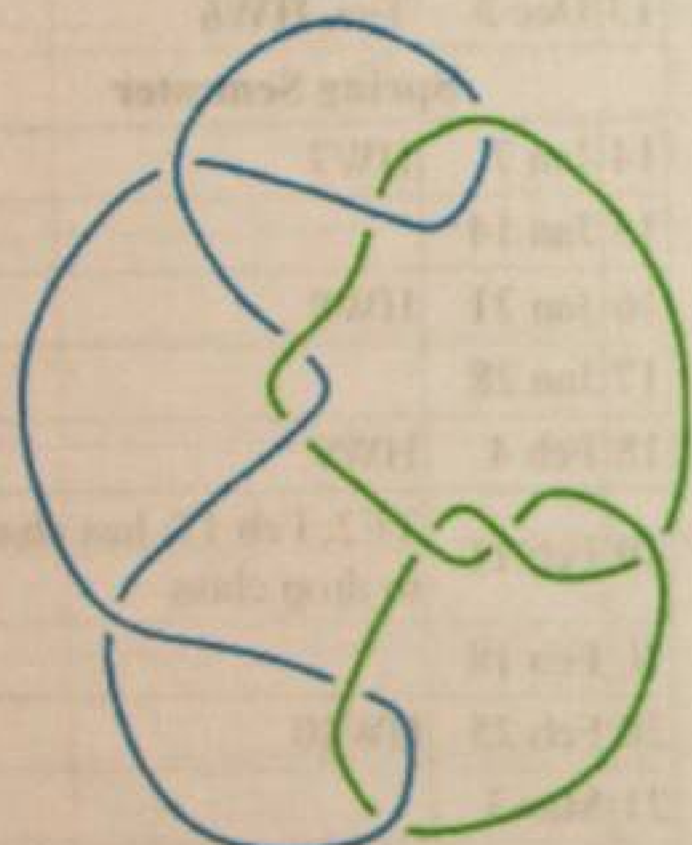
Problem 2. The standard volume form on S^2 is the form ω given by

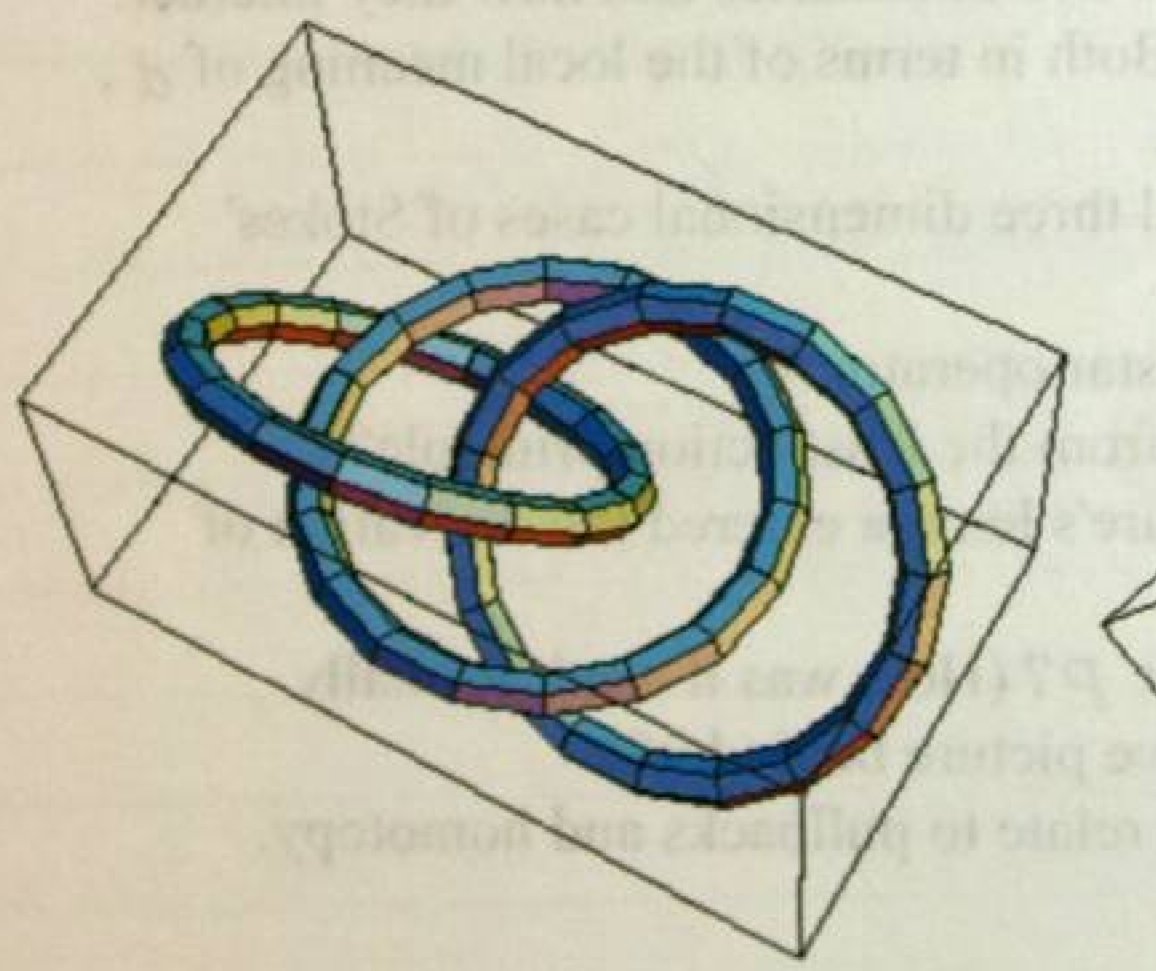
$\omega = \frac{1}{4\pi}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$. Show that $\int_{S^2} \omega = 1$.

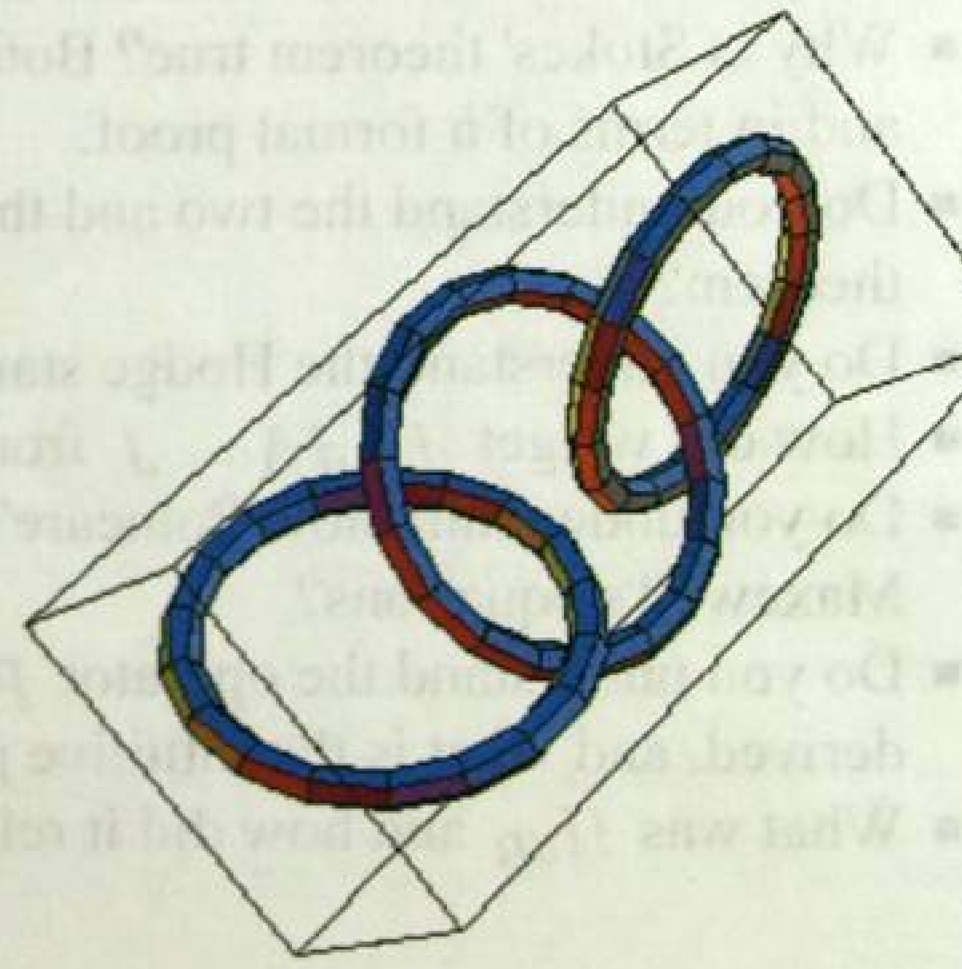
Problem 3. Show that if $\omega \in \Omega^2(S^2)$ satisfies $\int_{S_2} \omega = 0$, then ω is exact. Deduce that if $w_1 \in \Omega^2(S^2)$ and $w_2 \in \Omega^2(S^2)$ satisfy $\int_{S_2} \omega_1 = \int_{S_2} \omega_2$, then $[\omega_1] = [\omega_2]$ as elements of $H^2_{dR}(S^2)$. Deduce further that $\dim H^2_{dR}(S^2) = 1$.

Problem 4. A "link" in $\mathbb{R}^3$ is an ordered pair $\gamma = (\gamma_1, \gamma_2)$, in which γ_1 and γ_2 are smooth embeddings of the circle S^1 into $\mathbb{R}^3$, whose images (called "the components of γ") are disjoint. Two such links are called "isotopic", if one can be deformed to the other via a smooth homotopy along which the components remain embeddings and remain disjoint. Given a link γ, define a map $\Phi_\gamma : S^1 \times S^1 \to S^2$ by $\Phi_\gamma(t_1, t_2) := \frac{\gamma_2(t_2) - \gamma_1(t_1)}{||\gamma_2(t_2) - \gamma_1(t_1)||}$. Finally, let ω be the standard volume form of S^2, and define "the linking number of $\gamma = (\gamma_1, \gamma_2)$" to be $l(\gamma) = l(\gamma_1, \gamma_2) := \int_{S^1 \times S^1} \Phi^\star_\gamma \omega$. Show

1. If two links γ and γ' are isotopic, then their linking numbers are the same: $l(\gamma) = l(\gamma')$.
2. If ω' is a second 2-form on S^2 for which $\int_{S^2} \omega' = 1$ and if $l'(\gamma)$ is defined in the same manner as $l(\gamma)$ except replacing ω with ω', then $l(\gamma) = l'(\gamma)$. (In particular this is true if ω' is very close to a δ-function form at the north pole of S^2).
3. Compute (but just up to an overall sign) the linking number of the link L11a193 (http://katlas.org/wiki/L11a193) , displayed below:







The links L11a193, γ_3 and γ'_3.

Just for Fun

Prove that the two (3-component) links γ_3 and γ'_3 shown above are not isotopic, yet their complements are diffeomorphic. (See more at Classes: 2004-05: Math 1300Y - Topology: Homework Assignment 5 (http://www.math.toronto.edu/~drorbn/classes/0405/Topology/HW5/HW.html))

Due Date

This assignment is due in class on Thursday January 10, 2007.

Math 1300 Geom & Top, Dec 6 2007, hour 38.

* Finish homotopy proof
* H_{dR}, $H^2_{dR}(S^2)\neq 0$, homotopy on H_{dR}.
* Geometrical meaning of homotopy proof.

on board.

$$
\begin{tikzcd}
\Omega^{k-1}(N) \arrow[r, "d"] \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] & \Omega^{k}(N) \arrow[r, "d"] \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] \arrow[dl, "P"'] & \Omega^{k+1}(N) \arrow[d, shift right, "f^*"'] \arrow[d, shift left, "g^*"] \arrow[dl, "P"'] \\
\Omega^{k-1}(M) \arrow[r, "d"] & \Omega^{k}(M) \arrow[r, "d"] & \Omega^{k}(M)
\end{tikzcd}
$$

Thm If $f,g: M\to N$ are homotopic, then $\exists P: \Omega^k(N)\to\Omega^{k-1}(M)$ s.t. $f^*-g^*=dP+Pd$.

$\Omega^*(M\times I_t)$ with d raises, $\mathcal{L}=\frac{\partial}{\partial t}$ keep, ι lowers $=$ "feed" ∂t

$\mathcal{L}=d\iota+\iota d$

Geom

$\downarrow i_0^*$, $\downarrow i_1^*$, $\downarrow \mathrm{int}=\int_0^1 dt\, i_t^*$

$\Omega^*(\,=M)$

$\Rightarrow$ $i_1^*w-i_0^*w=d\,\mathrm{int}\,\iota w+\mathrm{int}\,\iota\, dw$

$\Rightarrow$ succes w/ $P=d\,\mathrm{int}\,h^*$.

* H_{dR}, $H^2_{dR}(S^2)\neq 0$, homotopy on H_{dR}
* Geometrical meaning.

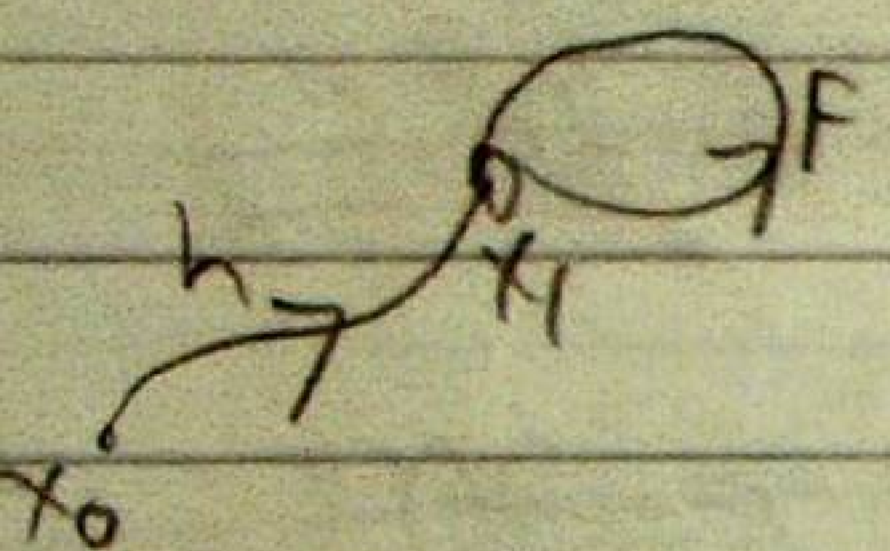
טופולוגיה אלגברית - 12 לינואר 2002

מסילה, הומוטופיה, הומוטופיה היא יחס שקילות, $[f]$

<u>הגדרה</u> $\pi_1(X,x_0)=\{[f:I\to X]: f(0)=f(1)=x_0\}$

<u>משפט</u> $\pi_1(X,x_0)$ היא חבורה יחסית לכפל

$$[f]\cdot[g]=[f\cdot g] \qquad f\cdot g(s)=\begin{cases} f(2s) & 0\le s\le 1/2 \\ g(2s-1) & 1/2\le s\le 1\end{cases}$$



<u>טענה</u> $\beta_h:\pi_1(X,x_1)\to\pi_1(X,x_0)$ היא איזומורפיזם

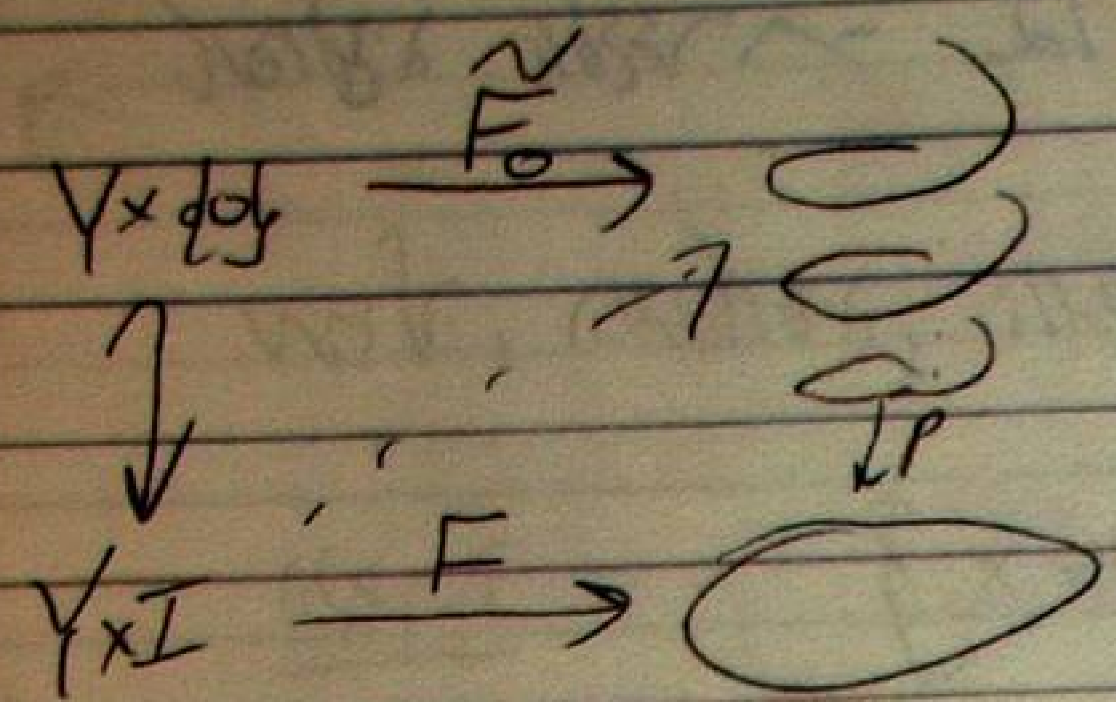
<u>מסקנה</u> $\pi_1(X)$ עבור X קשיר מסילתית
זה $\pi_1(X)$ נקבעת ביחידות עד כדי איזומורפיזם

$\varphi:(X,x_0)\to(Y,y_0)$ משרה פונקטוריאלית $\varphi_*:\pi_1(X)\to\pi_1(Y)$

<u>משפט</u> $\pi_1(S^1)=\mathbb{Z}$ <u>מסקנות:</u> 1. המשפט היסודי של האלגברה
2. אי קיום הסגה $D^2\to S^1$
3. נקודת השבת של בראואר
4. בורסוק-אולם.

<u>הוכחה:</u>

טופולוגיה אלגברית 14 לינואר 2002



<u>משפט</u> "תכונת ההרמה" $p(t)=e^{2\pi it}$
יש הרמה משותפת והיא יחידה.

<u>מסקנה</u> $\pi_1(S^1)=\mathbb{Z}$
<u>מסקנה</u> המשפט היסודי של האלגברה

$\varphi:X\to Y \Rightarrow \varphi_*:\pi_1(X)\to\pi_1(Y)$ (+ התנהגות יפה להרכבות)

~~מסקנה משפט נקודת השבת של בראואר~~

~~שיעורי בית: קראו על משפט בורסוק-אולם.~~

עוד תכונות של החבורה היסודית:
<u>הערה</u> התלות בנקודת הבסיס.

<u>טענה</u> $\pi_1(X\times Y)=\pi_1(X)\times\pi_1(Y)$

<u>טענה</u> אם φ_0,φ_1 הומוטופיות אזי $\varphi_{0*}=\varphi_{1*}$

<u>טענה</u> אם (X,x_0) (Y,y_0) שקולות הומוטופית אזי $\pi_1(X)\cong\pi_1(Y)$

<u>טענה</u> זה נכון אפילו אם השקילות היא לא דווקא במרחבים מנוקדים

הקדמה על משפט ואן-קאמפן

<u>שיעורי בית</u>: קראו עמודים 21-37 בספר.

טופולוגיה אלגברית 19 ליוני 2002

עוד תכונות של החבורה היסודית:

תלות בנקודת הבסיס; $\pi_1(X\times Y)=\pi_1(X)\times\pi_1(Y)$

<u>טענה</u>: אם $\varphi_{0,1}: X\to Y$ הומוטופיים אז $\varphi_{0*}=\varphi_{1*}$

<u>טענה</u>: אם $(X,x_0),(Y,y_0)$ שקולים הומוטופית אז: $\pi_1(X)\cong\pi_1(Y)$

<u>טענה</u>: זה נכון אפילו אם הנקודה הזו לא קבועה במהלך ההומוטופיה.

$\varphi: X\to Y$ שקילות $\Rightarrow$ $\varphi_*:\pi_1(X,x_0)\xrightarrow{\cong}\pi_1(Y,\varphi(x_0))$

$$\begin{tikzcd}
G_{\alpha_1} \arrow[dr] \arrow[drr, bend left, "\varphi_{\alpha_1}"] & & \\
G_{\alpha_2} \arrow[r] & G \arrow[r, dashed, "\exists!\Phi"] & H \\
\vdots & & \\
G_\alpha \arrow[uur] \arrow[uurr, bend right, "\varphi_\alpha"'] & &
\end{tikzcd}$$

המכפלה: $*_\alpha G_\alpha$

<u>משפט ואן קמפן</u>: אם $X=\bigcup A_\alpha$ הוא איחוד של פתוחות קשירות ~~מסילתית~~ מסילתית המכילות כולן את x_0 אזי:

1. אם $A_\alpha\cap A_\beta$ קשיר מסילתית אזי $\Phi: *_\alpha\pi_1(A_\alpha)\to\pi_1(X)$ הוא על

2. אם $A_\alpha\cap A_\beta\cap A_\gamma$ קשיר מסילתית, אזי הגרעין של Φ הוא החבורה הנורמלית בתוך $*_\alpha\pi_1(A_\alpha)$ הנוצרת ע"י

$$i_{\alpha\beta}(w)\cdot i_{\beta\alpha}(w)^{-1}$$

כאשר $i_{\alpha\beta}: A_\alpha\cap A_\beta\hookrightarrow A_\alpha$ ו- $w\in\pi_1(A_\alpha\cap A_\beta)$

<u>דוגמא</u>: החבורה היסודית של T^2; של גרף; של S^n

<u>ש"ב</u>: קראו מונקרס 40-52

מאי 30 ⇐ מאי 29
אני ... ו...

טופולוגיה אלגברית, 11 שעור 2002

משפט: $X=\bigcup A_\alpha$ הנ"ל, אזי קשירים,

$$\Phi: *_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$$

אזי $\ker\Phi$ נוצר ע"י: $i_{\alpha\beta}(w)=i_{\beta\alpha}(w)$
כאשר $i_{\alpha\beta}: A_\alpha\cap A_\beta \hookrightarrow A_\alpha$ ו- $w\in\pi_1(A_\alpha\cap A_\beta)$

מרחבי כיסוי: הגדרה $p:\tilde{X}\to X$, $\tilde{x}_0 \to x_0$; דוגמאות: $\mathbb{R}^1\to S^1$
$S^1 \xrightarrow{z^n} S^1$
$\mathbb{C}\setminus 0 \xrightarrow{z^n} \mathbb{C}\setminus 0$ (משטח רימן)
מקרה ...
הדוגמאות ...

תכונות הרמת ההומוטופיות:
בהינתן הומוטופיה $f_t:Y\to X$ והרמה
$\tilde{f}_0:Y\to\tilde{X}$ של f_0, קיימת הרמה יחידה $\tilde{f}_t:Y\to\tilde{X}$ של f_t
המשמרת את $\tilde{f}_0$
בפרט עבור מסלולים...

מסקנה: $p_*:\pi_1(\tilde{X},\tilde{x}_0)\to\pi_1(X,x_0)$ הוא חח"ע, ותמונתו המסלולים
שהרמתם מסלול

מסקנה: אם $X,\tilde{X}$ קשירים אזי מספר הכסויים של p הוא האינדקס של $p_*\pi_1(\tilde{X})$ בתוך $\pi_1(X)$

טענה: בהינתן $f:(Y,y_0)\to(X,x_0)$ אם Y קשיר וקשיר מקומית אזי קיימת הרמה $\tilde{f}$ של f אמ"מ $f_*(\pi_1(Y,y_0))\subset p_*\pi_1(\tilde{X},\tilde{x}_0)$

טענה: אם $\tilde{f}_{1,2}$ הרמות של $f:Y\to X$ המסכימות בנקודה אחת ואם Y קשיר, ההרמות זהות.

TEI: Tue Nov 11 6-8 PM
SS 1087

Math 1300 Topology, Nov 11 2004

Cont. as on Nov 9, 2004.

Web last handout!

Math 1300 Topology, Nov 16 2004.

1. Finish path lifting 2. Functoriality of π_1

3. Homotopy invariance of π_1 (under maps, under homotopy equiv.)

4 Brouwer.

Math 1300 Topology, Thursday Nov 18 2004.

The story with Q6. (see Munkres Thm 48.5)

Homotopy theory light

* Product spaces
* change of basepoint.
* Functoriallity.
* Homotopic maps.
* Homotopy equivalences
* No retract $r : D^2 \to S^1$
* Brouwer in D^2.
* The fundamental theorem of algebra.

תכנית החבורה השבוע:
שבוע 1. חבורה של חבורות(?) הגדרה ותכונות בסיסיות $\pi_1(S^1)$
שבוע 2. משפט נקודת השבת של ברואר. תכונות יסודיות יחסית לשקילות הומוטופיה.
שבוע 3. משפט ואן-קמפן ודוגמאות
שבוע 4. הוכחת ואן-קמפן.

מבוא לטופולוגיה 9 ביוני 1996

הגדרת חבורה

דוגמאות $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Q}^*$, זוויות (S^1)

מטריצות הפיכות מעל המרוכבים, S_n, החבורה החפשית

יוצרים ויחסים

דוגמאות: $\langle a \rangle = \mathbb{Z}$; $\langle a : a^2 = 1 \rangle =$ חבורה ציקלית מסדר 2

$\langle a,b ; a^2=b^2=1 ; aba=bab \rangle$: S_3

הומומורפיזם של חבורות. דוגמאות: det, ...

הגדרה מסילה; הומוטופיה של מסילות; זהו יחס שקילות.

הגדרה $\pi_1(X,b)$ בקבוצה; חוק הכפל.

דוגמאות $\{e\} = \pi_1(D^2,0)$; $\pi_1(\mathbb{R}^n,0)=0$

משפט $\pi_1(S^1,1)=\mathbb{Z}$

למה עיקרית: יהי $e:\mathbb{R}\to S^1$ ע"י $e(x)=e^{2\pi i x}$. לכל מסילה $\gamma:[0,1]\to S^1$ עם $\gamma(0)=1$ יש הרמה רציפה יחידה למסילה $\tilde{\gamma}:[0,1]\to\mathbb{R}$ עם $\tilde{\gamma}(0)=0$ וגם $\gamma = e\circ\tilde{\gamma}$.

הוכחת המשפט מהלמה: תהי $\gamma \in \pi_1(S^1,1)$. נגדיר $\mathrm{ind}\,\gamma = \tilde{\gamma}(1)$

טענות: 1. $\mathrm{ind}\,\gamma \in \mathbb{Z}$ עבור γ

2. $\mathrm{ind}\,\gamma$ לא תלוי בבחירת γ מתוך מחלקת השקילות (לצורך זה יש לנסח את הלמה להומוטופיה)
(לכן ind מוגדר היטב)

3. ind כפלי.

4. ind חח"ע ועל.

5. ind הוא הומומורפיזם של חבורות.

הוכחת הלמה העיקרית: יהי $\{U_i\}$ כיסוי של S^1 בפתוחות

[illegible] $\{\gamma^{-1}(U_i)\}$ כיסוי פתוח של $[0,1]$,

ועל פי הלמה של לבג קיים δ ... יהי M כך ש $\frac{1}{M}<\delta$ ונסתכל על

טענה. נניח יש הרמה יחידה עד k/M. אזי ... עד $(k+1)/M$

דרוש: זו ההתחלה של "תורת הכיסויים"

Math ~~0~~1300 Topology, Tues Nov 22 2005 (2 hours)

on board: Today's motto.

Evil is everywhere

Thm "Almost all" cont. functions are nowhere diff'able.

Thin sets, Baire spaces, Baire's Theorem, pf of Evil thm.

(compare with Nov 2, 2004)

Def locally compact

def αX Thm: it is compact.

Thm: A T_2 space is loc. compact iff it is a compact space with one point removed

Math 1300 Topology, Thursday Nov 24 2005 (1 hour)

The fundamental group

paths, homotopy of paths, this is equiv, $\pi_1(X,b)$ as ~~a~~ set, multiplication, group properties

Examples: $\pi_1(D^1,0)=0$ $\pi_1(\mathbb{R}^n,0)=0$

Thm $\pi_1(S^1,1)=\mathbb{Z}$

cont. as on June 9, 1996.

plan: 1. $\pi_1(S^1) = \mathbb{Z}$
2. corollaries
3. Homology Lite.

Math 1300 Topology, Tuesday Nov 29 2005 (2 hours)

π_1: A functor from based T.S to groups

$$\pi_1(X,b) = \{[\gamma]\} = \{\gamma : [0,1] \to X : \gamma(0)=\gamma(1)=b\}/\text{homotopy}$$

group under concatenation

Thm $\pi_1(S^1, 1) = \mathbb{Z}$

Pf as on June 9, 1996; except for general coverings

Corollaries: 1. The fundamental Thm of algebra
2. No retract $r : D^2 \to S^1$
3. Brouwer in D^2 !
4. Borsak-Ulam

Math 1300 Topology, Thursday ~~Nov 29~~ Dec 1 2005 (1 hour)

on board

A covering: $p : X \to B$ s.t. B is covered by open sets U s.t.

$p^{-1}(U) \cong U \times F$ "over U" i.e.,

$$\begin{tikzcd} & p^{-1}(U) \arrow[r, "\varphi", "\sim"'] \arrow[d, "p"] & U \times F \arrow[dl, "\pi_U"] \\ \text{comb} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & U & \end{tikzcd}$$

Problem: A comb $Y \times I$ maps to U via γ, a lift $\tilde{\gamma}$ is given on $Y \times \{0\}$, find a lift $\tilde{\gamma}$ in general.

Proven: 1. No retract $r : D^2 \to S^1$
2. If $f : D^2 \to D^2$ satisfies $f|_{S^1} = Id_{S^1}$, f is onto
3. Brouwer F.P. theorem: Any $f : D^2 \to D^2$ has a F.P.

Today: 4. The fundamental theorem of algebra.
5. Borsuk-Ulam: Any $f : S^2 \to \mathbb{R}^2$ maps identifies a pair of antipodes

Math 1300 Geom & Top, Jan 8 2008, hours 1-2.

The general philosophy of algebraic topology –

* Invariants & separation
* Functoriality & Brouwer.

Forget manifolds? Forget smooth?

The fundamental group.

Pointed spaces, paths, homotopy of paths, this is an equiv. rel.; $\pi_1(X, x_0)$ as a set, multiplication, the group properties.

Examples $\pi_1(\mathbb{R}^n, 0) = 0$; $\pi_1(S^1, 1) = \mathbb{Z}$

Functoriality. Cor Brouwer.

Thm $\pi_1(S^1, 1) = \mathbb{Z}$.

Lemma Let $e : \mathbb{R} \to S^1$ be given by $e(x) = e^{2\pi i x}$.

Then every path $\gamma : [0,1] \to S^1$ with $\gamma(0) = 1$ has a unique cont. lift $\tilde{\gamma} : [0,1] \to \mathbb{R}$ s.t.

$$\tilde{\gamma}(0) = 0 \quad \& \quad \gamma = e \circ \tilde{\gamma}$$

Pf of Thm from lemma: Let $[\gamma] \in \pi_1(S^1, 1)$. Set

$\operatorname{ind} \gamma = \tilde{\gamma}(1)$.

Claims
1. $\operatorname{ind} \gamma \in \mathbb{Z}$
2. $\operatorname{ind} \gamma$ depends only on $[\gamma]$.
3. ind is onto
4. ind is 1-1.
5. ind is a group homomorphism.

~~#~~ $\boxed{\text{HW7 is on web}}$

Math 1300 Geom & Top, Jan 10 2008, hour ~~II~~ / 3

Thm $\pi_1(S^1,1) \cong \mathbb{Z}$ via $\gamma \mapsto \tilde{\gamma}(1) =: \text{ind}[\gamma]$

Lemma $e:\mathbb{R}\to S^1$ is $e(x)=e^{2\pi i x}$. Then every path $\gamma:[0,1]\to S^1$ w/ $\gamma(0)=1$ has a unique cont. lift $\tilde{\gamma}:[0,1]\to\mathbb{R}$ w/

1. $\tilde{\gamma}(0)=0$ 2. $\gamma = e\circ\tilde{\gamma}$

Debts 1. pf of lemma
2. ind$[\gamma]$ well def
X3. ind a homomorphism.

Generalize to covering spaces.

Prove.

Generalize to "families"

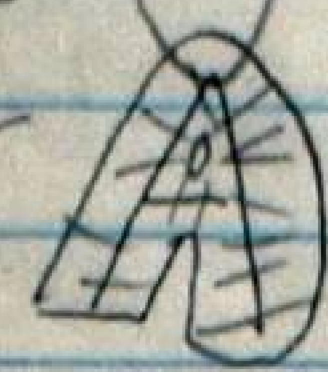
Complete Thm.

Math 1300 Topology, Tues Nov 23 2004

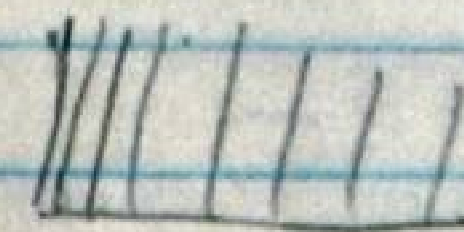
* TEI apology again
* Distribute TEI & HW3
* Godel's theorem?
* Groups defined by generators and relations?

Continue with Homotopy Theory Lite:

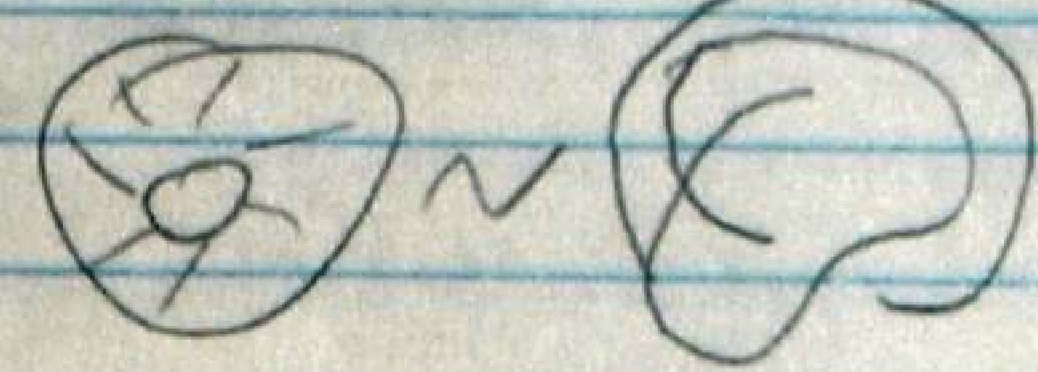
* Functoriality
* Homotopic maps
* Homotopy equivalences



"a deformation retract"



"comb space"



but not homeomorphic!

---

fun

* No retract $r: D^2 \to S^1$
* Brouwer in D^2 $\qquad \mathbb{R}^2 \neq \mathbb{R}^n$ $n \neq 2$
* The fundamental theorem of algebra.
* Borsuk-Ulam (So S^2 isn't a subspace of $\mathbb{R}^2$)

(If $S^2 = A_1 \cup A_2 \cup A_3$ (closed sets) then at least one of them contains antipodals)
(ask about 4)

---

pushouts?

*Grading issues

* $\pi_1(S^2)$

Math 1300 Topology, Thursday Nov 25 2004

If $\gamma: S^1 \to S^1$ is even, $\deg \gamma$ is even
is odd $\deg \gamma$ is odd.

Pf 1: lift $[0, 1/2]$

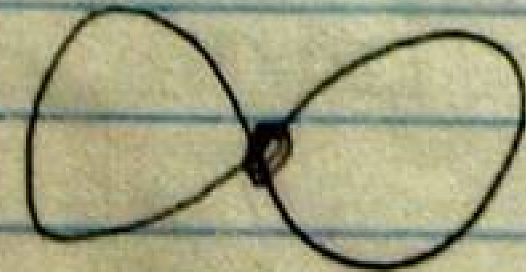
Pf 2 Use ~~group~~ Topological groups.

Borsuk-Ulam 1. S^2 is not a subspace of $\mathbb{R}^2$

2. $S^2 = A_1 \cup A_2 \cup A_3$ closed sets, at least one contains a pair of antipodes. (ask about 4)

Tues, Nov 30 1300:
Van-Kampen: under favorable conditions

$$\pi_1(U \cup V) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

Examples
1. 

2. T^2

3. $\mathbb{P}^2$

4. $S^n \quad n \geq 2$

5. $(T(p,q))^c$

Any finitely presented group is $\pi_1(X)$!

Math 1300 Topology, Thursday Dec 2, 2004

* Push outs in a general $\mathcal{C}$

$$\begin{tikzcd}
A \arrow[r, "i_1"] \arrow[d, "i_2"'] & B_1 \arrow[d, "j_1"] \arrow[ddr, bend left] & \\
B_2 \arrow[r, "j_2"] \arrow[drr, bend right] & C \arrow[dr, "\exists ! \phi"] & \\
 & & C'
\end{tikzcd}$$

<u>claim</u> C is unique up to isomorphism.

<u>Example 1</u> in Top: (First in Set)

$$\begin{tikzcd}
U_1 \cap U_2 \arrow[r, hook, "i_1"] \arrow[d, hook] & U_1 \arrow[d, "j_1"] \arrow[ddr, bend left] & \\
U_2 \arrow[r, "j_2"] \arrow[drr, bend right] & U_1 \cup U_2 & \\
 & & X
\end{tikzcd}$$

<u>Example 2</u> in Groups:

$$\begin{tikzcd}
H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d] \arrow[ddr, bend left] & \\
G_2 \arrow[r] \arrow[drr, bend right] & G_1 *_H G_2 \arrow[dr] & \\
 & & \Gamma
\end{tikzcd}$$

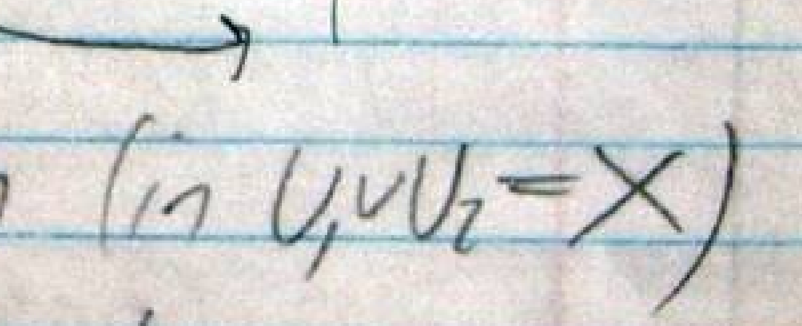
<u>Van Kampen</u> if U_1, U_2 are open (in $U_1 \cup U_2 = X$) and $b \in U_1 \cap U_2$ and $U_1 \cap U_2$ is pathwise connected, the

$$\pi_1(U_1 \cup U_2) = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$$
$$= \pi_1(U_1) * \pi_1(U_2) \Big/ \begin{matrix}\forall \gamma \in \pi_1(U_1 \cap U_2) \\ i_1\gamma = i_2\gamma\end{matrix}$$

<u>Pf</u> $\alpha : G_1 *_H G_2 \longrightarrow \pi_1(U_1 \cup U_2)$ — obvious.

$\beta : \pi_1(U_1 \cup U_2) \longrightarrow G_1 *_H G_2$ by mapping

$\gamma \longmapsto \gamma_p^b := \gamma_1 \beta_1^{-1} \cdot \beta_1 \gamma_2 \beta_2^{-1} \cdots$

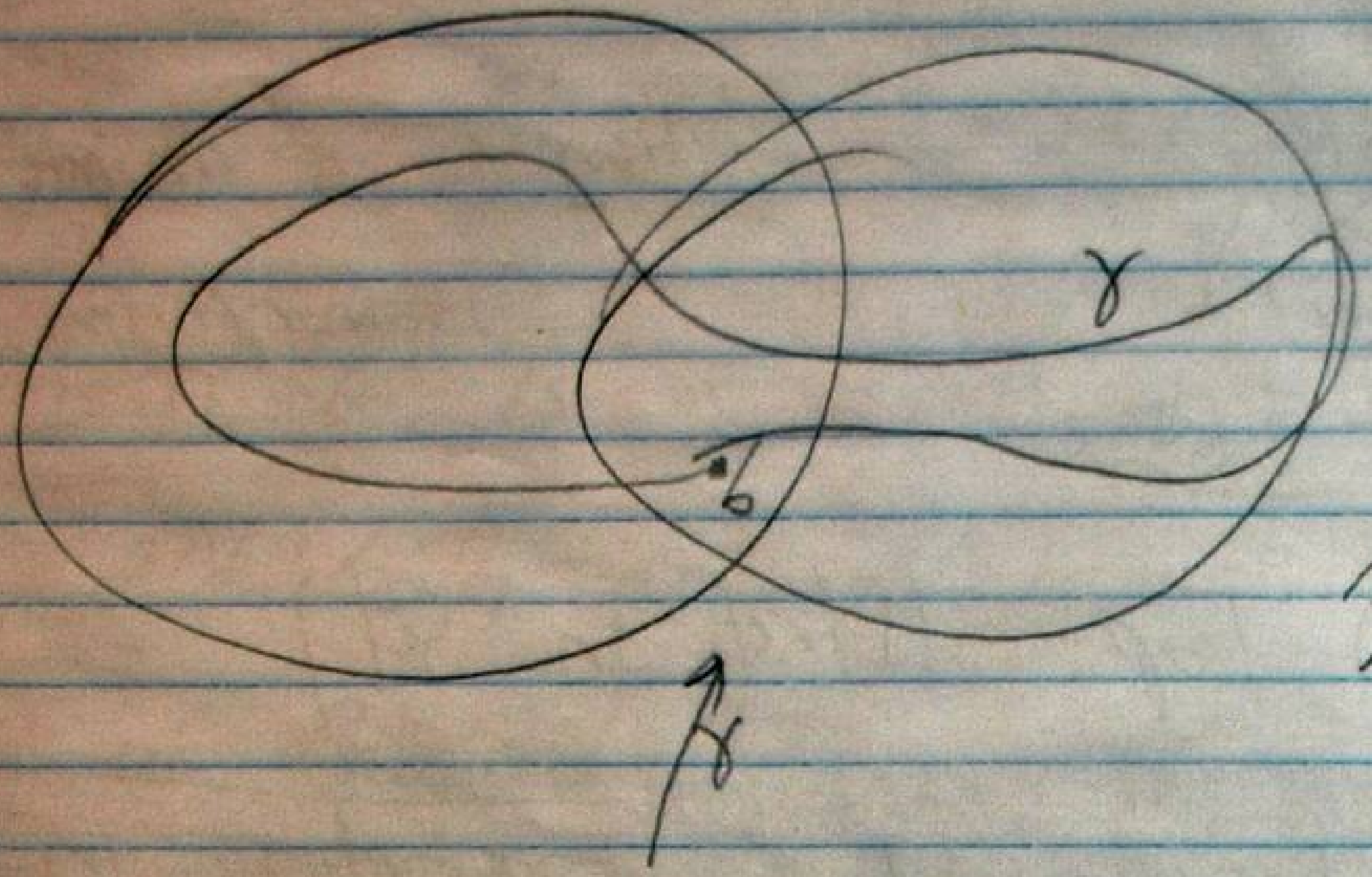


where γ^β is γ with "$\gamma(t_i)$ pulled to b along β_i"

1. dep on β.
2. dep on P.
3. dep on γ.

Math 1300 Topology, Tuesday Dec 7 2004

Goal: Prove Van-Kampen and go.



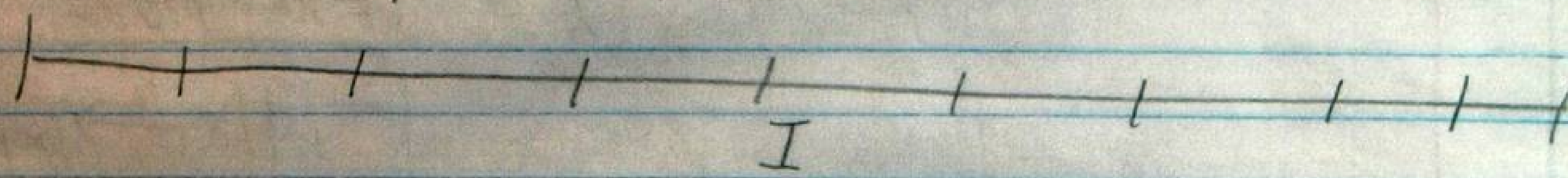
$G_1 = \pi_1(U_1)$

$G_2 = \pi_1(U_2)$

$H = \pi_1(U_1 \cap U_2)$

$\eta: G_1 *_H G_2 \to \pi_1(U_1 \cup U_2)$

the obvious.



Define $\sigma: \pi_1(U_1 \cup U_2) \to G_1 *_H G_2$ in steps.
$\gamma \in$

1. Partition $[0,1]$ so that $\gamma(\text{each part}) \subset U_1$ or U_2
2. Consolidate so $\gamma(t_i) \in U_1 \cap U_2$ & write $\gamma = \gamma_1 \gamma_2 \gamma_3 \ldots$
3. Pinch at each t_i choosing β_i w/ $\beta_i(0) = \gamma(t_i)$, $\beta_i(1) = b$ in $U_1 \cap U_2$
4. Set

$$\sigma_{\beta_i}(\gamma) = (\gamma_1 \beta_1)(\beta_1^{-1} \gamma_2 \beta_2)(\beta_2^{-1} \gamma_2 \beta_3) \ldots$$

<u>Claim</u> 1. Invariant under subdivision

2. Indep. ~~[illegible]~~ of β_i

3. homotopy invariant.

4. $\sigma \circ \eta = I_{G_1 *_H G_2}$

5. $\eta \circ \sigma = I_{\pi_1(U_1 \cap U_2)}$

Math 1300 Topology, Tuesday Dec 6 2005 (2 hours)

on board

Borsuk-Ulam: $f: S^2 \longrightarrow \mathbb{R}^2 \Rightarrow \exists\, x \in S^2$ s.t. $f(x) = f(-x)$

Thm $\gamma: S^1 \longrightarrow S^1$ ~~even~~ even $\Rightarrow [\gamma] = \deg \gamma$ even
odd $\Rightarrow \deg \gamma$ odd

Thm If G is a topological group, $[\gamma_1 \cdot \gamma_2] = [\gamma_1 * \gamma_2]$ and both are Abelian.

Cor. of Borsuk-Ulam If $S^1 = A_1 \cup A_2 \cup A_3$ is a union of 3 closed sets, at least one of them contains a pair of antipodes (4?)

---

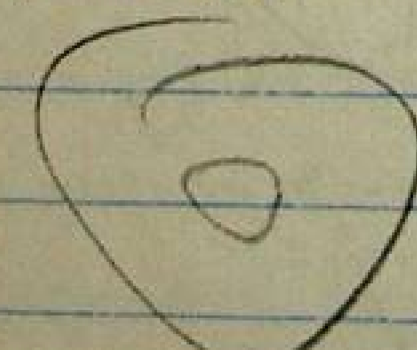
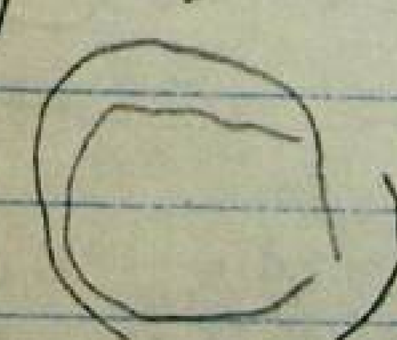
proofs of all of the above

Homotopy Lift: 1. change of base points

(Why not π_1: category of unbased spaces $\Rightarrow$ cat. of iso. classes of gps

2. homotopic maps.

3. homotopy equivalent spaces. (The category of spaces & homotopy equiv. classes of maps does make

Examples  

4. $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$

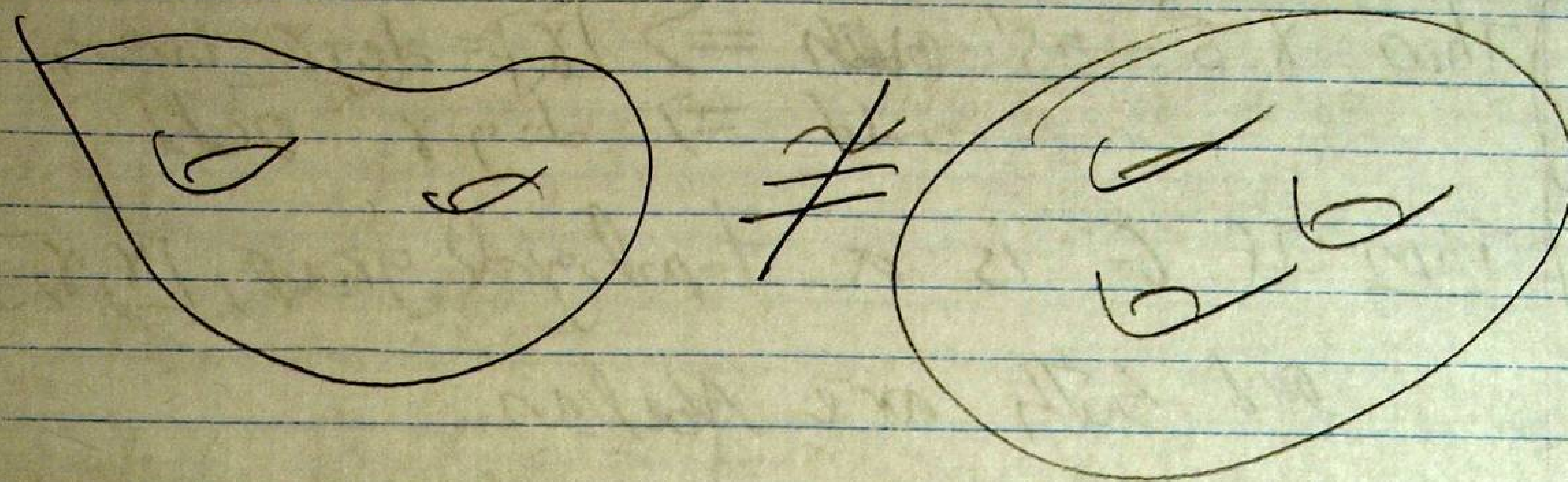
---

state Van-Kampen

Examples: 1. 
2. T^2
3. $\mathbb{RP}^2$
4. S^n
5. Σ_g
6. $T(PA)$

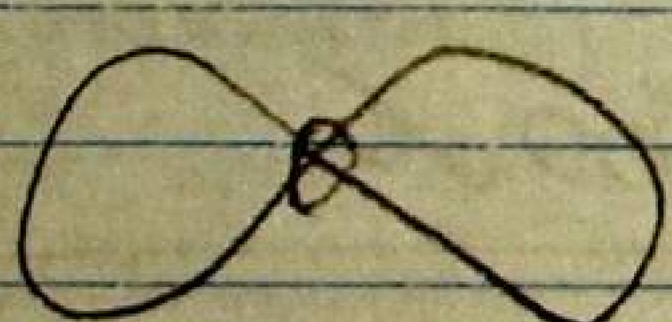
Math 1300 Topology, ~~Tuesday~~ Thursday Dec 8 2005 (1 hour)

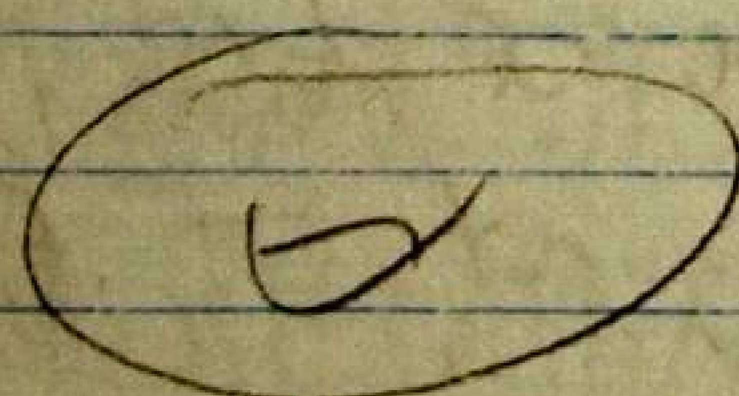
TE2 Monday 10AM, SS 2127. Results after break.

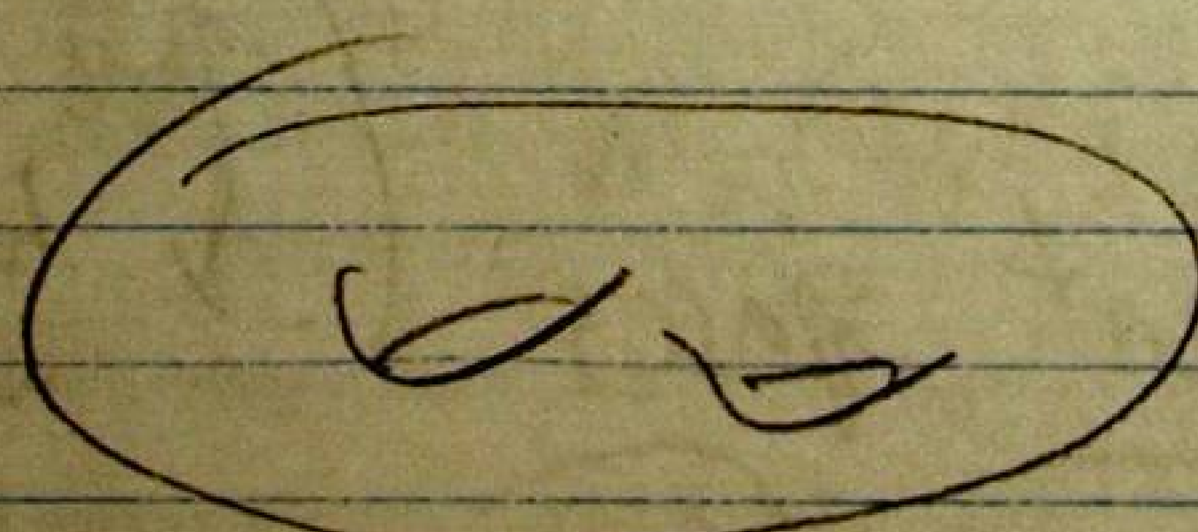


Via Van Kampen

1. statement / idea

2. 

3. 

4. 

5. Abelianization.

Math 1300 Geom & Top, Jan 15 2008 hours II/4-5.

Lemma Let $p:(X,x_0)\to(B,b_0)$ be a covering space.

The every (family of) path(s) $\gamma: \underbrace{Y\times}\, I \to B$ s.t. $\underbrace{\forall_y}\ \gamma(\underbrace{y,}\,0)=b_0$ has a unique lift

$\tilde{\gamma}: \underbrace{Y\times} I\to X$ s.t. $\underbrace{\forall_y}\ \tilde{\gamma}(\underbrace{y,}\,0)=x_0$ & $\gamma=p\circ\tilde{\gamma}$.

claim ind$[\gamma]$ is well def., hence $\pi_1(S^1,1)\cong\mathbb{Z}$

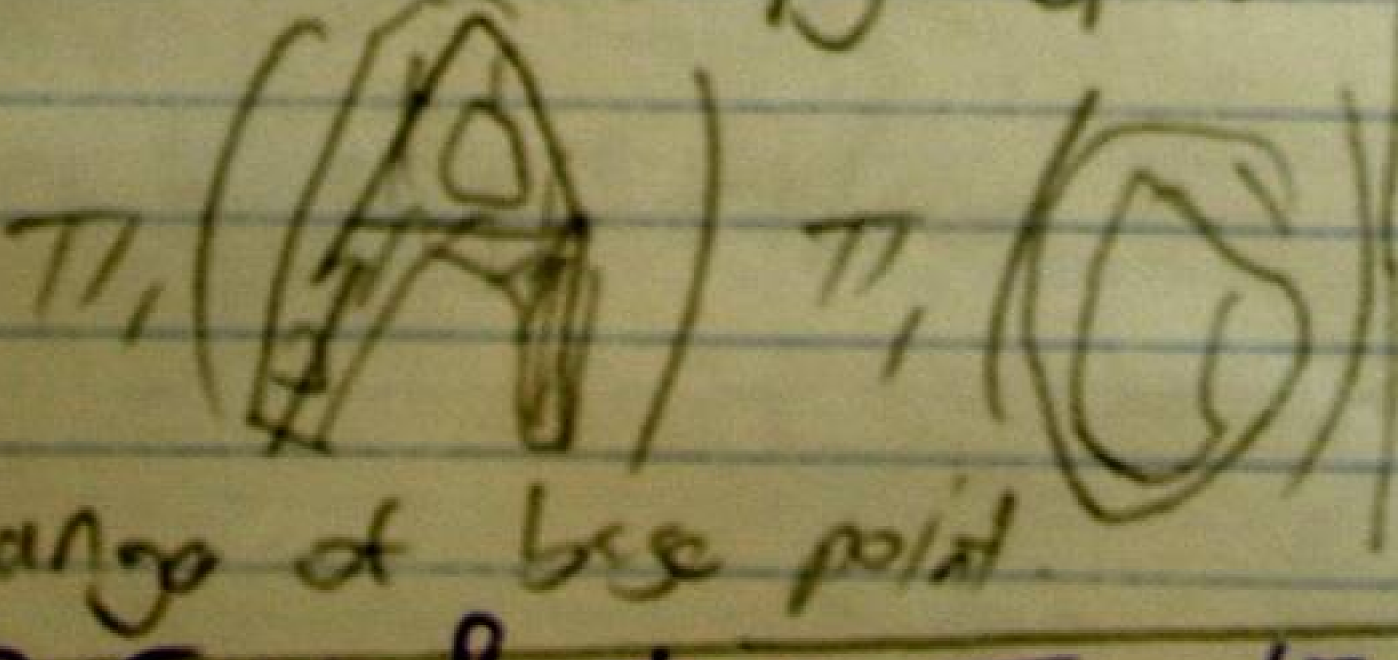
Pf of claim, pf of lemma.

Applications * No retract $r: D^2\to S^1$; Brouwer

* $\mathbb{R}^2\not\cong\mathbb{R}^n$ for $n\neq 2$.
* The Fundamental Thm of algebra.
* Borsuk Ulam * So S^2 isn't a subspace of $\mathbb{R}^2$
* If $S^2=A_1\cup A_2\cup A_3$, at least 1 contains a pair of antipodes
* w.l.o.t if 4?
* pf using π_1(even) = even, π_1(odd) = odd.

homotopy: * homotopic maps

* homotopy equiv



* Change of base point

Math 1300 Geom & Top, Jan 17 2008 hour II/6.

Van-Kampen as on Jan 10, 2006.

Math 1300 Geom & Top, Jan 22 2008, hours II/7–8.

Continue Van-Kampen examples

Σ_g , $\mathbb{RP}^2$

State Van-Kampen.

Puncturing a mfd of dim ≥ 3

$\pi_1(S^3)$ in two ways

$\pi_1(T_{q,3}^c)$

Push forwards as on Dec 2, 2004

Proof of Van-Kampen as on Jan 17, 2006.

(rug on a bed of nails.....)

<u>On board</u>. If $X = U_1 \cup U_2$, U_i open, $b \in U_1 \cap U_2$ connected, Then

$$\pi_1(X) = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2) = \left\{ \begin{matrix} \text{words in} \\ \pi_1(U_1) \cup \pi_1(U_2) \end{matrix} \right\} \Big/ \begin{matrix} \text{obvious} \\ \text{reduction} \\ \text{and} \\ i_{1*}(\gamma) = i_{2*}(\gamma), \\ \gamma \in \pi_1(U_1 \cap U_2) \end{matrix}$$

$$\begin{tikzcd} & U_1 \arrow[dr] & & \\ U_1 \cap U_2 \arrow[ur] \arrow[dr] & & U_1 \cup U_2 & \pi_1(U_1 \cap U_2) \\ & U_2 \arrow[ur] & & \end{tikzcd}$$

Today: 1. More examples (blowing into tori)

2. More diagrams

3. Proof: "Throw a handkerchief on a bed of nails"

Math 1300 Geom & Top, Jan 24 2008, hour II/9

HW8

Pf of Van Kampen as on Jan 12, 2006.

Math 1300 Topology, Tuesday Jan 10 2006 (2 hours)

in
oard

http://katlas.math.toronto.edu/0506-Topology

<u>Van-Kampen' Theorem</u> If $X=U_1\cup U_2$, U_1 & U_2 are open, $b\in U_1\cap U_2$ and $U_1\cap U_2$ is connected, then

$$\pi_1(X)\cong \pi_1(U_1) *_{\pi_1(U_1\cap U_2)} \pi_1(U_2)$$

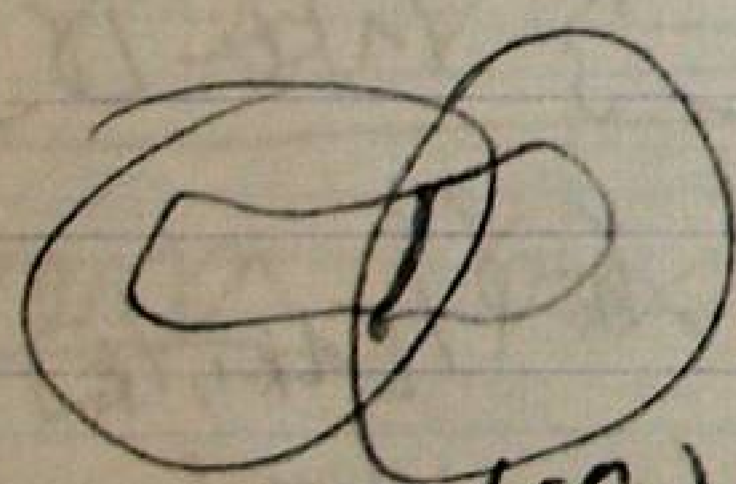
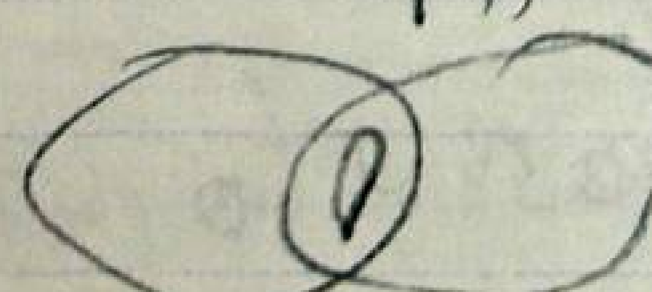
$$\begin{tikzcd} & U_1 \arrow[dr, "j_1"] & \\ U_1\cap U_2 \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & U_1\cup U_2 \\ & U_2 \arrow[ur, "j_2"'] & \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} & \pi_1(U_1) \arrow[dr] & \\ \pi_1(U_1\cup U_2) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \\ & \pi_1(U_2) \arrow[ur] & \end{tikzcd}$$

If $H \xrightarrow{\varphi_1} G_1$, $H \xrightarrow{\varphi_2} G_1$ then $G_1 * G_2 = \left(\begin{array}{l}\text{Words in } G_1\sqcup G_2 = \{g : g\in G_1\}\cup\{\hat{g} : g\in G_2\} \\ \text{under concat, mod } e=\hat{e}=1,\ g_1g_2=\overline{g_1g_2},\ \hat{g}_1\hat{g}_2=\end{array}\right.$

$$G_1 *_H G_2 = G_1 * G_2 \Big/ \overline{\varphi_1(h)=\varphi_2(h)}\quad \forall h\in H$$

<u>Idea</u>  yet 

<u>Examples</u>

0. $\pi_1(S^2)$
1. $\pi_1(T^2)$
2. $\pi_1(\Sigma_g)$ / Abelimization
4. Puncturing a 3-d nbd in X
3. $\pi_1(S^3)$ via S^3 = Union of two solid tori
5. $\pi_1(\mathbb{R}^3)$
6. $\pi_1(T_{8,3}^c)$

Math 1300 Topology, Thursday Jan 12 2006 (1 hour)

on board. { Re-state Van-Kampen $X = U_1 \cup U_2$ U_1, U_2 open, $U_1 \cap U_2$ connected
$\Rightarrow \pi_1(X) = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$

Compute $\pi_1(T_{8,3}^c)$

Puncturing a 3-d nbd in X.

$\mathbb{RP}^2$

K (Klein bottle)

A word about HVZ

Math 1300 Topology, Tuesday Jan 17, 2006 (2 hours)

Proof of Van Kampen. $G_i = \pi_1(U_i)$, $H = \pi_1(U_1 \cap U_2)$

1. $\Phi : G_1 *_H G_2 \longrightarrow \pi_1(X)$ obvious.

2. $\Psi : \pi_1(X) \longrightarrow G_1 *_H G_2$; take $[\gamma] \in \pi_1(X)$
   a. choose $0 = t_0 \ldots t_n = 1$ s.t. $\gamma_k = \gamma|_{[t_{k-1}, t_k]} \subset U_{i_k}$ $i_k \in \{1, 2\}$
   b. choose paths η_i in $U_1 \cap U_2$ from $\gamma(t_i)$ to
   c. set $\Psi([\gamma]) = \underbrace{\bar{\eta}_0 \gamma_1 \eta_1}_{G_{i_1}} \underbrace{\bar{\eta}_1 \gamma_2 \eta_2}_{G_{i_2}}$

3. $\Psi([\gamma])$ indep. of: (η_i, t_i),

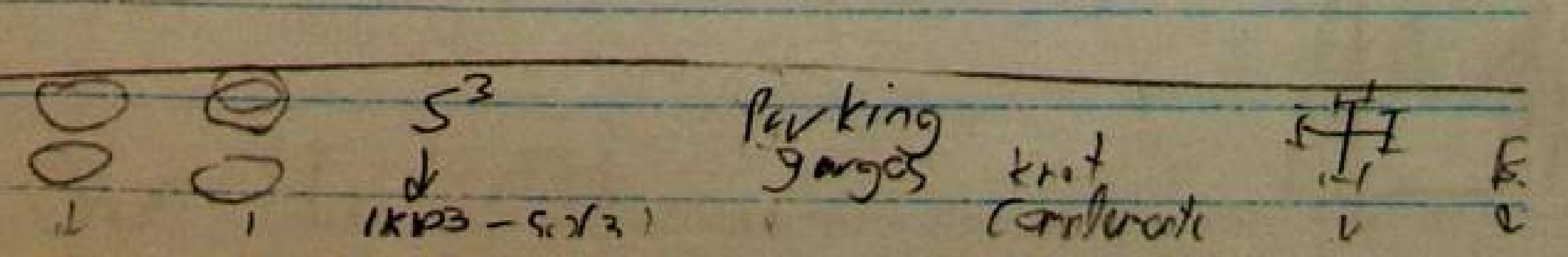
4. $\Psi([\gamma])$ inv. under homotopies of γ

5. Φ & Ψ are inverses of each other

The "multiple" case.

Pushouts

Examples of covering spaces:

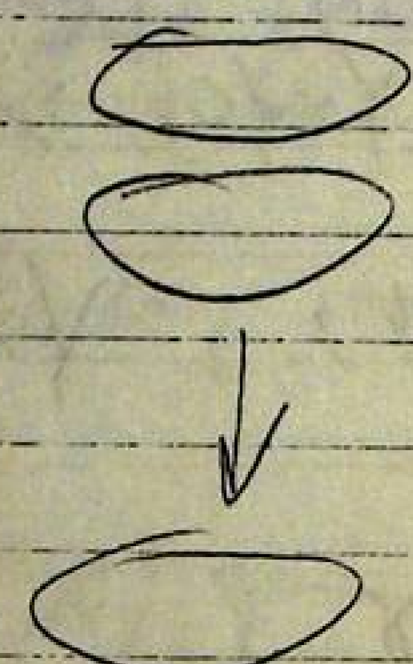


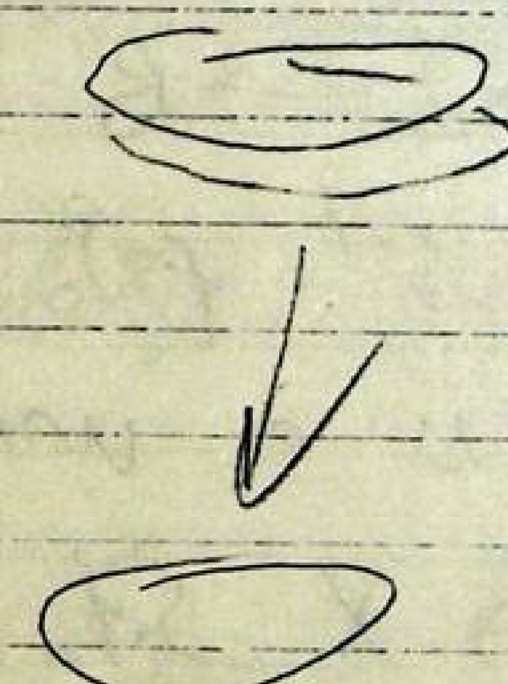
Math 1300 Topology, Thursday Jan 19, 2006 (1 hour)

Def $p: X \to B$ is a "covering map", and "X is a cover of B", if for some fixed set F ("the fiber"), taken with a discrete topology, every $b \in B$ has a nbd U s.t. $p^{-1}(U) \cong F \times U$ "over U"

$$\begin{tikzcd} p^{-1}(U) \arrow[r, phantom, "\cong"] \arrow[dr, "p"'] & F \times U \arrow[d, "\pi_2"] \\ & U \end{tikzcd}$$

Examples



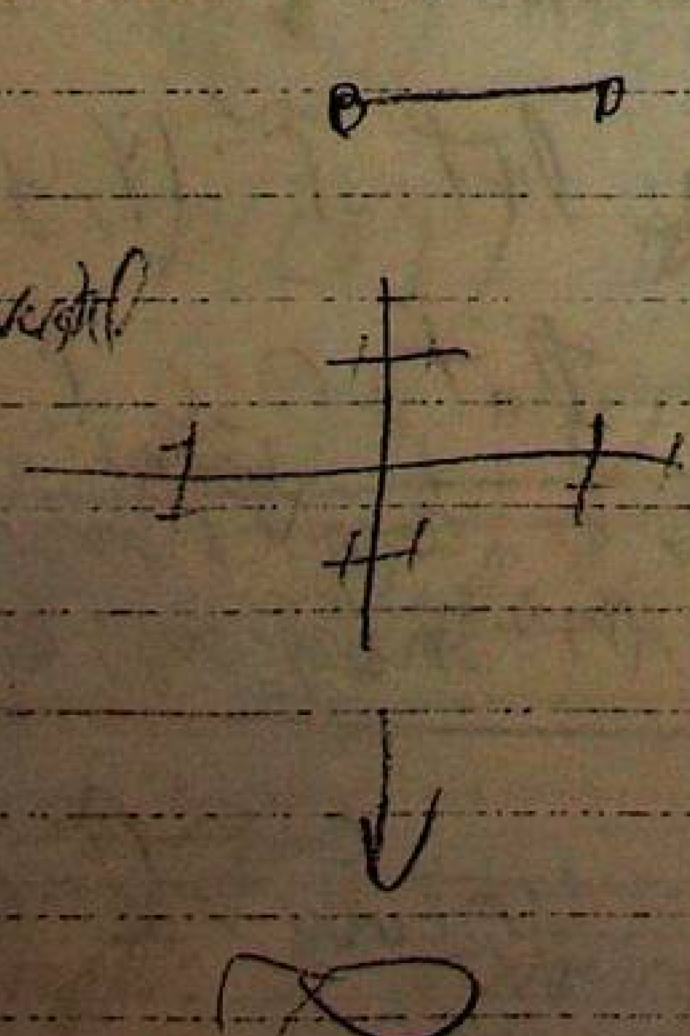


$S^3 \longrightarrow \mathbb{RP}^3 = SO(3)$ $\quad$ $S^n \longrightarrow \mathbb{RP}^n$

Parking garage

knot complements & Seifert covers

the "universal cover"



Hatcher's page 58

Math 1300 Topology, Tuesday Jan 24, 2006 (2 hours)

1. Good deed certificates.

2. Web demo, HW7 hint, Klein bottle, Hatcher's page 58

on board

<u>Definition</u> $p: X \to B$ a "covering" if X is locally a product $F \times B$. "Based covering": choose basepoints $x_0 \in X$, $b_0 \in B$ s.t. $p(x_0) = b_0$.

<u>Thm</u> I. $p: X \to B$ is a covering, and $\gamma: Y \times I \to B$ & $\tilde{\gamma}_0 : Y \times \{0\} \to X$ are given s.t. $p \circ \tilde{\gamma}_0 = \gamma|_{Y \times \{0\}}$, then there is a unique $\tilde{\gamma}: Y \times I \to X$ s.t. $p \circ \tilde{\gamma} = \gamma$ & $\tilde{\gamma}|_{Y \times \{0\}} = \gamma_0$

$$\begin{tikzcd} Y \times \{0\} \arrow[r, hook] \arrow[rr, bend left, "\tilde{\gamma}_0"] & Y \times I \arrow[r, dashed, "\exists ! \tilde{\gamma}"] \arrow[dr, "\gamma"'] & X \arrow[d, "p"] \\ & & B \end{tikzcd}$$

<u>Lemma</u> $p: X \to B$ based $\Rightarrow p_*$ is injective;

$$p_* \pi_1(X, x_0) = \left\{ \begin{array}{l} \text{paths in } B \\ \text{whose lift to } X \text{ is closed} \end{array} \right\}$$

<u>Prop</u> ("the lifting criterion") $p: X \to B$ based, (Y, y_0) connected & locally connected. A map $f: (Y, y_0) \to (B, b_0)$ has a lift $\tilde{f}: (Y, y_0) \to (X, x_0)$ iff $f_* \pi_1(Y) \subset p_* \pi_1(X)$. In that case $\tilde{f}$ is unique.

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

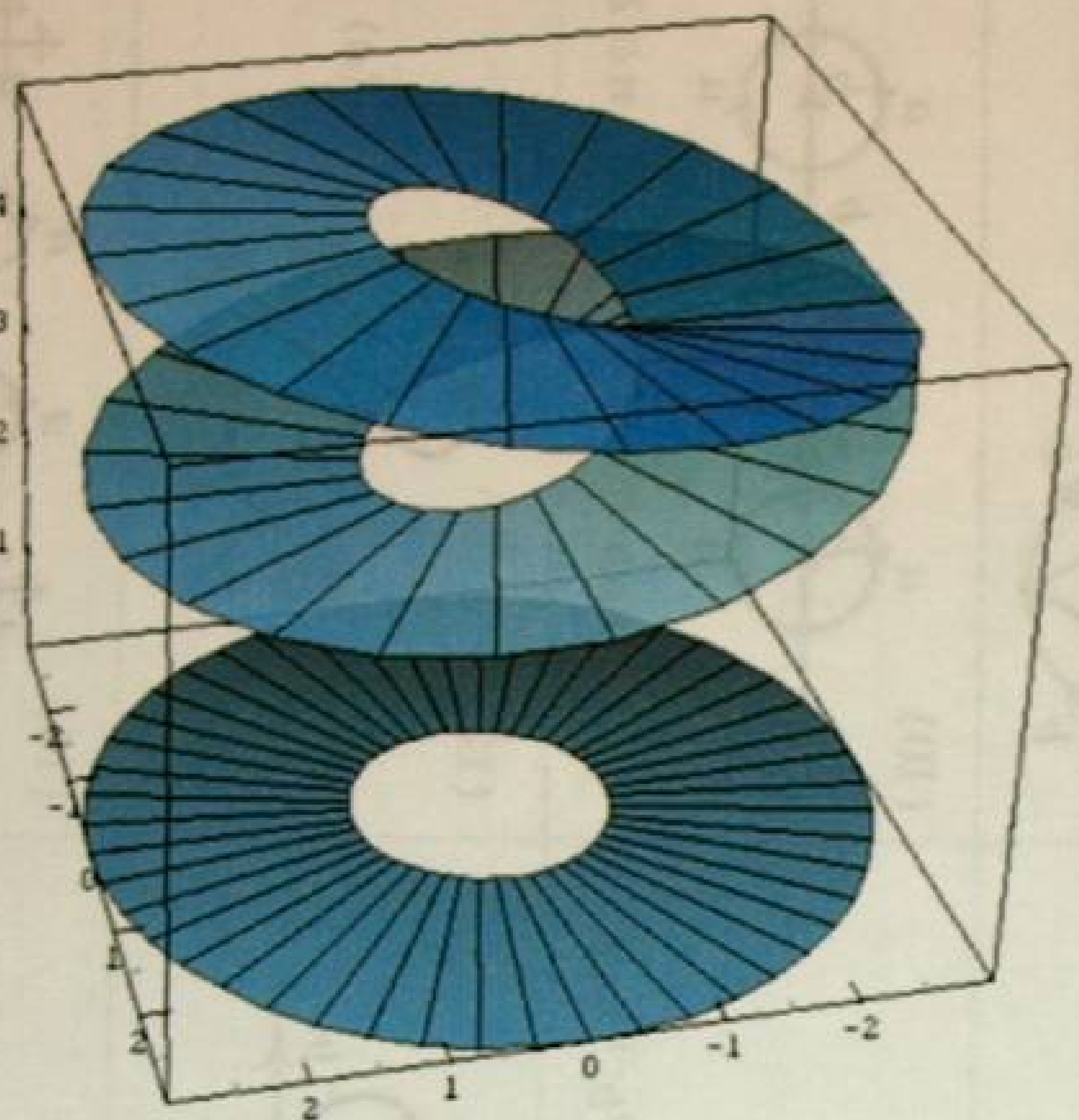
<u>Corollary</u> Connected & lo. conn. based coverings of B are in 1-1 corres. with subgroups of $\pi_1(B)$.

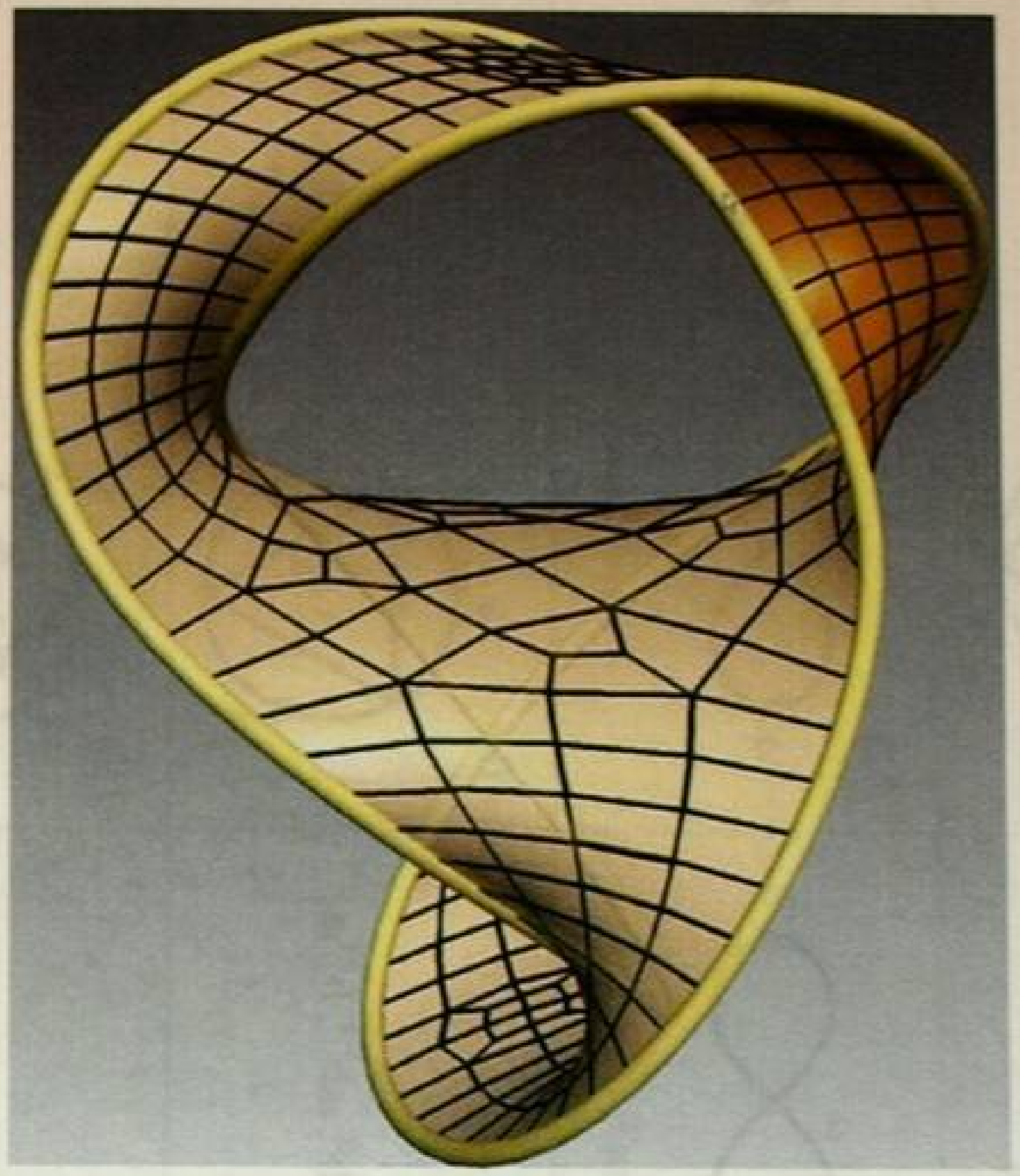
Start the construction of a universal covering space.

0708-1300/Class notes for Tuesday, January 29

Two Covering Spaces

[edit]





Mathematica code:

```
ParametricPlot3D[
  {
    {r Cos[2t], r Sin[2t], 4 + Cos[t]},
    {r Cos[t], r Sin[t], 0},
    PlotPoints -> {49, 2}
  },
  {t, 0, 2Pi}, {r, 1, 3}
]
```

Seifert surface of the Figure Eight Knot, drawn using Jack van Wijk's amazing SeifertView.

14 Covering Spaces

[edit]

14 further covering spaces can be found on page 58 of Hatcher's book.

0708-1300/Navigation Panel [Hide]

Add your name / see who's in!

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	Fall Semester	
1	Sep 10	About, Tue, Thu
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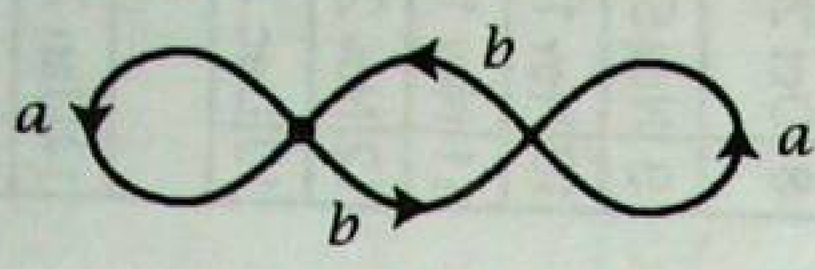
3	Sep 24	Tue, Photo, Thu
4	Oct 1	Questionnaire, Tue, HW2, Thu
5	Oct 8	Thanksgiving, Tue, Thu
6	Oct 15	Tue, HW3, Thu
7	Oct 22	Tue, Thu
8	Oct 29	Tue, HW4, Thu, Hilbert sphere
9	Nov 5	Tue,Thu, TE1
10	Nov 12	Tue, ~~Thu~~
11	Nov 19	Tue, HW5
12	Nov 26	Tue, Thu
13	Dec 3	Tue, HW6
	Spring Semester	
14	Jan 7	Tue, Thu, HW7
15	Jan 14	Tue, Thu
16	Jan 21	Tue, HW8
17	Jan 28	**Tue**
18	Feb 4	HW9
19	Feb 11	TE2; Feb 17: last chance to drop class
R	Feb 18	

Dror Bar-Natan: classes: 0708: GeomTop:

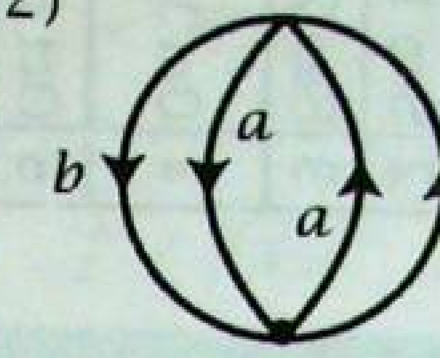
Some Covering Spaces of $S^1 \vee S^1$

(1)



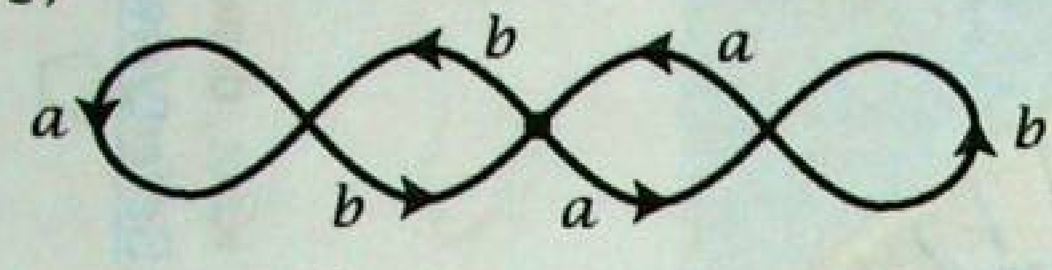
$\langle a, b^2, bab^{-1} \rangle$

(2)



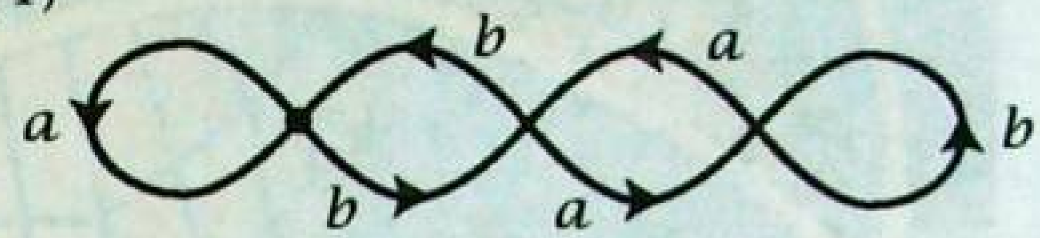
$\langle a^2, b^2, ab \rangle$

(3)



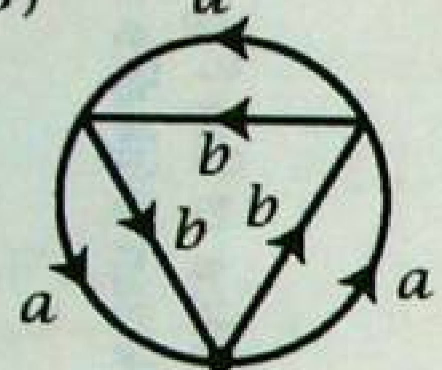
$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

(4)



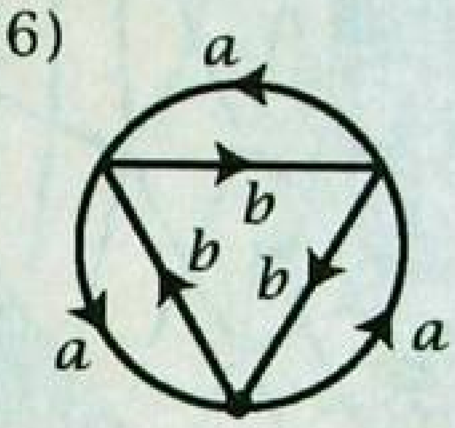
$\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$

(5)



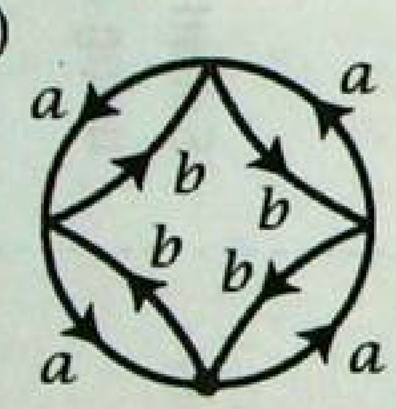
$\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$

(6)



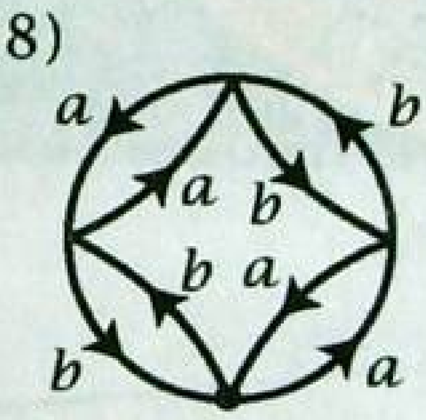
$\langle a^3, b^3, ab, ba \rangle$

(7)



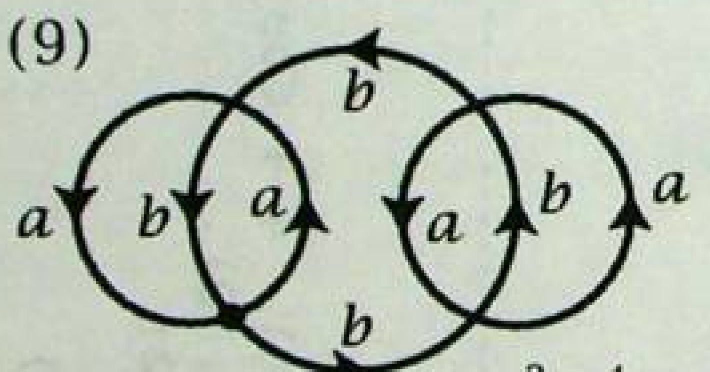
$\langle a^4, b^4, ab, ba, a^2b^2 \rangle$

(8)



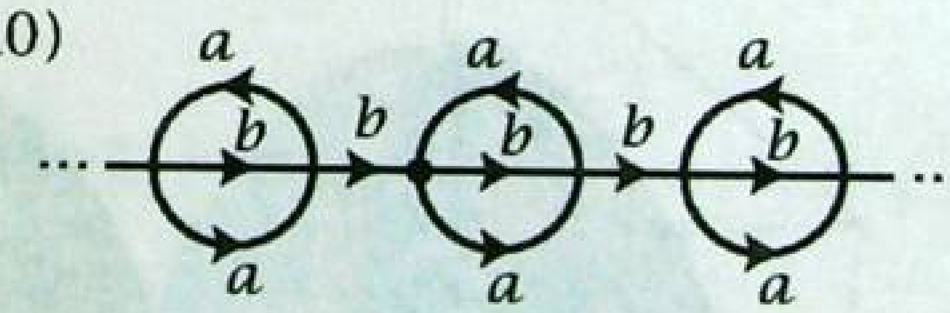
$\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$

(9)



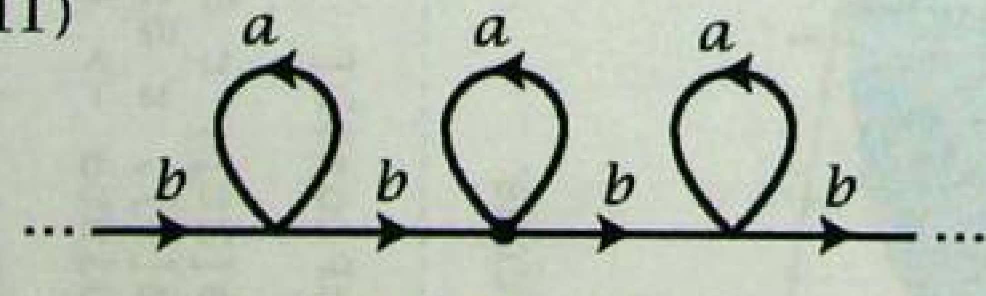
$\langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle$

(10)



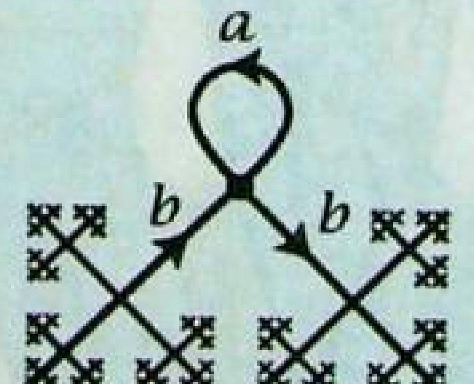
$\langle b^{2n}ab^{-2n-1}, b^{2n+1}ab^{-2n} \mid n \in \mathbb{Z} \rangle$

(11)



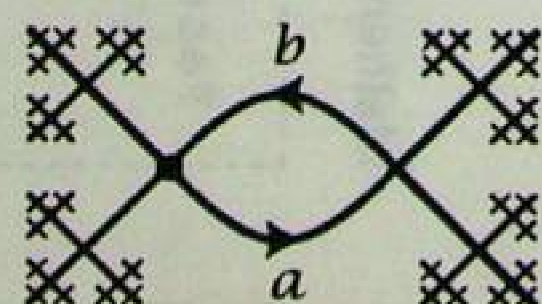
$\langle b^nab^{-n} \mid n \in \mathbb{Z} \rangle$

(12)



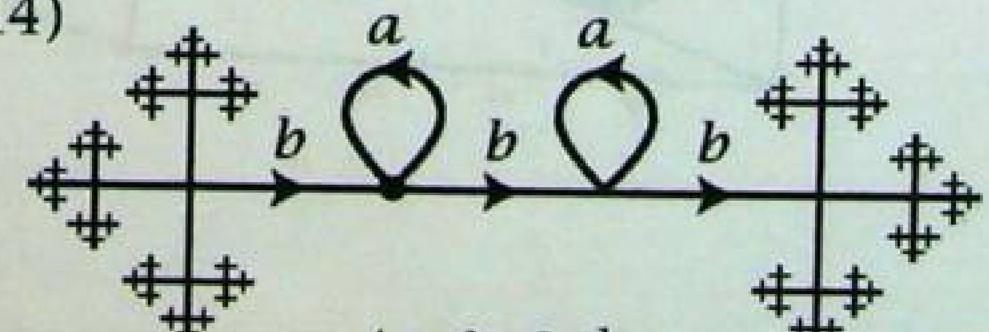
$\langle a \rangle$

(13)



$\langle ab \rangle$

(14)



$\langle a, bab^{-1} \rangle$

Hatcher's page 58.

Van Kampen for $\bigcup U_i$

Math 1300 Topology & Geometry, Jan 29 2008, hours II/10-11

Coverings: ~~Roughly little~~ like Jan 19, 2006,

Def $p: X \to B$ is a "covering map" and "X is a cover of B", if for some fixed set F ("the fiber"), taken w/ a discrete topology, every $b \in B$ has a nbd U s.t

$$\begin{array}{ccc} p^{-1}(U) & \cong & F \times U \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi_2} & \\ & U & \end{array}$$

Dream Thm for decent B's, {Coverings of B} $\longleftrightarrow$ Subgroups of $\pi_1(B)$

$X \longrightarrow p_*\pi_1(X)$

Later we'll fulfill the dream but also realize it's not quite the right dream.

Examples as on Jan 19, 2006

$p_*\pi_1(X)$ in Hatcher's examples.

Lemma $p: X \to B \Rightarrow p_*$ is injective.

Prop "The lifting criteria" $p: X \to B$ given, Y is connected & locally connected. Then $f: Y \to B$ has a lift $\tilde{f}: Y \to X$ with $p \circ \tilde{f} = f$ iff $f_*\pi_1(Y) \subset p_*\pi_1(X)$. In that case, the lift is unique.

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[r, "f"] \arrow[ur, "\exists!"] & B \end{tikzcd}$$

Cor p_* is injective in another sense - If $p_*X_1 = p_*X_2$, the $X_1 \simeq X_2$.

start constructing the universal covering.

HW7 due, dist. HW8.

Math 1300 Topology, Thursday Jan 26 (1 hour)

on board

$p: X \to B$ based covering

Lemma $p_*: \pi_1(X) \hookrightarrow \pi_1(B)$, image $= \{$paths whose lift ~~are~~ closed$\}$

Prop $Y \xrightarrow{f} B$, with $\exists! \tilde{f}: Y \to X$, $p: X \to B$. Y connected & loc. connected; given f, $\tilde{f}$ exists & unique iff $f_*\pi_1(Y) \subset p_*\pi_1(X)$

Corollary If $X_1 \xrightarrow{p_1} B$, $X_2 \xrightarrow{p_2} B$ are two connected, loc. connected coverings and

$p_{1*}\pi_1(X_1) = p_{2*}\pi_1(X_2)$, then X_1 & X_2 are equivalent as based coverings of B.

Q: Does every $H < G := \pi_1(B)$ corresponds to a connected, loc. connected cover?

$H = G$: ✓

$H = \{e\}$: "The Universal cover"

Thm If B is connected, locally connected and semi-locally simply connected, then B has a (unique) Universal cover.

Pf $X = \{[\gamma] : \gamma: [0,1] \to B,\ \gamma(0) = b_0\}$ $[\gamma]$ equiv classes mod endpt preserving homotopies

$p[\gamma] = \gamma(1)$

Given a connected $U \subset X$ (s.t. $i_*\pi_1(U) = \{e\}$) and $[\gamma]$ w/ $p[\gamma] \in U$, set $U_{[\gamma]} = \{[\gamma\eta] : \eta \in U\}$

Such U's form a basis for the top. of B.

claim 1. If $[\gamma_1] \in U_{[\gamma]}$, then $U_{[\gamma_1]} = U_{[\gamma]}$

2. $\{U_{[\gamma]}\}$ form a "basis for a topology"

3. $p: U_{[\gamma]} \to U$ is a homeomorphism.

4. p is a covering map.

5. $\pi_1(X) = \{e\}$.

* wiki show / handout.
* Good deed certificates.
* Return H/W 6

Math 1300 Topology, Tuesday Feb 1 (2 hours)

B: connected & locally connected.

$\left(\begin{array}{c}\text{connected based coverings}\\ \text{of } B\end{array}\right) \xrightarrow[1-1]{} \left(\begin{array}{c}\text{subgroups of}\\ G = \pi_1(B)\end{array}\right)$

on board

Thm If B is also semi-locally simply connected (SLSC), there exists a "universal cover" of B, a connected cover whose fundamental group is trivial.

Pf Set $X_u = \{\text{spelunkers in } B\} = \{[\gamma] : \gamma : [0,1] \to B,\ \gamma(0) = b_0\}$

$p : X_u \to B$ via $\gamma \mapsto \gamma(1)$

For "good" U & $[\gamma]$ w/ $p(\gamma) \in U$ set $U_{[\gamma]} = \{[\gamma\eta] : \eta \subset U\}$

Define slsc

Examples The Hawaiian Earrings

Cone over the Hawaiian Earrings

Claims 1. If $[\gamma_1] \in U_{[\gamma]}$, then $U_{[\gamma]} = U_{[\gamma_1]}$

2. $\{U_{[\gamma]}\}$ form a basis for a topology

3. $p : U_{[\gamma]} \to U$ is a homeomorphism

4. p is a covering map (with fiber $\pi_1(B)$)

5. $\pi_1(X_u) = \{e\}$ Pf 1 / Pf 2 highly recommended - read the book.

Claim X_u carries a left G-action

Definition Given $H < G$, set $X_H = X_u / \begin{array}{c} x \sim hx \\ \forall x \in X_u, h \in H\end{array}$

Claim $\pi_1(X_H) = H$

Def / Proposition For any $\begin{array}{c}X\\ \downarrow p\\ B\end{array}$, $p^{-1}(b_0)$ is a "right G-set", $\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "p_X"'] & Y \arrow[d, "p_Y"] \\ & B \end{tikzcd}$

induces a morphism between right G-sets

0708-1300/Class notes for Tuesday, February 5

From Drorbn

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0708-1300/Navigation Panel [Hide]

Add your name / see who's in!

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R	Feb 18	
20	Feb 25	HW10
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25	Mar 31	

On Term Exam 2

Term Exam 2 will take place on Monday February 11, 2008, in room GB217 of the Galbraith Building (across St. George from Bahen), starting at 6:10PM sharp and ending at 8PM. The material is everything since the last exam - differential forms and Stokes' theorem from last semester, and the fundamental group and covering spaces from this semester. The general style and form of the exam will be **exactly the same** as the style and form of Term Exam 1. but harder, perhaps.

Imprecise Definition. "Sketch the proof of a major theorem" means that you should write an outline of the proof, omitting details that any graduate student taking this class could have considered as exercises while studying the proof for the first time, while not omitting anything that really requires creative thinking.

Example. A sketch of the proof that every smooth manifold M carries a proper smooth function h into the positive reals would be "let $\lambda_k(x)$ be a partition of unity subordinate to a cover of M by open sets with compact closures, and take $h(x) := \sum_k k\lambda_k(x)$". The formula $\sum_k k\lambda_k(x)$ in the quoted statement above requires creativity and is hard to come by; so every sketch must contain it or something equivalent to it. The proof that the resulting function h is indeed smooth and proper is an exercise and may well be omitted.

Unbased Covering Spaces

Let B be a topological space and let $\mathcal{C}(B)$ be the category of covering spaces of B: The category whose objects are (unbased!) coverings $X \to B$ and whose morphisms are maps between such coverings that commute with the covering projections - a morphism between $p_X : X \to B$ and $p_Y : Y \to B$ is a map $\alpha : X \to Y$ so that the diagram below is commutative:

26	Apr 7
Register of Good Deeds	
Errata to Bredon's Book	

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & B & \end{tikzcd}$$

Every topologists' highest hope is to find that her/his favourite category of topological objects is equivalent to some category of easily understood algebraic objects. The following theorem realizes this dream in full in the case of the category $\mathcal{C}(B)$ of covering spaces of any reasonable base space B:

Theorem 1. *(Classification of covering spaces)*

- If B is connected and locally connected with base point b_0 and fundamental group $G = \pi_1(B, b_0)$, then the map which assigns to every covering $p : X \to B$ its fiber $p^{-1}(b_0)$ over the basepoint b_0 induces a functor $\mathcal{F}$ from the category $\mathcal{C}(B)$ of coverings of B to the category $\mathcal{S}(G)$ of G -sets - sets with a right G -action and set maps that respect the G action.
- If in addition B is semi-locally simply connected then the functor $\mathcal{F}$ is an equivalence of categories. (In fact, this is iff).

If indeed the categories $\mathcal{C}(B)$ and $\mathcal{S}(G)$ are equivalent, one should be able to extract everything topological about a covering $p : X \to B$ from its associated G -set $\mathcal{F}(X) = p^{-1}(b_0)$. The following theorem shows this to be right in at least two ways:

Theorem 2.

- The set of connected components of X is in a bijective correspondence with the set of orbits of G in $\mathcal{F}(X)$.
- Let $x_0 \in \mathcal{F}(X) = p^{-1}(b_0)$ be a basepoint for X that covers the basepoint b_0 of B. Then the fundamental group $\pi_1(X, x_0)$ is isomorphic via the projection $p_\star$ into $G = \pi_1(B, b_0)$ to the stabilizer group $\{h \in G : xh = x\}$ of x in $x_0 \in \mathcal{F}(X)$.

(Both assertions of this theorem can be sharpened to deal with morphisms as well, but we will not bother to do so).

Based Covering Spaces

There are similar theorems (call them **theorem 1'** and **theorem 2'**) relating the category of based covering spaces with the category of based G -sets.

The Main Point

Ok. Every math technician can spend some time and effort and understand the statements and (only then) the proofs of these two theorems. Your true challenge is to digest the following statement:

All there is to know about covering spaces follows from these two theorems

In particular, the following facts are all simple algebraic corollaries of these theorems:

Corollary 1. If X is connected then its covering number (="number of decks") is equal to the index of $H = p_\star\pi_1(X)$ in $G = \pi_1(B)$, and the decks of X are in a non-canonical correspondence with the left cosets $H\backslash G$ of H in G.

Corollary 2. If B is semi-locally simply connected, there exists a unique (up to base-point-preserving isomorphism) "universal covering space U of B" (a connected and simply connected covering U).

Corollary 3. The group of automorphisms of the universal covering U is equal to $G = \pi_1(B)$.

Corollary 4. $\pi_1(S^1) = \mathbb{Z}$.

Corollary 5. $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$.

Corollary 6. If B is semi-locally simply connected, then for every $H < G = \pi_1(B)$ there is a unique (up to base-point-preserving isomorphism) connected covering space X with $p_\star\pi_1(X) = H$.

Corollary 7. If X_i for $i = 1, 2$ are connected coverings of B with groups $H_i = p_{i\star}\pi_1(X_i)$ and if $H_1 < H_2$ then X_1 is a covering of X_2 of covering number $(H_2 : H_1)$.

Corollary 8. If B is semi-locally simply connected there is a bijection between conjugacy classes of subgroups of $G = \pi_1(B)$ and unbased connected coverings of B.

Corollary 9. A connected covering X is normal (for any $x_1, x_2 \in p^{-1}(b)$ theres an automorphism τ of X with $\tau x_1 = x_2$) iff its group $p_\star\pi_1(X)$ is normal in $G = \pi_1(B)$.

Corollary 10. If X is a connected covering of B and $H = p_\star\pi_1(X)$, then $\mathrm{Aut}(X) = N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G.

Proposition 11. If we forgot anything, it follows too.

Steps in the proofs of Theorem 1 and 2

1. Use path liftings to construct a right action of G on $p^{-1}(b_0)$.
2. Show that this is indeed a group action and that morphisms of coverings induce morphisms of right G-sets.
3. Start the construction of an "inverse" functor $\mathcal{G}$ of $\mathcal{F}$: Use spelunking (cave exploration) to construct a universal covering U of B, if B is semi-locally simply connected.
4. Show that $\mathcal{F}(U) = G$.
5. Use the construction of U or the general lifting property for covering spaces to show that there is a left action of G on U.
6. For a general right G-set S set $\mathcal{G}(S) = S \times_G U = \{(s, u) \in S \times U\}/(sg, u) \sim (s, gu)$ and show that $\mathcal{G}(S)$ is a covering of B and $\mathcal{F}(\mathcal{G}(S)) = S$.
7. Show that $\mathcal{G}$ is compatible with maps between right G-sets.
8. Understand the relationship between connected components and orbits.
9. Prove Theorem 2.
10. Use the existence and uniqueness of lifts to show that $\mathcal{G} \circ \mathcal{F}$ is equivalent to the identity functor (working connected component by connected component).

A Deep Thought Question

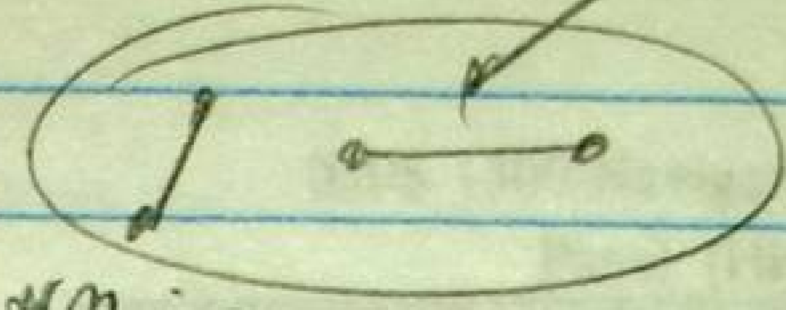
What does it at all mean "$\mathcal{G} \circ \mathcal{F}$ is equivalent to the identity functor" (and first, why can't it simply *be* the identity functor)? And even harder, what does it at all mean for two categories to be "equivalent"? If you answer this question correctly, you'll probably re-invent the notions of "natural transformation between two functors" and "natural equivalence", that gave the historical impetus for the development of category theory.

From the Wikipedia entry for Natural Transformation (http://en.wikipedia.org/wiki/Natural_transformation) :

> Saunders Mac Lane, one of the founders of category theory, is said to have remarked, "I didn't invent categories to study functors; I invented them to study natural transformations." Just as the study of groups is not complete without a study of homomorphisms, so the study of categories is not complete without the study of functors. The reason for Mac Lane's comment is that the study of functors is itself not complete without the study of natural transformations.
>
> The context of Mac Lane's remark was the axiomatic theory of homology. Different ways of constructing homology could be shown to coincide: for example in the case of a simplicial complex the groups defined directly, and those of the singular theory, would be isomorphic. What cannot easily be expressed without the language of natural transformations is how homology groups are compatible with morphisms between objects, and how two equivalent homology theories not only have the same homology groups, but also the same morphisms between those groups.

TE2 options: Mon Feb 11 ~~Tue Wed~~ Thu Feb 14 Fri Feb 15

Math 1300 Geometry & Topology, Jan 31 2008, hour II/12



1. A word about analytic continuations

2. The lifting criterion:

on board

assuming $f_*\pi_1(Y) \subset p_*\pi_1(X)$.

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[ur, "\exists!\tilde{f}"] \arrow[r, "f"] & B \end{tikzcd}$$

connected & locally connected.

Lemma: If η in B lifts to closed path and $\gamma \sim \eta$, then $\tilde{\gamma}$ is closed too.

Pf of lifting lemma:

Cor: $p_{1*}\pi_1(X_1) = p_{2*}\pi_1(X_2) \Rightarrow X_1 \cong X_2$

---

construct universal coverings as on Feb 1, 2007.

Math 1300 Geom&Top, Feb 5 2008 hours II/13-14

* Complete the construction of the universal covering.
* The left G-action of X_U.
* Construction of arbitrary covers.
* The true main theorem.

Math 1300 Geom&Top, Feb 7 2008, hours II/15.

philosophy day! (first, ~~warm~~ questions) TE2 OH: Mon 12-2

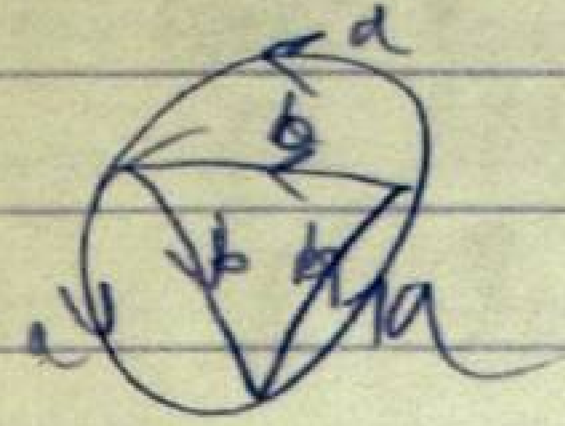
$$\left\{\begin{array}{c}\text{coverings}\\ \text{of}\\ B\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\text{subgroups}\\ \text{of}\\ G=\pi_1(B)\end{array}\right\}$$

1. Not good enough; the real main thm....
2. philosophy of H_* as dual of H^*_{dR}
3. H_* as the domain of intersection theory.

טופולוגיה אלגברית, 2 ליוני 2002

המשך מרחבי כיסוי ע"פ הקלף.

<u>שאלה</u> מהי ה-set המתאימה לכיסוי של ∞ המתאים? לגרף



<u>הומולוגיה</u> הרעיון הוא "אוטומטי" של $H_*(S^2), H_*(T^2)$ ושל $H_*(\mathbb{R}P^2)$ $H_*(S^n)$

סימפ. <u>הגדרה</u> $\Delta^n = \{(t_0, \dots t_n) \in \mathbb{R}^{n+1} : \sum t_i = 1,\ t_i \geq 0\}$, וכדומה כללי

$[v_0 \dots v_n] \rightarrow \mathrm{conv}\{v_i\} = \{\sum t_i v_i : \bar{t} \in \Delta^n\}$

$[e_0 \dots e_n] \longmapsto \Delta^n$

<u>הגדרה</u> n-סימפלקס סינגולרי ב-X: $\sigma : \Delta^n \rightarrow X$

$\{\sum k_i \sigma_i\} = C_n(X) = $

$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$

ע"י

$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[e_0 \dots \hat{e}_i \dots e_n]}$

<u>טענה</u> $\partial^2 = 0$.

<u>הגדרה</u> $H_n(X) = \ker \partial_n / \mathrm{Im}\, \partial_{n+1}$

<u>טענה</u> התנהגות על רכיבי קשירות

<u>טענה</u> $H_0(X) = \mathbb{Z}$

<u>חישוב</u> $H_*(pt)$ <u>הגדרה</u> $\tilde{H}_*(X)$ עבור X לא ריק

<u>הערה</u> מיפוי בין מרחבים טופולוגיים משרה מיפוי בין שרשראות וזה משרה מיפוי בין חבורות ההומולוגיה וכל אלה פונקטוריים.

הרצאה: 14,79 תרגילי כיתה
16 שברים (לא בחומר)
21 ... (לא בחומר)
23,28 ...
30 ...

טופולוגיה אלגברית, 7 במאי 2002.

<u>הומולוגיה</u> חזרה על הרעיון של $H_*(S^2), H_*(T^2), H_*(S^n), H_*(\mathbb{RP}^2)$

<u>סימפלקס: הגדרה</u> $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} : \sum t_i = 1; t_i \geq 0\}$ (ה-n סימפלקס)

באופן כללי אם $v_i \in V$

$[v_0, \ldots v_n] \to \mathrm{Conv}\{v_i\} = \{\sum t_i v_i : \bar{t} \in \Delta^n\}$

הזיהוי V ו-Δ^n ע"י $\bar{t} \mapsto \sum t_i v_i$

$[e_0 \ldots e_n] \xrightarrow{\text{כלומר}} \Delta^n$.

<u>הגדרה</u> n-סימפלקס סינגולרי ב-X: $\sigma : \Delta^n \to X$

$C_n(X) = \{\sum k_i \sigma_i : k_i \in \mathbb{Z}\} = \langle n\text{-סימפלקסים ב-}X \rangle$ (n-שרשראות)

$\partial_n : C_n(X) \to C_{n-1}(X)$ ע"י $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ [e_0 \ldots \hat{e}_i \ldots e_n]$

<u>טענה</u> $\partial^2 = 0$ <u>הגדרה</u> $H_n(X) = \ker \partial_n / \mathrm{im}\, \partial_{n+1} = \frac{n\text{-מחזורים}}{n\text{-שפות}}$

<u>טענה</u> אם $X = \dot{\cup} X_i$ אז $H_n(X) = \bigoplus H_n(X_i)$ לכל n

<u>טענה</u> $H_0(X) = \mathbb{Z}^{\#\text{רכיבי קשירות}}$

<u>טענה</u> $H_n(pt) = 0 \ldots$

<u>הגדרה</u> $\tilde{H}_n(X)$ וחישוב בדוגמה.

<u>פונקטוריאליות</u>:

$$\text{מ"ט} \xrightarrow{\cancel{\Delta}C_*} \text{שרשראות (chain complexes)} \xrightarrow{\cancel{H_n}\, H_n^{\text{חב'}}} \text{חב'}$$

$$\text{מ"ט} \xrightarrow{\cancel{\Delta}\, H_n^{\text{טופ'}}} \text{חב'}$$

דיאגרמה מתחלפת של פונקטורים.

<u>משפט</u> אם $f \sim g$ אז $H_n(f) = H_n(g)$

<u>מסקנה</u> אם $X \sim Y$ אזי $H_*(X) = H_*(Y)$

שלבי ההוכחה: דוגמה "הומוטופיה" בין chain complexes - בין ההומומורפיזמים.

טופולוגיה אלגברית, 9 למאי 2002

$C_n^X = \{\sum k_i \sigma_i : \sigma_i : \Delta^n \to X\}$

~~$C_n = \{\sum k_i \sigma_i$~~

תזכורת: $[v_0 \dots v_n] : t \in \Delta^n \longmapsto \sum t_i v_i$

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ [e_0 \dots \hat{e_i} \dots e_n]$$

טענה: $\partial^2 = 0$ ונשאר להגדיר

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

טענה: $H_n(pt) = \begin{cases} \mathbb{Z} & n=0 \text{ אם} \\ 0 & n>0 \text{ אם} \end{cases}$

טענה: אם $X = \sqcup X_i$ אז $H_n(X) = \oplus H_n(X_i)$

טענה: $H_0(X) = \mathbb{Z}^{\#\text{מרכיבי הקשירות}}$

הגדרה: $H_n(X)$;

פונקטוריאליות:

$$\begin{tikzcd} \text{מ"ט} \arrow[r, "C_*"] \arrow[rr, bend right, "H_n^{top}"'] & \text{סדרות אלגבריות (chain complex)} \arrow[r, "H_n^{alg}"] & \text{חב'} \end{tikzcd}$$

[illegible] 2 פונקטורים.

משפט: אם $f \sim g$ אז $H_n(f) = H_n(g)$

מסקנה: אם $X \sim Y$ אז $H_*(X) = H_*(Y)$ ולכן $H_*(\mathbb{R}^k) = \{$

TE3 Mon Feb 6 6-8PM @ MS 3163

~~Point Set Topology~~ (start hiking at 5:30, bring water bottle)

Math 1300 Topology, Thursday Feb 2, 2006 (1 hour)

on board:

B: connected, loc. connected, s.l.s.c, based.

$$\begin{pmatrix}\text{connected based}\\ \text{coverings of}\\ B\end{pmatrix} \underset{X_H \leftarrow H}{\overset{X \mapsto p_*\pi_1(X)}{\Longleftrightarrow}} \begin{pmatrix}\text{Subgroups } H\\ \text{of } G=\pi_1(B)\end{pmatrix}$$

1. In what sense is "bigger H smaller X_H"?
2. Drop "connected"?
3. Drop "based"?
4. Morphisms?

__claim__ $S=p^{-1}(b_0)$ is a right G-set;

__claim__ This induces a functor

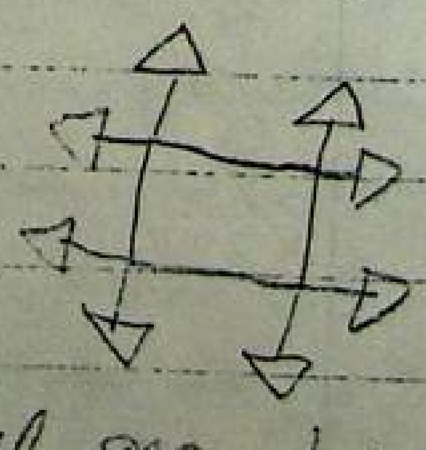
$$\begin{pmatrix}\text{cov's}\\ \text{of } B\end{pmatrix} \Longrightarrow \begin{pmatrix}\text{right}\\ G\text{-sets}\end{pmatrix}$$

__claim__ (connected components) $\Longleftrightarrow$ orbits

---

__Examples__ Find the coverings of ∞ corresponding to

1. $\langle a \rangle$
2. $\langle a^3 \rangle$
3. $\langle aba^{-1}b^{-1} \rangle$:



4. The normal subgroup gen. by $aba^{-1}b^{-1}$

* Wordy class today
* HW 7
* Wiki

Math 1300 Topology, Tuesday Feb 7 2006 (2 hours)

* Coverings like Feb 2.

* philosophy of homology.

Math 1300 Topology, Thursday Feb 9 2006 (1 hour)

on board

Idea: $H_n(X) = \left\{ \begin{matrix} k\text{-dim } C \\ \text{s.t. } \partial C = \phi \end{matrix} \right\} \Big/ \left\{ \begin{matrix} \partial C' \\ \dim C' = k+1 \end{matrix} \right\}$ HW7

better be $\partial(\partial C') = \phi$ or $\partial^2 = 0$

The domain space for intersection theory, which tells us when equations ~~must~~ have solutions

Continue as on May 7, 2002

Math 1300 Topology, Tuesday Feb 14 2006 (2 hours)

Follow Jan 25, 2005

Also on board:

~~Don't read:~~ Latecomers

The best part of every class is the first five minutes!

* Efficient review [don't tell me you don't need it]
* Statement of purpose, ideology, summary.

HW9 on Thursday.

Math 1300 Geom & Top, Feb 12 2008, hours II/16-17

Philosophy of homology:

1. boundriless submanifolds / boundaries.

2. Dual of de-Rham

3. The domain of intersection Theory.

$\Delta_p = \left\{ \sum_{i=0}^{p} \lambda_i e_i : \sum \lambda_i = 1 \right\}$ $\quad [v_0, \dots, v_p] : \Delta_p \to \mathbb{R}^n$

F_i^p = "The ith face map of Δ_p" $= [e_0 \dots \hat{e}_i \dots e_p]$

The face of a face

$\Delta_p(X) = C_p(X)$ "p-chains"

$\partial_p : C_p(X) \to C_{p-1}(X)$

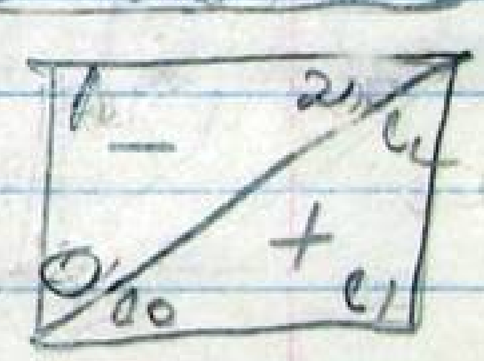
<u>Lemma</u> $\partial_p \partial_{p+1} = 0$ $(\partial^2 = 0)$

<u>Def</u> Z_p, B_p, H_p

$H_*(pt)$; H_*(disjoint union)

H_0(anything)

ϕ well def



<u>Thm</u> Let X be connected. then $H_1(X) \cong \pi_1^{ab}(X)$.

<u>Pf</u> Define $\phi : [\gamma] \mapsto [\gamma]$ in the obvious manner & show
$\qquad\qquad\quad \pi_1^{ab}(X) \qquad H_1(X)$

0. well def. on $\pi_1(X)$

1. $[\gamma_1 \gamma_2] \mapsto [\gamma_1] + [\gamma_2]$ 2. $[\bar{\gamma}]_\pi \to -[\gamma]_{H_1}$

3 $[e] \mapsto 0$ 4. ϕ descends to π^{ab}

5. ϕ has an inverse ψ ~~[illegible]~~ well def, ... /cont.

cont.

$$\psi(\sigma) = \gamma_{\sigma(0)}\,\sigma\,\overline{\gamma_{\sigma(1)}}$$

1. well def.

2. $\psi\circ\phi = Id$ (on the nose)

3. $\phi\circ\psi = Id.$ (up to homology) $\square$

So

X	H_1	
S^1	$\mathbb{Z}$	
S^n	0	$n>1$
P^n	$\mathbb{Z}/2$	$n\geq 2$
T^n	$\mathbb{Z}^n$	
∞	$\mathbb{Z}\oplus\mathbb{Z}$	

Math 1300 Geom & Top, Thursday Feb 14 2008, hour II/18

on board

$\Delta_p := \{\sum_{i=0}^{p} \lambda_i e_i : \sum\lambda_i = 1\}$ $[V_0,\dots,V_p]:\Delta_p \to \mathbb{R}^N$ by $\sum\lambda_i e_i \mapsto \sum\lambda_i V_i$

$C_p(X) = \langle \sigma : \Delta_p \to X\rangle$ $\partial : C_p \to C_{p-1}$ by

$\sigma \mapsto \sum_{i=0}^{p} (-)^i \sigma\circ[e_0 \dots \hat{e_i} \dots e_p]$ $\partial^2=0$

$\Rightarrow \quad \to C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow$

$C_{p+1} \supset Z_p$, $C_p \supset Z_p = \ker\partial_p \supset B_p = \operatorname{im}\partial_{p+1}$

$H_p = Z_p / B_p$

1. complete proof
2. on to philosophy

Thm $H_1(X)$ (X connected) $\cong \pi_1^{ab}(X)$

via $\phi : \pi_1 \to H_1$ by $[\gamma] \mapsto [\gamma]$

Do not turn this page until instructed.

Math 1300 Geometry and Topology

Term Exam 2

University of Toronto, February 11, 2007

Solve the 4 problems on the other side of this page.
Each problem is worth 30 points.
You have an hour and fifty minutes to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Advance apology: It may take me a while to grade this exam; sorry.

Good Luck!

Solve the following 4 problems. Each problem is worth 30 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1 "Compute". Let $\omega \in \Omega^2(\mathbb{R}^3_{xyz})$ be $\omega = y\,dx \wedge dz$, and orient $\mathbb{R}^3_{xyz}$ using the order (x, y, z). Let R be the rectangle $[-\frac{\pi}{2}, \frac{\pi}{2}]_\theta \times [0, 2\pi]_\phi$ oriented using the order (θ, ϕ). Let $\lambda : R \to \mathbb{R}^3_{xyz}$ be given by $\lambda(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta, \cos\theta\sin\phi)$.

1. Compute $\lambda^\star\omega$. $-\sin^2\theta\cos\theta\,d\theta d\phi$
2. Compute $\int_R \lambda^\star\omega$. $-\frac{4}{3}\pi$
3. Compute $d\omega$. $-dxdydz$
4. Compute $\int_{[x^2+y^2+z^2\leq 1]} d\omega$. $-\frac{4}{3}\pi$

Problem 2 "Reproduce". State precisely and prove in detail the theorem about existence and uniqueness of lifts of maps $f : Y \to B$, where B is the basis of a covering $p : X \to B$.

only Y need be connected & loc. conn.

Problem 3 "Think". Let Y be the space obtained from a triangle by identifying its three edges, where all three edges are oriented counterclockwise. (Alternatively, $Y = \{z \in \mathbb{C} : |z| \leq 1\}/(z \sim e^{2\pi i/3}z$ whenever $|z| = 1))$.

21 1. Compute $\pi_1(Y)$, quoting the theorems you use along the way.

9 R 2. Prove that every map $Y \to \mathbb{R}P^2$ lifts to a map $Y \to S^2$, or find one that doesn't.

Problem 4 "Sketch". Sketch the derivation of the four Maxwell equations, along with the necessary condition on the charge-current, using differential forms and starting from the least action principle.

Problem 3
4 Van Kampen quote
6 cutting up Y
8 computation
3 result
9 Part 2

~~Good Luck!~~

Problem 4
5 1. statement of least action.
5 2. def of $\star$/discussion of relativity
5 3. derivation of E–L: $d\star dA = \star J$
5 4. The charge-current condition.
5 5. Poincaré & $F = dA$
5 6. Interpretation of $dF=0$, $d\star F = J$.

Part overall is more important than details.

רשימת תרגילים: 18 באפריל
2 במאי
7 במאי

טופולוגיה אלגברית 14 למאי 2002.

1. תזכורת על אינווריאנטיות תחת הומוטופיה.

2. מבנה המנסרה

$$P\sigma = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times Id) \circ [e_0 \dots e_i f_i f_{i+1} \dots f_n]$$

$$\partial P\sigma - P\partial\sigma = F_*\sigma - g_*\sigma$$ <u>טענה</u>

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3. <u>משפט</u> אם A סגורה לא ריקה שהיא כווץ דפורמציה של סביבה שלה ב-X
אזי יש סדרה מדויקת ארוכה

$$A \xrightarrow{i} X \xrightarrow{j} X/A$$

$$\to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \dots$$

מוטיבציה: מה זה סדרה מדויקת?

$0 \to A \to 0$

$0 \to A \to B$ $\quad 0 \to A \to B \to C \to 0$

$A \to B \to 0$

$0 \to A \to B \to 0$

ג. מסקנה: חישוב $F_*(S^n)$ $\quad (X,A) = (D^n, S^{n-1})$

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4. קו-נוס אלגברי: סדרה מדויקת קצרה של קומפלקסים
$0 \to A_* \to B_* \to C_* \to 0$ משרה סדרה מדויקת
ארוכה של חבורות הומולוגיה.

2 hrs

Goals: 1. YPIS
2. Homology lite
3. Bigwords "Homotopy invariance of homology"

Math 1300 Topology, Tue Jan 25 2005

on board.

$\Delta^n = \{ t \in \mathbb{R}^{n+1} : \sum_0^n t_i = 1 \}$ $[v_0, \ldots v_n] : \Delta^n \to V,\ t \mapsto \sum t_i v_i$

$C_n(X) = \langle \sigma : \Delta^n \to X \rangle$ $\partial_n : C_n \to C_{n-1}$ $\sigma \mapsto \sum (-)^i \sigma \circ [\ \hat{e}_i\ ]$

<u>Lemma</u> $\partial^2 = 0$ "$\to C_{n+1} \to C_n \to C_{n-1} \to \cdots$ is a complex"

$Z_n = \ker \partial_n$ $B_n = \operatorname{im} \partial_{n+1}$ $H_n(X) \stackrel{def}{=} Z_n / B_n$

YPIS pf of $\partial^2 = 0$

HomLite: $H_n(pt)$, $X = \bigsqcup X_i \Rightarrow H_n(X) = \bigoplus H_n(X_i)$

$H_0(X) = \mathbb{Z}^{\#(\text{connected comps})}$

$\tilde{H}_n(X)$

$$
\begin{tikzcd}
\text{spaces} \arrow[r, "C_*"] \arrow[dr, "H_n^{top}"'] & \text{chain complexes} \arrow[d, "H_n^{alg}"] \\
 & \text{Abelian groups}
\end{tikzcd}
$$

lift to a diagram of functors.

Homotopy invariance of homology:

<u>Thm</u> $f \sim g : X \to Y \Rightarrow \left(H_n(f) = H_n(g)\right) : H_n(X) \to H_n(Y)$

<u>Cor</u> $X \sim Y \Rightarrow H_n(X) = H_n(Y)$; $H_*(\mathbb{R}^k) = \ldots$

Intuitive idea, realization:

$$P\sigma = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times Id) \circ [f_0, \ldots f_i g_i g_{i+1} \ldots g_n]$$

<u>claim</u> $\partial P \sigma - P \partial \sigma = f_* \sigma - g_* \sigma$

1 hr.

Math 1300 Topology, Thu Jan 27 2005

on board

$f \sim g : X \to Y$ i.e. $H : X \times I \to Y$, $H|_{t=0} = f$ $H|_{t=1} = g$

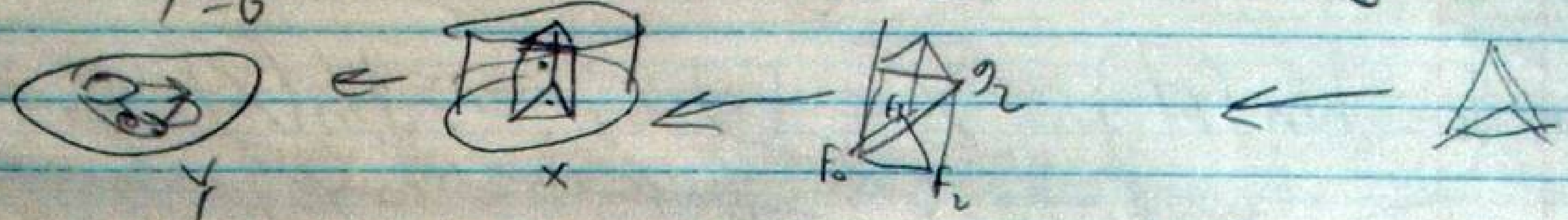
$$\begin{tikzcd}
\cdots C^{n+1}(X) \arrow[r, "\partial"] \arrow[d, shift right, "f_*"'] \arrow[d, shift left, "g_*"] & C^n(X) \arrow[r, "\partial"] \arrow[dl, "P"'] \arrow[d, shift right, "f_*"'] \arrow[d, shift left, "g_*"] & C^{n-1}(X) \cdots \arrow[dl, "P"'] \arrow[d, shift right, "f_*"'] \arrow[d, shift left, "g_*"] \\
\cdots C^{n+1}(Y) \arrow[r, "\partial"] & C^n(Y) \arrow[r, "\partial"] & C^{n-1}(Y) \cdots
\end{tikzcd}$$

$$\partial\sigma = \sum (-1)^i \sigma \circ [e_0 \ldots \hat{e}_i \ldots e_n]$$

$$P\sigma = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times Id) \circ [f_0 \ldots f_i g_i g_{i+1} \ldots g_n]$$

dream: $\partial P - P\partial = f_* - g_*$

start here



claim Indeed $\partial P - P\partial = f_* - g_*$

Continue as on May 14, 2002 —

* If $A \subset X$ is non-empty & closed, and a deformation retract of a neiborhood of it in X, then there is a "long exact sequence"

$$\to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A) \to \ldots$$

def 1. def. retract.

2. Top on X/A: a the biggest for which $X \xrightarrow{\pi} X/A$ is cont.

$\boxed{\text{Aside } X/A \text{ could be ugly}}$

$$\begin{tikzcd}
X \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\
X/A \arrow[r, "f"] & Y
\end{tikzcd}$$

π cont, $\tilde{f}$ cont $\Rightarrow f$ cont

3. exact sequences:

$0 \to A \to 0$, $0 \to A \to B$, $A \to B \to 0$,

$0 \to A \to B \to 0$, $0 \to A \to B \to C \to 0$

4. $A \overset{i}{\hookrightarrow} X \xrightarrow{j} X/A$

5. Compute $\tilde{H}_*(S^n)$ using $(X, A) = (D^n, S^{n-1})$

0708-1300/Class notes for Tuesday, February 26

From Drorbn

A Homology Theory is a Monster

6.1. Definition. A *homology theory* (on the category of all pairs of topological spaces and continuous maps) is a functor H assigning to each pair (X, A) of spaces, a graded (abelian) group $\{H_p(X, A)\}$, and to each map $f:(X,A)\to(Y,B)$, homomorphisms $f_*: H_p(X,A)\to H_p(Y,B)$, together with a natural transformation of functors $\partial_*: H_p(X,A)\to H_{p-1}(A)$, called the *connecting homomorphism* (where we use $H_*(A)$ to denote $H_*(A,\varnothing)$, etc.), such that the following five axioms are satisfied:

(1) (Homotopy axiom.)

$$f\simeq g:(X,A)\to(Y,B) \quad\Rightarrow\quad f_*=g_*:H_*(X,A)\to H_*(Y,B).$$

(2) (Exactness axiom.) For the inclusions $i: A\hookrightarrow X$ and $j: X\hookrightarrow (X,A)$ the sequence

$$\cdots\xrightarrow{\partial_*}H_p(A)\xrightarrow{i_*}H_p(X)\xrightarrow{j_*}H_p(X,A)\xrightarrow{\partial_*}H_{p-1}(A)\xrightarrow{i_*}\cdots$$

is exact.

(3) (Excision axiom.) Given the pair (X, A) and an open set $U\subset X$ such that $\bar{U}\subset \operatorname{int}(A)$ then the inclusion $k:(X-U,A-U)\hookrightarrow(X,A)$ induces an isomorphism

$$k_*: H_*(X-U,A-U)\xrightarrow{\approx}H_*(X,A).$$

(4) (Dimension axiom.) For a one-point space P, $H_i(P)=0$ for all $i\neq 0$.

(5) (Additivity axiom.) For a topological sum $X = +_\alpha X_\alpha$ the homomorphism

$$\bigoplus(i_\alpha)_*:\bigoplus H_n(X_\alpha)\to H_n(X)$$

is an isomorphism, where $i_\alpha: X_\alpha\hookrightarrow X$ is the inclusion.

The statement that ∂_* is a "natural transformation" means that for any map $f:(X,A)\to(Y,B)$, the diagram

$$\begin{array}{ccc} H_p(X,A) & \xrightarrow{\partial_*} & H_{p-1}(A) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ H_p(Y,B) & \xrightarrow{\partial_*} & H_{p-1}(B) \end{array}$$

is commutative. The statement that H is a functor means that for maps $f:(X,A)\to(Y,B)$ and $g:(Y,B)\to(Z,C)$ we have $(g\circ f)_*=g_*\circ f_*$, and also $1_*=1$, where 1 stands for any identity mapping.

Page 183 of Bredon's book

0708-1300/Navigation Panel [Hide]

Add your name / see who's in!

#	Week of...	Links
	Fall Semester	
1	Sep 10	About, Tue, Thu
2	Sep 17	Tue, HW1, Thu
3	Sep 24	Tue, Photo, Thu
4	Oct 1	Questionnaire, Tue, HW2, Thu
5	Oct 8	Thanksgiving, Tue, Thu
6	Oct 15	Tue, HW3, Thu
7	Oct 22	Tue, Thu
8	Oct 29	Tue, HW4, Thu, Hilbert sphere
9	Nov 5	Tue,Thu, TE1
10	Nov 12	Tue, ~~Thu~~
11	Nov 19	Tue, HW5
12	Nov 26	Tue, Thu
13	Dec 3	Tue, Thu, HW6
	Spring Semester	
14	Jan 7	Tue, Thu, HW7
15	Jan 14	Tue, Thu
16	Jan 21	Tue, Thu, HW8
17	Jan 28	Tue
18	Feb 4	Tue
19	Feb 11	TE2, HW9, Thu, Feb 17: last chance to drop class
R	Feb 18	Reading week
20	Feb 25	**Tue**, HW10
21	Mar 3	
22	Mar 10	HW11
23	Mar 17	
24	Mar 24	HW12
25	Mar 31	
26	Apr 7	
	Register of Good Deeds	
	Errata to Bredon's Book	

Bredon's Plan of Attack: State all, apply all, prove all.

Our Route: Axiom by axiom - state, apply, prove. Thus everything we will do will be, or should be, labeled either "State" or "Prove" or "Apply".

Math 1300 Geom & Top, Feb 26 2008 hours II / 19-20

On board: summary, TE2 at end.

Finish $H_1 = \pi_1^{ab}$

~~Homology & intersection theory:~~ later

Our / Bredon's approach.

Functoriality

homotopy invt: statement, applications, proof using prisms.

$$X \xRightarrow{C_*} \cdots \to C_{p+1}(X) \longrightarrow \underset{\{\sigma:\Delta_p \to X\}}{C_p(X)} \xrightarrow{\partial} C_{p-1}(X) \to \cdots \xRightarrow{H_p} H_p(X) = \frac{\ker \partial}{\operatorname{im} \partial}$$

$\partial\sigma = \sum (-1)^i \sigma \circ [e_0 \ldots \hat{e}_i \ldots]$

Reminders $H_p(\dot\cup X_i) = \oplus H_p(X_i)$, $H_p(pt) = \begin{cases} \mathbb{Z} & p=0 \\ 0 & \text{otherwise} \end{cases}$

$$H_0(X) = \mathbb{Z}^{\#\text{connected compnts}}, \quad H_1(X) \overset{\text{connected}}{=} \pi_1^{a.b.}(X)$$

Via $\phi : [\gamma]_{\pi_1^{ab}} \longmapsto [\gamma]_{H_1}$ $\quad \psi : [\sigma \longmapsto \gamma_{\sigma(0)} \sigma \bar{\gamma}_{\sigma(1)}$

$\psi \circ \phi = Id$ is trivial. Compute $\phi \circ \psi$

Our / Bredon's approach.

Functoriality of everything in sight,

Homotopy invariance idea level

Detail level: $P\sigma := \sum^n (-1)^i H \circ (\sigma \times Id) \circ [f_0 \ldots f_i g_i g_{i+1} \ldots g_n]$

Claim $\partial P\sigma = P\partial\sigma = f_*\sigma - g_*\sigma$

Schedule final to
Fri May 2 2–5

Math 1300 Geom & Top, Thursday Feb 28 2008, hour II/21

On board: Homology is a functor,

$$\begin{array}{c} X \\ \downarrow \\ Y \end{array} \implies \begin{array}{ccccccc} \circ & \to & \circ & \to & \circ & \to & \circ \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \circ & \to & \circ & \to & \circ & \to & \circ \cdots \end{array} \implies \begin{array}{c} H_p(X) \\ \downarrow \\ H_p(Y) \end{array}$$

well-behaved under homotopy

Today: The exactness axiom: given (X,A), we have a "long exact sequence"

$$\cdots \to H_p(A) \to H_p(X) \to H_p(X,A) \xrightarrow{\partial} H_{p-1}(A) \to \cdots$$

Questions to answer:

1. What's $H_p(X,A)$?
2. Who cares?
3. What does it mean?
4. Why is it true
5. Some real apps

Preview

1. $H(X,A) \sim H(X/A)$ & "if you know this, you know the third".
   Take $X = D^n$ $A = S^{n-1}$
2. $H(X,A) = H\left(C(X)/C(A)\right)$
3. homology & intersection theory

4. exact & short exact sequences

$0 \to A \to 0$ $\qquad$ $A \to B \to 0$

$0 \to A \to B$ $\qquad$ $0 \to A \to B \to 0$

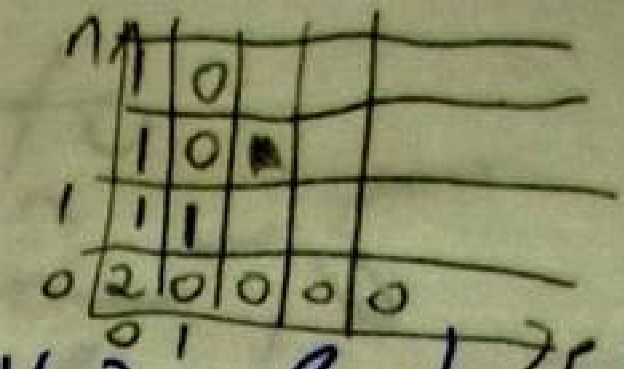
$0 \to A \to B \to C \to 0$

5. Long exact sequences from short exact sequences of chain complexes.

Final exam

Last class:	Thu Apr 10	us:
"study period":	Apr 14–18	"study period" or
real Andy's	Tue Apr 22	Wed Apr 30 →
401	Mon Apr 28	Fri May 2
complex	We May 7	

Math 1300 Geom & Top, Tuesday March 4 2008, hours II/22-3



on board

The sequence

$$\begin{array}{c} H(C_*(X)/C_*(A)) \\ \parallel \end{array}$$

$$H_p(A) \longrightarrow H_p(X) \longrightarrow H_p(X,A) \to H_{p-1}(A)$$

is long exact and natural.

Future plans

hours	
2	1. Long exact sequences
1	2. $H_p(S^n) \cong H_{p-1}(S^{n-1})$ for large p.
3	3. proof of excision.
2	4. basing inductions.

---

1. short exact sequences:

$$0 \to A \to 0 \qquad A \to B \to 0$$

$$0 \to A \to B \qquad 0 \to A \to B \to 0$$

$$0 \to A \to B \to C \to 0$$

2. Long exact sequences from short exact sequences of chains.

4. Naturality. (review in advance)

5. Excision

3 Examples: (D^n, S^{n-1})

(S^n, D^n_+)

6. Example $(S^n, D^n_+) \sim (D^n_-, S^{n-1})$

Why is U open?

Final: Fri May 2 2-5 PM
Bahen 2159

Math 1300 Geom & Top, Thu March 6 2008, hour II/24

on board:

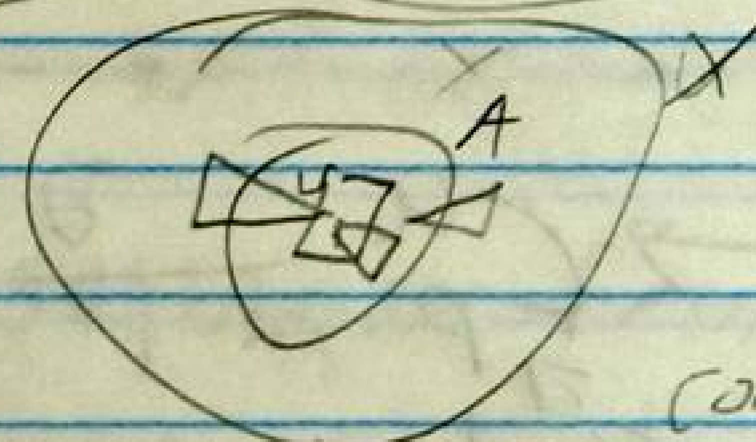
1. Homotopy $f \sim g : (X,A) \to (Y,B) \Rightarrow f_* = g_* : H_p(X,A) \to H_p(Y,B)$
2. L.E.S: Given (X,A), there is a functorial L.E.S:
$$\cdots \to H_p(A) \to H_p(X) \to H_p(X,A) \xrightarrow{\partial} H_{p-1}(A) \to \cdots$$
3. Excision: If $\bar{U} \subset \operatorname{int} A$, U open, Then
$$i_* : H_*(X-U, A-U) \xrightarrow{\sim} H_*(X,A)$$

$\rightsquigarrow$ for $p>1$, $H_p(S^n) \cong H_{p-1}(S^{n-1})$

$H_p(S^n)$ for singular homology (can't do it yet axiomatically)

Cor: The Brouwer f.p. Theorem

Proof of Excision



We have $i_* : C_*(X-U, A-U) \hookrightarrow C_*(X,A)$; constructing an inverse for "small" σ's is no problem.

Generalization "singular homology with small simplices"

Let $\mathcal{U}$ be an open cover of X; define $C_p^{\mathcal{U}}(X,A)$, $H_p^{\mathcal{U}}$

Thm The obvious $i : C_p^{\mathcal{U}} \to C_p$ induces an isomorphism $: H_p^{\mathcal{U}} \to H_p$

(Proves excision by taking $\mathcal{U} = \{A^\circ, X - \bar{U}\}$)

Solving the problem the American way:

"If you don't like something, smash it to pieces".

... cont.

For any $\epsilon > 0$ & any ~~V~~ normed V

We'll construct $S_\epsilon : C_p^{\text{affine}}(V) \to C_p^{\text{aff}}(V)$

and $T_\epsilon : C_p^{\text{affine}}(V) \to C_{p+1}^{\text{aff}}(V)$

s.t.
1. if $p < 1000$, the diameter of every simplex appearing in $S\sigma$ ~~will~~ is at most ϵ time $\operatorname{diam}(\sigma)$, and they all lie within the support of σ)
2. If $\phi : V \to W$ is affine linear, $S \circ \phi_* = \phi_* \circ S$
3. $S\partial = \partial S$
4. $S - I = T\partial + \partial T$ (Geometrically, $\partial S = \partial + \partial T \partial$) (& all simplices in $T\sigma$ lie inside σ)

Given S & T, Thm follows.

$$\begin{tikzcd}
C^{U}_{p+1} \arrow[r, "\partial"] \arrow[d, "i"] & C^{U}_{p} \arrow[r, "\partial"] \arrow[d, "i"] & C^{U}_{p-1} \arrow[d, "i"] \\
C_{p+1} \arrow[r] \arrow[u, dashed, bend left, "S_\epsilon"] & C_p \arrow[r, "\partial"] \arrow[l, bend right, "T_\epsilon"'] \arrow[u, dashed, bend left, "S_\epsilon"] & C_{p-1} \arrow[l, bend right, "T_\epsilon"'] \arrow[u, dashed, bend left, "S_\epsilon"]
\end{tikzcd}$$

Math 1300 Geom & Top, Tue March 11 2008, hours II/25-26

on board. $\Bigg\{$ U need not be open in $H_*(X-U, A-U) \xrightarrow{\sim} H_*(X,A)$
"Wanted" poster

* What we do with them:

$$
\begin{tikzcd}
C^U_{p+1} \arrow[r, "\partial"] \arrow[d, shift left] & C^U_p \arrow[r, "\partial"] \arrow[d, shift left] & C^U_{p-1} \arrow[d, shift left] \\
C_{p+1} \arrow[r, "\partial"] \arrow[u, dashed, shift left, "T"] & C_p \arrow[r, "\partial"] \arrow[u, dashed, shift left, "S"] \arrow[l, bend right, "T"'] & C_{p-1} \arrow[u, dashed, shift left, "S"] \arrow[l, bend right, "T"']
\end{tikzcd}
$$

---

construction Given $v \in \mathbb{R}^n$, $\sigma \longmapsto C_v\sigma$ by
$[v_0, \dots, v_p] \mapsto [v, v_0, \dots, v_p]$

$$\partial C_v \sigma = \sigma - C_v \partial \sigma$$

If $\sigma = [v_0 \dots v_p]$, let $b\sigma = \frac{1}{p+1} \sum v_i$

and set

$$S'\sigma = \begin{cases} \sigma & p=0 \\ C_{b\sigma} \cdot S'\partial\sigma & p>0 \end{cases}$$

and

$$T^{(1)}\sigma = \begin{cases} 0 & p=0 \\ C_{b\sigma}(S\sigma - \sigma - T^{(1)}\partial\sigma) & p>0 \end{cases}$$

<u>Claim</u> $S^{(1)}, T^{(1)}$ satisfy 1, 2, 4.

$\partial S = S\partial$: $\partial S\sigma = \partial C_{b\sigma} S\partial\sigma = S\partial\sigma - C_{b\sigma}\partial S\partial\sigma = S\partial\sigma$

$S - I = \partial T + T\partial$: $\partial T\sigma = \partial C_{b\sigma}(S\sigma - \sigma - T\partial\sigma) =$
$= S\sigma - \sigma - T\partial\sigma - C_{b\sigma}(\partial S\sigma - \partial\sigma - \partial T\partial\sigma)$ cont.

Continued.

Proof: claim If $B \subset A$ are finite non-empty ~~sets~~ sequences of vectors,

$$|bB - bA| \leq \frac{|A|-1}{|A|} \max_{v,w \in A} |v-w|$$

Proof
* replace B by $|B|$ times bB; lhs shrinks
* replace $A-B$ by $|A|-|B|$ times $b(A-B)$
* think.

claim if τ appears in $S\sigma$, then $\operatorname{diam} \tau \leq \frac{n}{n+1} \operatorname{diam} \sigma$

Finally, Let $S^k = S^k$ & $T^{(k)} = \text{"}\frac{S^k - 1}{S-1}\text{"}\, T^{(1)}$

Now $\tilde{H}$. and the last two axioms.

4 Dimension axiom $H_i(\{pt\}) = 0$ for $i \neq 0$.

5. Additivity: if $i_\alpha : X_\alpha \hookrightarrow \dot{\bigcup} X_\alpha$, then $\oplus i_\alpha : \oplus H_p(X_\alpha) \to$
$\to H_p(\dot{\bigcup} X_\alpha)$ is an iso.

Singular homology satisfies those!

HW11 will be available within a few days.

Today's agenda: 1. booting up homology
2. Degrees of maps $S^n \to S^n$

Math 1300 Geom & Top, Thu March 13 2008, hour II/27

The remaining axioms:

4. $H_p(pt) = 0$ for $p \neq 0$ (Declare $G := H_0(pt)$)

5. If $i_\alpha : X_\alpha \hookrightarrow \dot\bigcup X_\alpha$ then

$$\oplus i_\alpha : \oplus H_p(X_\alpha) \longrightarrow H_p(\dot\bigcup X_\alpha) \text{ is an iso}$$

Example $H_0(S^0) = G \oplus G$,
$x \mapsto -x$ acts by switching comps.

Def $\tilde{H}_0(X) := \ker(\epsilon_* : H_0(X) \longrightarrow H_0(pt))$

So

$0 \to \tilde{H}(X) \to H(X) \to H(pt) \to 0$

is exact

also set $\tilde{H}(X,A) = H(X,A)$

$\tilde{H}$ is a functor!

So $H_0(X) \cong \tilde{H}_0(X) \oplus G$ (not canonical)

Aside $0 \to A \to B \to C \to 0$
means $C = B/A$

"split" $0 \to A \underset{\beta}{\leftrightarrows} A \oplus C \underset{\alpha}{\leftrightarrows} C \to 0$

claim Given $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$
if either α or β exist, (α s.t. $p\circ\alpha = I$ / β s.t. $\beta\circ i = I$)
then the other exists too & $B = A \oplus C$.

Pf: diagram chase.

Cor $\tilde{H}_0(pt) = 0$, $\tilde{H}(S^0) = G$

but if $f : x \mapsto -x$ acts on $\tilde{H}_0(S^0)$ as $g \mapsto -g$.

So f induces non-commutative "

$$\begin{array}{ccc} \tilde{H}_0(S^0) \oplus G & \leftarrow & H_0(S^0) \\ \downarrow f_* & & \downarrow f_* \\ \tilde{H}_0(S^0) \oplus G & \leftarrow & H_0(S^0) \end{array}$$

Thm $\tilde{H}$ satisfies axioms 1, 2, 3 & furthermore $\tilde{H}_p(D^n) = 0$ for all p & n.

$\Rightarrow \tilde{H}_p(S^n) \cong \tilde{H}_{p-1}(S^{n-1}) \quad \forall p, n$

$\Rightarrow$ "homology has booted up".

Aside $\tilde{H}$ is the "right" singular homology

Thm $\deg(\text{reflection}) = -1$ for all n.

Math 1300 Geom & Top, Tue March 18 2008, hours II/28-29

$\epsilon : X \longrightarrow \{pt\}$ $\tilde H(X) := \ker \epsilon_*$

$H_p(X) = \begin{cases} \tilde H_p(X) & p \neq 0 \\ G \oplus \tilde H(X) & p = 0 \end{cases}$ $\tilde H(X,A) := H(X,A)$

Today's agenda
1. Odds & ends re. $\tilde H$
2. Degrees lite.
3. Degrees pro.

Satisfies 1. homotopy
2. (Exactness) for $X, A \neq \emptyset$
3. (Excision
4. (Dimension) $\tilde H_p(pt) = 0$
5. (Additivity) → under good conditions

$$\tilde H(X \vee Y) \underset{i_*}{\overset{p_*}{\rightleftarrows}} \tilde H(X) \oplus \tilde H(Y)$$

Further notes:
1. $\tilde H$'s kinda natural.

2. All axioms for $p \in \mathbb{Z}$, though in practice, $p \geq 0$.

Def of deg, deg(reflection) $= -1$ for all n.

$$H_{n-1}(S^{n-1}) \xleftarrow{\sim} H_n(D^n, S^{n-1}) \xrightarrow{\sim} H_n(S^n, D^n_-) \xleftarrow{\sim} H_n(S^n)$$

Prop deg is homotopy invariant.

Prop $\deg f \circ g = \deg f \cdot \deg g$ Cor $\deg(\underset{\text{antipode}}{a}) = (-1)^{n+1}$

Cor if n is even, every $f : S^n \to S^n$ has a f.p.

Cor Every v.f. on S^2 has a zero.

---

remember that every f. can be approx.

Thm If $f : S^n \to S^n$ is smooth, y_0 is a regular value (plenty exist.)
and $f^{-1}(y_0) = \{x_1 \dots x_k\}$, then

$$\deg f = \sum_{i=1}^{k} \underset{\text{dep on orientation of } f \text{ near } x_i}{\pm 1} = \sum_{i=1}^{k} \operatorname{sign} \det df_{x_i}$$

(using special orthogonal trans to compare coords.

.../cont.

Math 1300 3/18/2009 cont.

<u>Examples</u> $z \mapsto z^k$; rolled wrapping, twist wrapping.

<u>Proof</u> 1. $T: S^n \to S^n$ a rotation

2. $A: \mathbb{R}^n \to \mathbb{R}^n$ non singular, defines $\tilde{A}: S^n \to S^n$

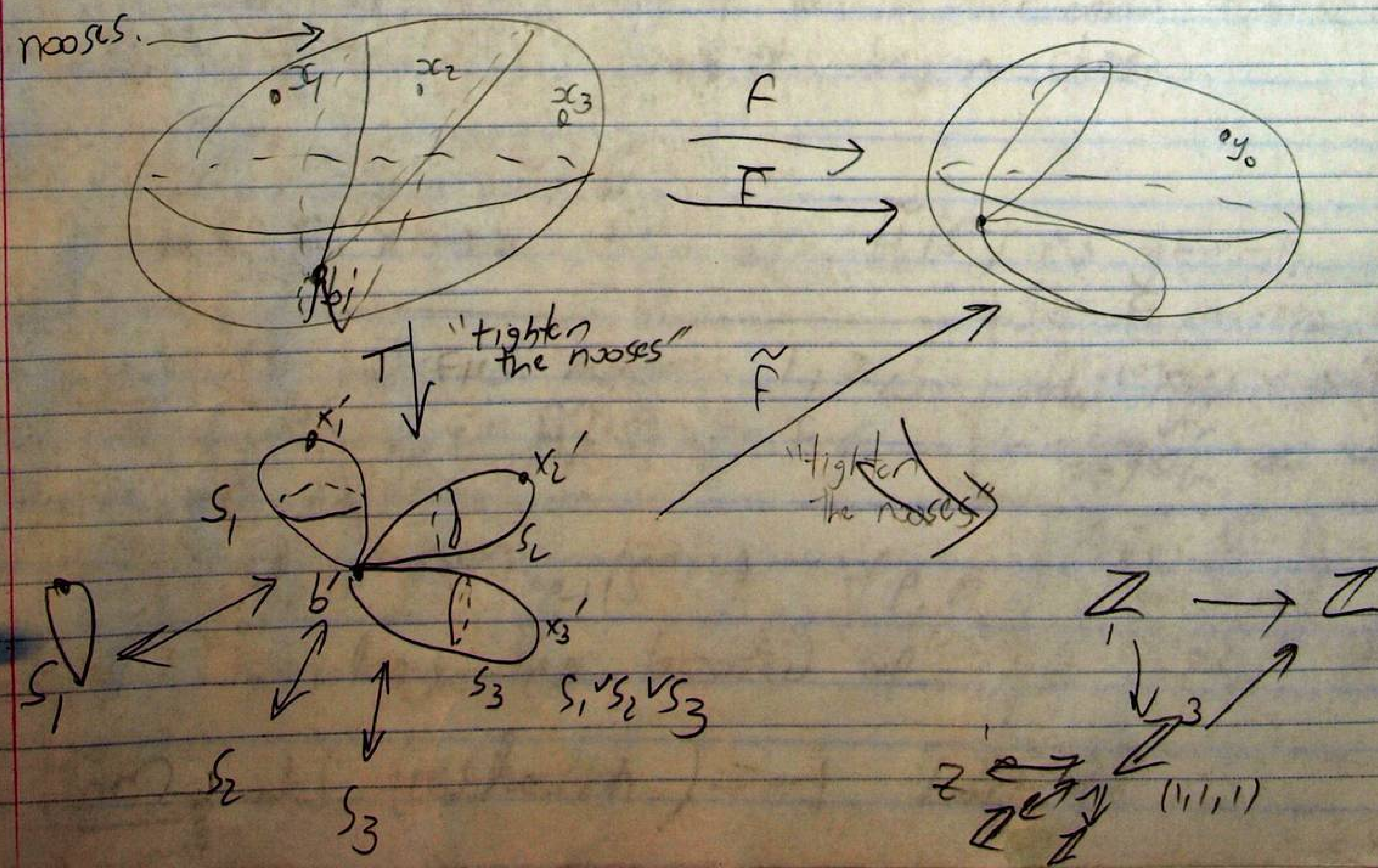
3. $F: \mathbb{R}^n \to \mathbb{R}^n$ smooth, with $F^{-1}(0) = \{0\}$ and $dF_0 = A$ non-sing

and $F(\infty) = \infty$

(Proof: take $F_t(x) = tF\left(\frac{x}{t}\right)$; then $F_1 = F$ and $F_0 = \tilde{A}$

4. Same w/ $F^{-1}(y_0) = x_0$

5. The general case:



Math 1300 Geom & Top, Thu March 20, 2008, cont.

<u>Def</u> A CW space K is a union $K = \bigcup_{n=0}^{\infty} K^{(n)}$ of nested spaces, taken with the weak topology, s.t.

* $K^{(0)}$ is a discrete set of points
* $K^{(n)}$ is obtaind from $K^{(n-1)}$ by glueing n-cells:

$K_n = \{\sigma\}$ $\{f_{\partial\sigma} : \partial D^n_\sigma = S^n_\sigma \to K^{(n-1)}\}$,

$$K^{(n)} = K^{(n-1)} \,\dot\cup\, \dot{\bigcup_\sigma} D^n_\sigma \Big/ {\scriptstyle \text{if } x \in S^n_\sigma,\ x \sim f_\sigma(x)}$$

<u>Def</u> $f_\sigma : D^n_\sigma \to K$ is The obvious ~~not-quite-~~inclusion

$\tau \in K_n$ $p_\tau : K^{(n)} \to S^n = B^n \cup \{\infty\}$ by mapping $K^{(n)}$ except the interior of D^n_τ to the south and int $D^n_\tau \to B^n$ in'

<u>Def</u> given $\tau \in K_{n-1}$, $\sigma \in K_n$ $[\tau : \sigma] = \deg(p_\tau f_{\partial\sigma}) : S^{n-1} \to S^{n-1}$

construct on board whils stating defs.

$$C^{CW}_n(K) = \langle K_n \rangle, \quad \partial\sigma = \sum_\tau [\tau : \sigma]\tau$$

$$C^{CW}_n(K, A) = \langle K_n - A_n \rangle$$

<u>Thm</u> $H_*(C^{CW}_*(K)) = H_*(K)$

<u>Examples</u>: All surfaces, $\mathbb{R}P^n$.

Math 1300 Geom & Top, Thu March 20 2008, hour II/30

Leftovers:

1. $F: S^{2n} \to S^{2n}$ has $f(x)=x$ or $f(x)=-x$
2. In last/step 3, why $F^{-1}(0)=\{0\}$ is necessary?
3. complete last/step 5.

Agenda:

1. Leftovers
2. CW-homology (statement)
3. Examples

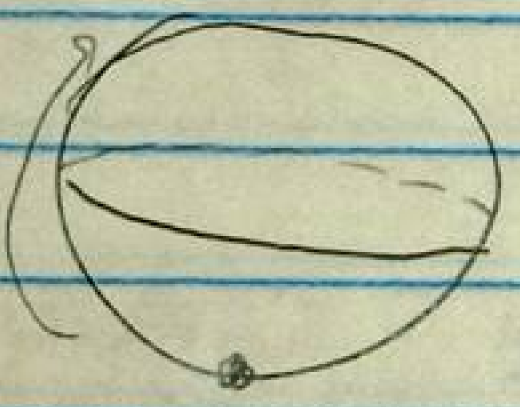
on board

1. $f: S^{2n} \to S^{2n}$ has $f(x)=x$ or $f(x)=-x$
2. The case $F^{-1}\{0\}=\{0\}$; i.e.,

$F: \mathbb{R}^n \to \mathbb{R}^n$, $dF|_0 = A$ non-sing, $F(\infty)=\infty$

I said take $F_t(x) = tF(\frac{x}{t})$ for $t\in[1,\infty]$

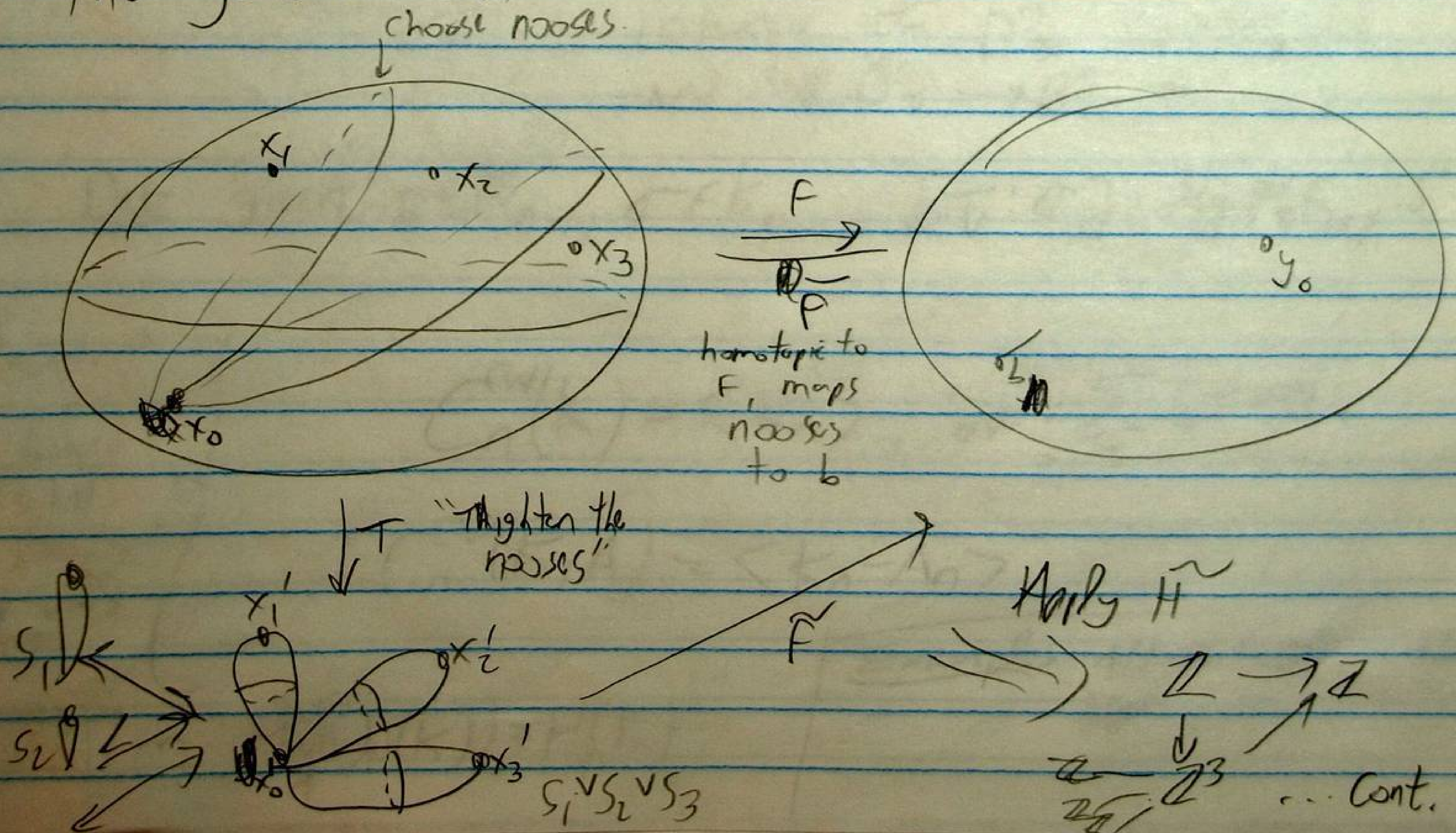
I wrong:



pushing the trouble to ∞ doesn't make it go away

Thm $f: S^n \to S^n$ smooth, y_0 regular, $f^{-1}(y_0)=\{x_1 \dots x_k\}$, $\deg f = \sum_{i=1}^{k} \operatorname{sign} \det df|_{x_i}$

The general case



... Cont.

Math 1300 Geom&Top, Tue March 25 2008, hours II/31-32

on board

CW-space: $K = \bigcup_{n=0}^{\infty} K^n$ "K^n "the n-th skeleton" s.t.

* K^0 is a discrete set of points.
* $K^n = K^{n-1} \,\dot\cup\, \dot\bigcup_{\sigma\in K_n} D^n_\sigma \Big/ \begin{array}{l}\text{if } x\in\partial D^n_\sigma = S^n_\sigma \\ x\sim f_\sigma(x)\end{array}$ $\Big|$ given "gluing maps" $\{f_{\partial\sigma} : S^n_\sigma \to K^{n-1}\}$

<u>Def</u> $f_\sigma : D^n_\sigma \longrightarrow K$ the obvious not-quite-inclusion

$\tau\in K_n$ $p_\tau : K^n \to S^n = B^n\cup\{\infty\}$ by mapping int $D^n_\tau \to B^n$, rest to ∞

<u>Def</u> given $\tau\in K_{n-1}$, $\sigma\in K_n$, $[\tau:\sigma] := \deg(p_\tau f_{\partial\sigma}) : S^{n-1} \to S^{n-1}$

<u>Def</u> $C^{CW}_n(K) = \langle K_n\rangle$ $\quad \partial\sigma := \sum_\tau [\tau:\sigma]\tau$

<u>Thm</u> C^{CW}_* is a chain complex an $H^{CW}_*(K) = H_*(K)$.

<u>Examples</u> $S^n, T^2, K^2, \Sigma_g, \mathbb{RP}^n, \mathbb{CP}^n$.

$\mathbb{RP}^n$:

$\sigma_n \xrightarrow{1+(-1)^n} \sigma_{n-1}$ $\qquad \sigma_4 \xrightarrow{2} \sigma_3 \xrightarrow{0} \sigma_2 \xrightarrow{2} \sigma_1 \xrightarrow{0} \sigma_0$

$\qquad\qquad\qquad\qquad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z}$

For $n>1$

So $H_p(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & p=0 \\ \mathbb{Z}/2 & 0<p<n \text{ odd} \\ 0 & 0<p<n \text{ even} \\ \mathbb{Z} & p=n \text{ odd} \\ 0 & p=n \text{ even} \end{cases}$ ... /cont.

Math 300 Geom & Top, March 25, 2008, Cont.

Lemma

$$H_p(K^n, K^{n-1}) \cong \bigoplus_{\sigma \in K_n} H_p(D^n_\sigma, S^{n-1}_\sigma) \quad \left(= \begin{cases} \langle K_n \rangle & p = n \\ 0 & \text{otherwise, } p > 0 \end{cases} \right)$$

Precisely, $H_p(K^n, K^{n-1}) \underset{f'_\sigma}{\overset{p'_\sigma}{\rightleftarrows}} H_p(D^n_\sigma, S^{n-1}_\sigma)$ induces the iso.

$$H_p(D^n_\sigma, S^{n-1}_\sigma) = H_p(S^n_\sigma, D^n_{\sigma-}) = H_p(S^n_\sigma)$$

Pf of theorem: Assume K is f.d., use "genetic recombination" & diagram chase

Precisely,

$(K^n_\sigma, K^{n-1}_\sigma)$:

$$\to H_{n+1}(K^{n-1}) \to H_{n+1}(K^n) \to 0 = H_{n+1}(K^n, K^{n-1}) \to$$
$$\to H_n(K^{n-1}) \to H_n(K^n) \to \langle K_n \rangle \to$$
$$\to H_{n-1}(K^{n-1}) \to H_{n-1}(K^n) \to 0 \to$$
$$\to H_{n-2}(K^{n-1}) \to H_{n-2}(K^n) \to 0$$

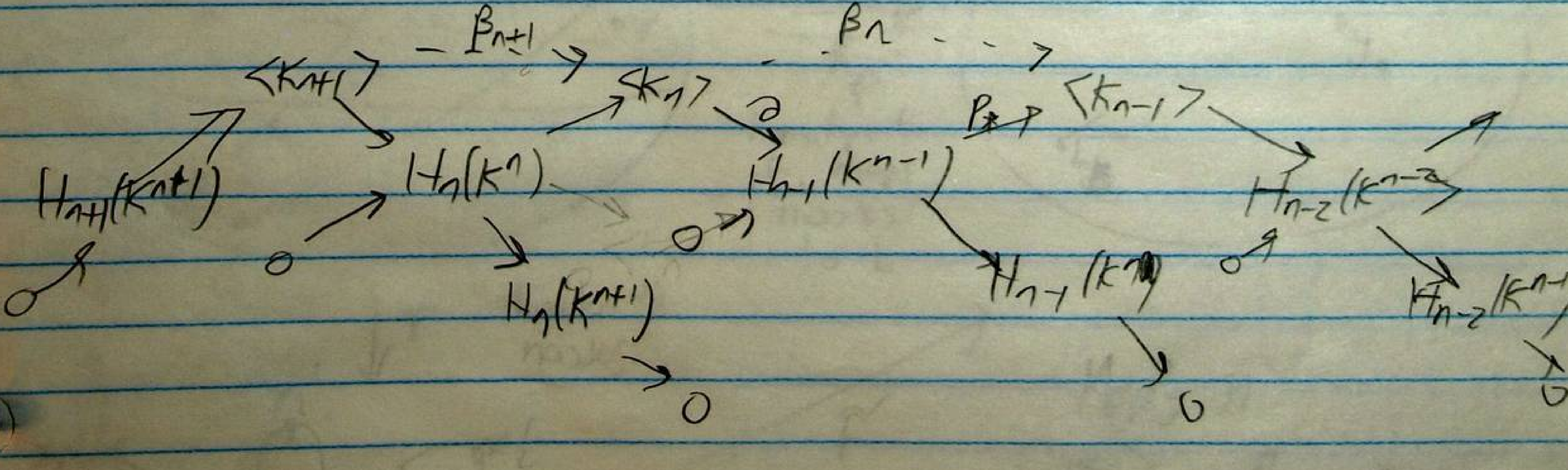
Conclusions:

$p > n \Rightarrow H_p(K^n) = 0$

$p < n \Rightarrow H_p(K^n) = H_{p}(K^{n-1})$

So we only need to compute $H_{n-1}(K^n)$

$$\forall n \Rightarrow 0 \to \underline{H_n(K^n)} \xrightarrow{p_*} \langle K_n \rangle \xrightarrow{\partial} \underline{H_{n-1}(K^{n-1})} \to H_{n-1}(K^n) \to 0$$



claim $H_n(K^{n+1}) \cong H(\beta)$ **Proof** diagram chase.

0708-1300/Homework Assignment 12

From Drorbn

Reading

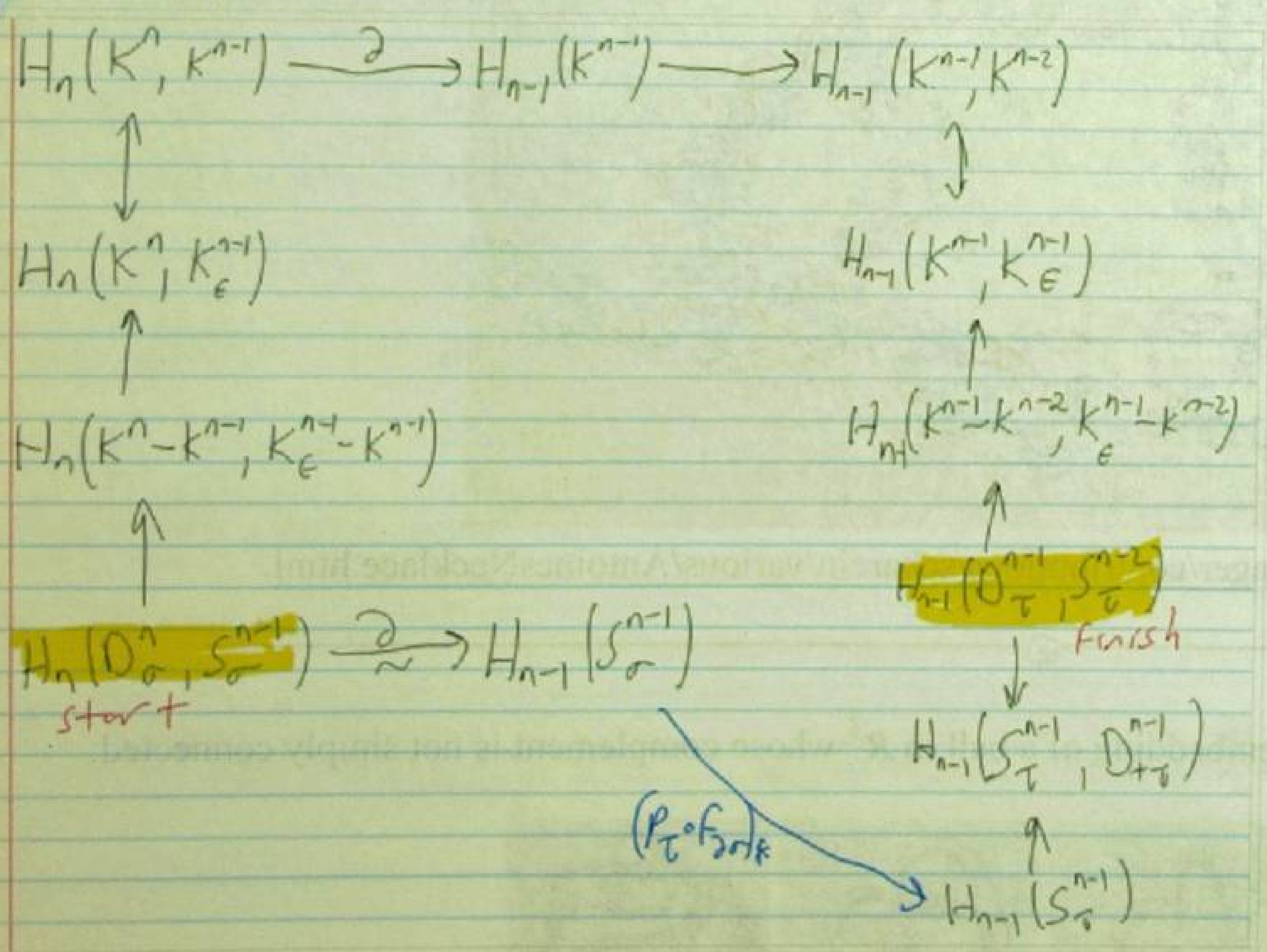
Read, reread and rereread your notes to this point, and make sure that you really, really really, really really really understand everything in them. Do the same every week! Also, read section 8-11, 13 and 18-20 of chapter IV of Bredon's book (three times, as always).

Doing

Solve all the problems in pages 206-207 of Bredon's book, but submit only your solutions of problems 1, 5, 9, and all the problems in pages 230 but submit only problem 1. Also, solve and submit the following:

Problem 12. Given a CW-space K with n-cells indexed by K_n and skeleta denoted K^n, show that the map $\partial_1 : \langle K_n \rangle \to \langle K_{n-1} \rangle$ given by the composition $H_n(K^n, K^{n-1}) \longrightarrow H_{n-1}(K^{n-1}) \longrightarrow H_{n-1}(K^{n-1}, K^{n-2})$ is equal to the one defined using degrees: $\partial_2 \sigma = \sum_{\tau \in K_{n-1}} [\tau : \sigma]\tau$, where $[\tau : \sigma] := \deg p_\tau \circ f_{\partial\sigma}$ and the $f_{\partial\sigma}$'s are the gluing maps defining K.

Hint. Dror's notes on the subject are:



Divide and conquer!

Due Date

This assignment is due in class on Thursday April 10, 2008.

0708-1300/Navigation Panel [Hide]

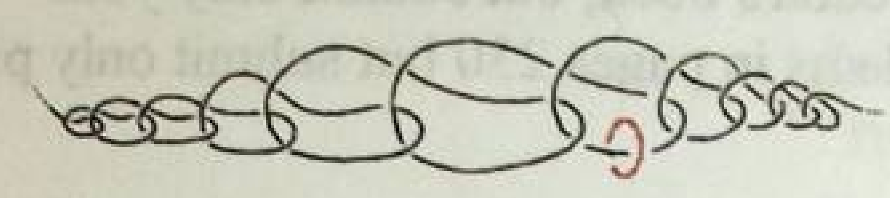
Add your name / see who's in!

#	Week of...	Links
	Fall Semester	
1	Sep 10	About, Tue, Thu
2	Sep 17	Tue, HW1, Thu
3	Sep 24	Tue, Photo, Thu
4	Oct 1	Questionnaire, Tue, HW2, Thu
5	Oct 8	Thanksgiving, Tue, Thu
6	Oct 15	Tue, HW3, Thu
7	Oct 22	Tue, Thu
8	Oct 29	Tue, HW4, Thu, Hilbert sphere
9	Nov 5	Tue,Thu, TE1
10	Nov 12	Tue, ~~Thu~~
11	Nov 19	Tue, HW5
12	Nov 26	Tue, Thu
13	Dec 3	Tue, Thu, HW6
	Spring Semester	
14	Jan 7	Tue, Thu, HW7
15	Jan 14	Tue, Thu
16	Jan 21	Tue, Thu, HW8
17	Jan 28	Tue
18	Feb 4	Tue
19	Feb 11	TE2, HW9, Thu, Feb 17: last chance to drop class
R	Feb 18	Reading week
20	Feb 25	Tue, HW10
21	Mar 3	
22	Mar 10	Tue, HW11
23	Mar 17	Tue
24	Mar 24	Tue, **HW12**, Thu
25	Mar 31	
26	Apr 7	
R	Apr 14	
R	Apr 21	
F	Apr 28	Final (Fri, May 2)
	Register of Good Deeds	
	Errata to Bredon's Book	

Topological Pathologies in R^3

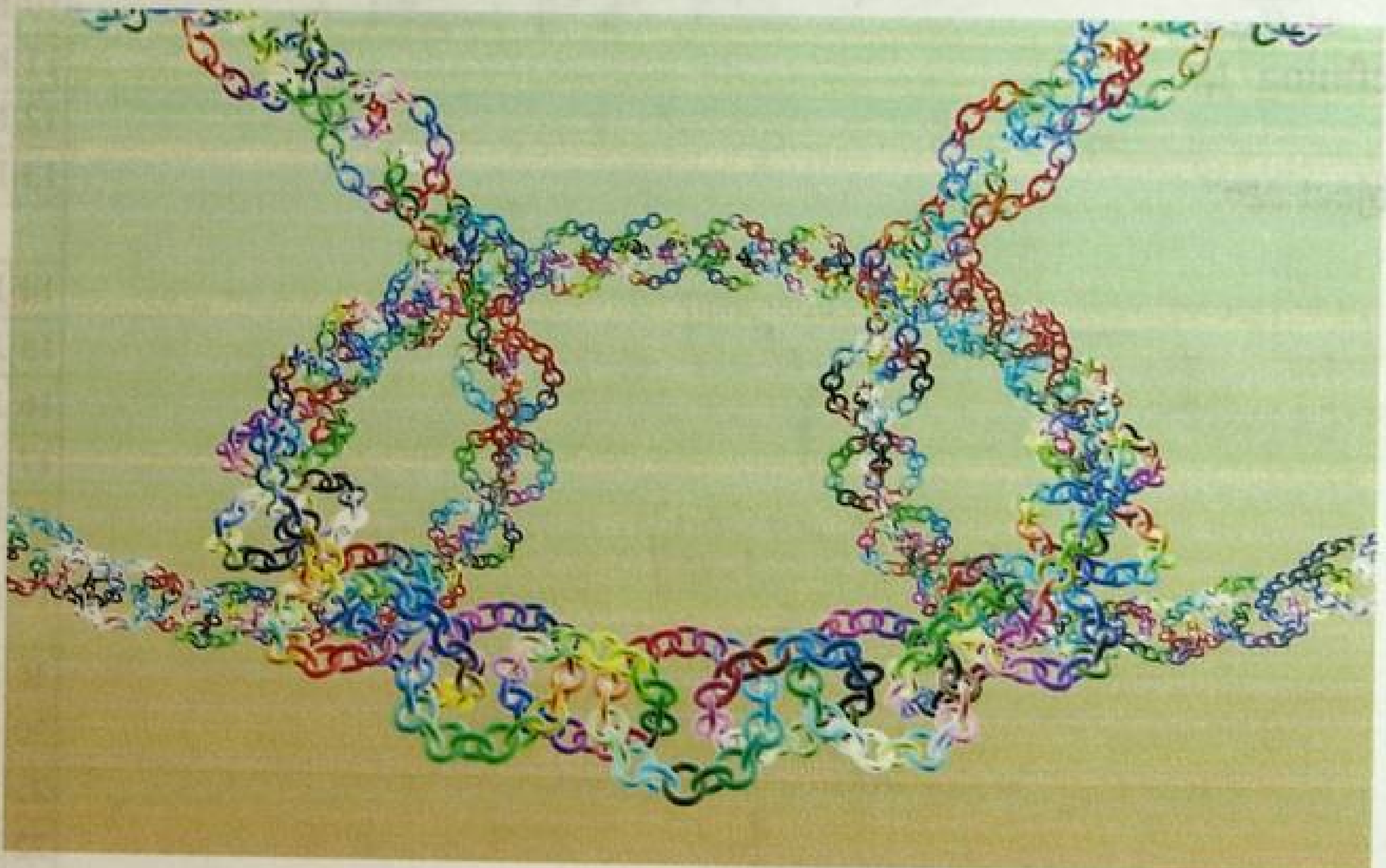
An embedding of an interval in R^3 whose complement is not simply connected:

See Hocking and Young's *Topology* pp. 176-177.



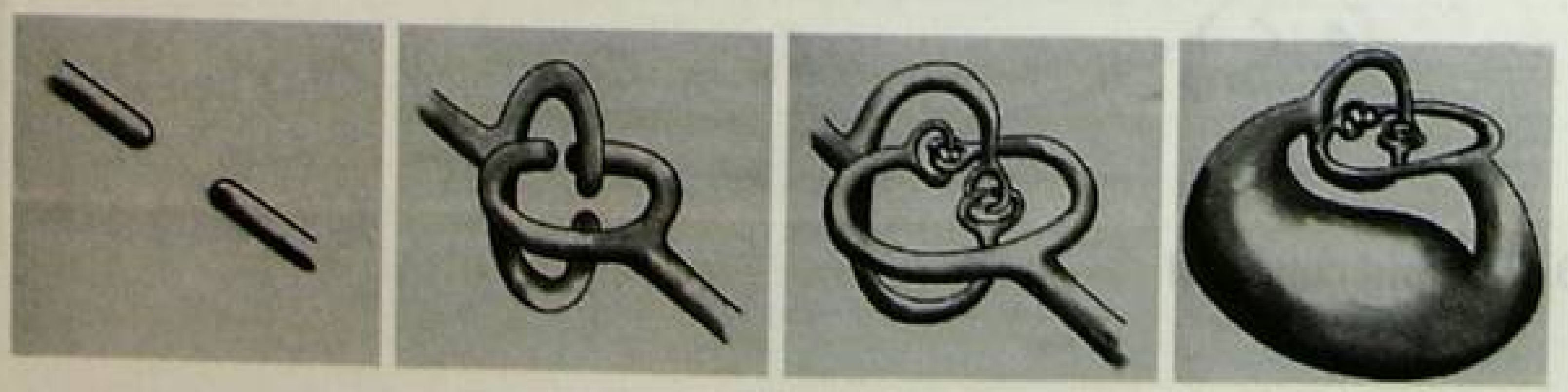
See http://www.math.ohio-state.edu/~fiedorow/math655/Jordan.html.

Antoine's necklace - an embedding of a Cantor set in R^3 whose complement is not simply connected:



See http://www.cs.ubc.ca/nest/imager/contributions/scharein/various/AntoinesNecklace.html.

The Alexander horned sphere - a continuous embedding of a ball in R^3 whose complement is not simply connected:



See http://users.math.uni-potsdam.de/~oeitner/EIGENES/RAEUME/hornsph.htm.

Most math manuscripts are like open source computer programs without any comments.
They are open (at least that), they compile and run, but they are impossible to understand.

The basic exact sequence for a CW-space $K = \bigcup K^{(n)}$, using the pair $(K^{(n)}, K^{(n-1)})$:

can be shared.

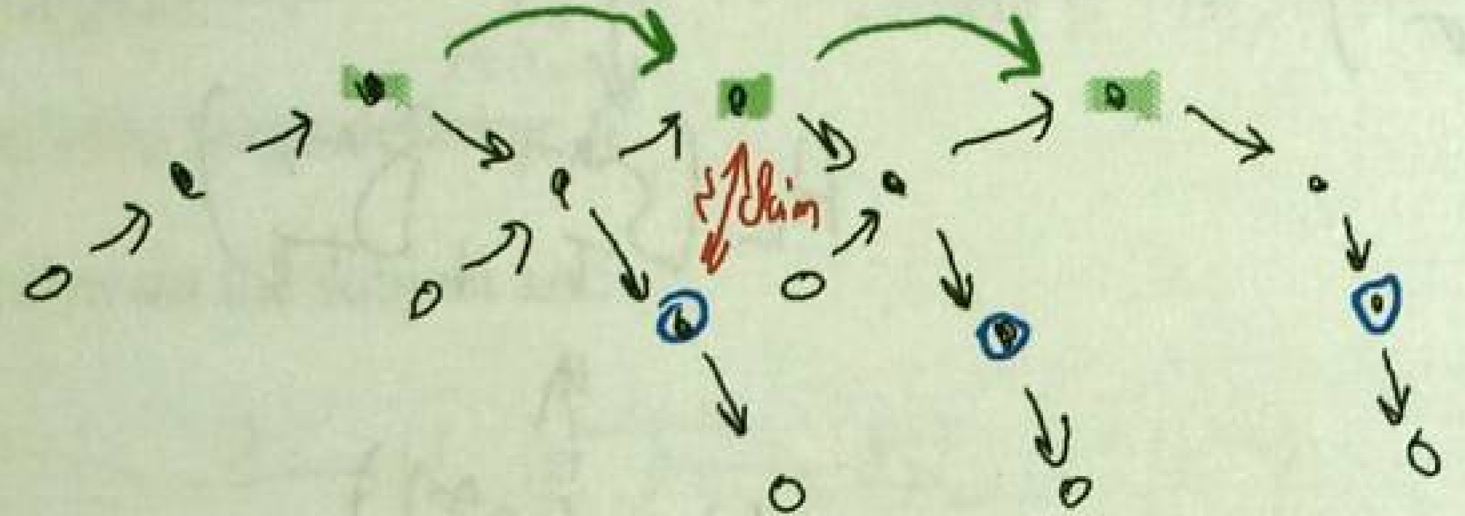
$$0 \longrightarrow H_n(K^{n-1}) \longrightarrow H_n(K^n) \longrightarrow \underset{\text{understood}}{H_n(K^n, K^{n-1})} \longrightarrow H_{n-1}(K^{n-1}) \longrightarrow$$

$$\longrightarrow \underbrace{H_{n-1}(K^n)}_{\text{goal}} \longrightarrow 0$$

$(H_{n+1}(K^n, K^{n-1}))$

$H_n(K^{n-1}) \cong H_n(K^{n-2}) \cong$ etc. $= 0$

$\cong H_{n-1}(K^{n+1}) \cong \dots$



Agenda for March 26:

1. Show a large commutative diagram to justify the degrees appearing in the differential.
2. Not to be justified claims:
   1. The same works in the infinite case
   2. The relative case.
   3. Cellular maps
   4. The cellular approximation theorem.
3. The exact seq of a triple.

4. The Mayer-Vietoris sequences.

$$\begin{tikzcd}
H_n(K^n, K^{n-1}) \arrow[r, "\partial"] \arrow[d, leftrightarrow] & H_{n-1}(K^{n-1}) \arrow[r] & H_{n-1}(K^{n-1}, K^{n-2}) \arrow[d, leftrightarrow] \\
H_n(K^n, K^{n-1}_\epsilon) & & H_{n-1}(K^{n-1}, K^{n-1}_\epsilon) \\
H_n(K^n - K^{n-1}, K^{n-1}_\epsilon - K^{n-1}) \arrow[u] & & H_{n-1}(K^{n-1} - K^{n-2}, K^{n-1}_\epsilon - K^{n-2}) \arrow[u] \\
H_n(D^n_\sigma, S^{n-1}_\sigma) \arrow[u] \arrow[r, "\partial"] & H_{n-1}(S^{n-1}_\sigma) & H_{n-1}(D^{n-1}_\tau, S^{n-2}_\tau) \arrow[u] \arrow[d] \\
 & & H_{n-1}(S^{n-1}_\tau, D^{n-1}_{+\tau}) \\
 & & H_{n-1}(S^{n-1}_\tau) \arrow[u]
\end{tikzcd}$$

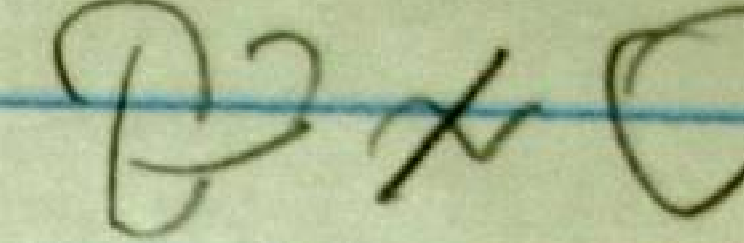
Math 1300 Geom & Top, Thu March 27 2008, hour II/33

1. The large commutative diagram.

2. Things we skip:
   0. deg formula (HW)
   1. The same works in the infinite case
   2. The relative case
   3. Cellular maps
   4. The cellular approx Thm
3. Euler's formula
4 ~~8~~. The exact seq. of a triple
5 ~~0~~. The Mayer Vietoris seq.
6 ~~8~~. Topological pathologies in $\mathbb{R}^3$.

on board:

Theorem 1



(as continuous embeddings of circles, modulo homotopy of such

Thm 2 An embedding of ~~an~~ closed intrvl in $\mathbb{R}^3$ has a simply connected complement

Thm 3 Same for a closed ball B^3

Thm 4 Same for a Cantor set.

Math 300 Topology, Thu March 10 2005

* deg reminder, linking clarification
* CW complexes & homology
* Examples
* Mayer Vietoris
* $H_1(X) = \pi_1(X)^{ab}$

* Return HW.

Math 300 Topology, Tue March 15 2005

* Exam on April 29, 2-5PM (Friday)
* Finish $\mathbb{RP}^n$
* $H_1(X) = \pi_1(X)^{ab}$
* Mayer Vietoris
* $\mathbb{R}^n$ Theorems 5&6

---

on board

$\mathbb{RP}^n = S^n/\pm 1 = e^n \cup_F \mathbb{RP}^{n-1}$, $F: \partial e^n = S^{n-1} \xrightarrow{proj} \mathbb{RP}^{n-1}$

C^{cw}: $\underset{n}{\mathbb{Z}} \longrightarrow \underset{n-1}{\mathbb{Z}} \longrightarrow \underset{n-2}{\mathbb{Z}} \to \xrightarrow{2} \underset{3}{\mathbb{Z}} \xrightarrow{0} \underset{2}{\mathbb{Z}} \xrightarrow{2} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}}$

H^{cw}: $\mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z}$

So $H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2 & k<n \text{ k odd} \\ 0 & k<n \text{ k even} \\ \mathbb{Z} & k=n \text{ odd} \\ 0 & k=n \text{ even} \end{cases}$

$H_k^b(X)$ for path connected X.

$\pi_1(X,b)^{ab} \cong H_1^b(X)$

Mayer Vietoris

$\mathbb{R}^n$ theorems 5&6.

Math 1300 Topology, Thu March 17 2005

* Mayer Vietoris
* $\mathbb{R}^n$ thms 5 & 6 (Pathologies in $\mathbb{R}^n$)

Special date April 11/12 Mon/Tue for Final.

Math 1300 Topology, Tue March 22, 2005

QE exams on Thursday.

on board

M-V: $X = A^\circ \cup B^\circ \Rightarrow$ exact is

$\to H_{n+1}(X) \to H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A \cap B) \to$

Theorem 5 If D is an embedded closed k-disk in S^n, then $\tilde{H}_i(S^n \setminus D) = 0$

Theorem 6 If S is an embedded k-sphere in S^n, Then $\tilde{H}_i(S^n \setminus S) = \mathbb{Z}$ for $i = n-k-1$ & 0 otherwise.

* Proof of theorem 5. (excellent exercise: Get the tube of last class from this proof)

* Proof ~~of~~ of theorem 6.
* Cor: Jordan's curve theorem & $S^{n-1} \subset S^n$
* Cor: invariance of domain

Borsuk-Ulam thm For every $g: S^n \to \mathbb{R}^n$ there's an $x \in S^n$ s.t. $g(x) = g(-x)$

Lemma $f: S^n \to S^n$ is odd $\Rightarrow$ deg f is odd.

Lemma $f: S^n \to S^n$ is odd $\Rightarrow$ (let $\bar{f}: \mathbb{RP}^n \to \mathbb{RP}^n$) $\deg_{\mathbb{Z}/2} \bar{f}$ = odd.

Math 1300 Topology, Thu March 24, 2005

Prove Borsuk+Ulam

Math 1300 Topology, Thursday March 30 (1 hour)

on board:

Thms 5,6
cor 7 } of handout.
Thm 8

The "knotted interval.

state theorems; go over pathologies handout, prove theorems.

Math 1300 Topology, Tuesday April 4 (2 hours)

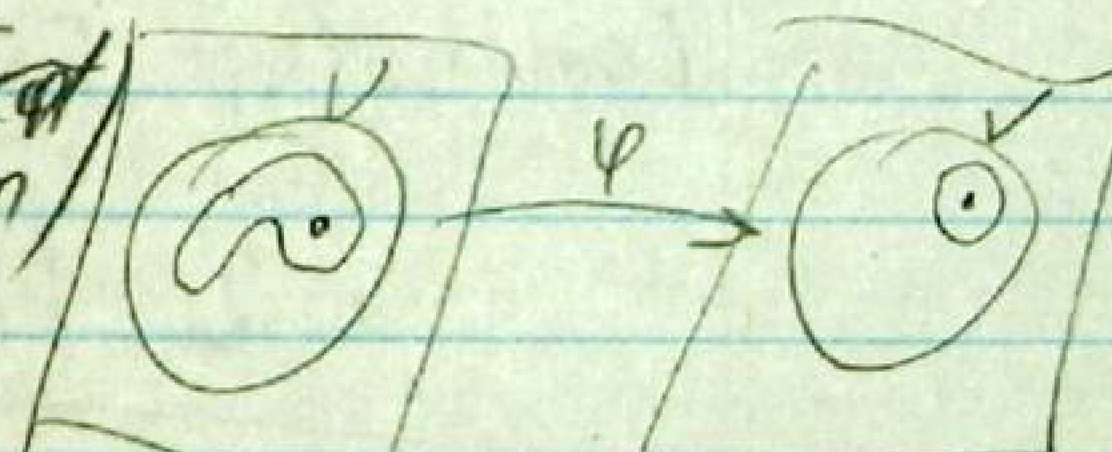
If a set in $\mathbb{R}^n$ is homeomorphic to a compact set in $\mathbb{R}^n$, is it compact in $\mathbb{R}^n$?

dense? closed? open?

on board:
* If a set in $\mathbb{R}^n$ feels open, it open
  "open (in $\mathbb{R}^n$) is an intrinsic property
* Homology with coefficients.
* The salad bowl theorem.
* The Borsuk-Ulam theorem.

Thm if $U \subset \mathbb{R}^n$ is homeomorphic to $V \subset \mathbb{R}^n$ & V is open, then so is U

~~Homology with coefficients - all is same (except $\mathbb{RP}^n$)~~



Statement of the salad bowl theorem & proof from Borsuk-Ulam.

If $g: S^n \to \mathbb{R}^n$, then $\exists x$ s.t. $g(x) = g(-x)$.

Borsuk-Ulam Prop: If $f: S^n \to S^n$ is odd, then $\deg f$ is odd.

If X is a double cover of B,

$$0 \to C_n(B, \mathbb{Z}/2) \xrightarrow{T} C_n(X, \mathbb{Z}/2) \to C_n(B, \mathbb{Z}/2) \to 0$$

Thus $0 \to H_n(\mathbb{RP}^n) \to H_n(S^n) \to H_n(\mathbb{RP}^n) \to H_{n-1}(\mathbb{RP}^n) \to 0 \to$

$\to 0 \to H_i(\mathbb{RP}^n) \to H_{i-1}(\mathbb{RP}^n) \to$

$0 \to H_1(\mathbb{RP}^n) \to H_0(\mathbb{RP}^n) \to H_0(S^n) \to H_0(\mathbb{RP}^n) \to 0.$

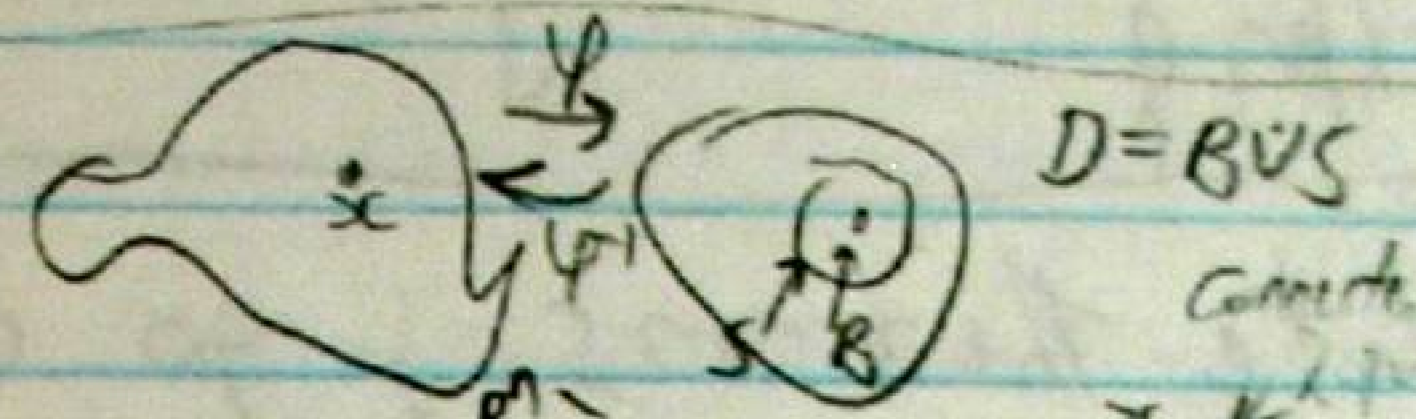
Math 1300 Topology, Thursday April 6 (1 hour)

* Plans

Thu	Tue	Thu
loose ends dreams on (co)homology	Cohomology & Cup.	loose ends, final, post-mortem

on board.

* Invariance of domain



$$C_1 \cup C_2 = \varphi^{-1}(S) = \mathbb{R}^n \setminus \varphi^{-1}(D) \cup \varphi^{-1}(B)$$

both connected, open

connected

* Borsuk-Ulam

---

comments 1. F_* act by mult. by $(\deg f)$ on $H_n(S^n;G)$

2. $H^{CW}(X;G)$

---

dreams on (Co)homology of Manifolds.

Math B00 Geom & Top, April 1 2008, cont.

Invariance of domain: "Openness in $\mathbb{R}^n$ is intrinsic"
"If V is homeomorphic to an open set in $\mathbb{R}^n$, it is an open set in $\mathbb{R}^n$."

If a set in $\mathbb{R}^n$ is homeomorphic to a compact set in $\mathbb{R}^n$, is it compact in $\mathbb{R}^n$?

dense? closed? open?

Proof

Borsuk-Ulam If $g: S^n \to \mathbb{R}^n$, then $\exists x$ s.t. $g(x) = g(-x)$

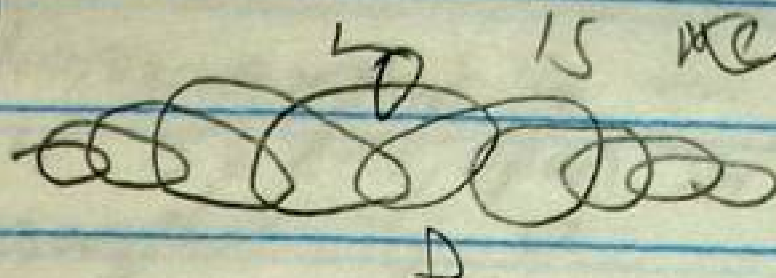
Salad Bowl If $\mu_1 \ldots \mu_n$ are ~~non-atomic~~ densities on $\mathbb{R}^n$, then $\exists$ ~~half space~~ hyperplane H s.t. $\mu_i(H_+) = \mu_i(H_-)\ \forall i$.

Prop If $f: S^n \to S^n$ is odd, then $f_*: H(S^n; \mathbb{Z}/2) \circlearrowleft$ is the identity.

Read Hatcher before class

Math 1300 Geom & Top, Tue April 1 2008, hours II/34-35

on board:
z_i best exercise: Find a surface whose boundary is ~~KE~~ L. in D^c



* Go over ~~[illegible]~~ referendum.

* **Thm** If $\phi: D^k \to S^n$ is an embedding, then $\tilde{H}_p(S^n - \phi(D^k))$ is what you think it is (i.e. 0).

Proof for $k=0$ - easy.

Now for $k>0$, assume $\alpha \in \tilde{H}_p(S^n - \phi(D^k))$ is non-trivial

$D^k = \boxed{D^k_L \mid D^k_R} \times D^{k-1}$ $\quad S^n - \phi(D^k) = (S^n - \phi(D^k_L)) \cup S^n - \phi(D^k_R))$

write Mayer-Vietoris

$\Rightarrow$ There is a sequence I_j of subintervals of I, st. $\cap I_j = \{x\}$

α is non-trivial in $\tilde{H}_p(S^n - \phi(I_j \times D^{k-1}))$ contradicting compactness & induction

Asides
1. The connecting homo. for M-V

Thm If $\phi: S^k \to S^n$ is an embedding,

Then $\tilde{H}_p(S^n - \phi(S^k))$ is what you think it is.

Pf $S^k = D^k_+ \cup D^k_-$ so $S^n - \phi(S^k) = (S^n - \phi(D^k_+)) \cup (S^n - \phi(D^k_-))$

Again use M-V. & induction.

(Notes 1. Induction starts at $k=0$, so to get Jordan, we had to go through all intermediates.
2. Claim In an open subset of $\mathbb{R}^n$, connected comp = path-connected comp.

cont.

3 Volunteers needed! 1. To count referendum votes
2-3 to rob a jewellery store.

Math 800 Geom & Top, Thu April 3 2008, hour II/36

Today: Borsuk-Ulam: $f: S^n \to \mathbb{R}^n \Rightarrow \exists x \in S^n \; f(x) = f(-x)$

$(\Longleftrightarrow \nexists g: S^n \to S^{n-1} \; g(-x) = -g(x))$

Thm If $g: S^n \to S^n$ is odd, then $f_*: H_n(S^n, \mathbb{Z}/2) \circlearrowleft$ is the identity. (implies B-U)

If $X \xrightarrow{p} B$ a 2-sheeted cover $\quad 0 \to C_n(B, \mathbb{Z}/2) \xrightarrow{l} C_n(X, \mathbb{Z}/2) \xrightarrow{p_*} C_n(B, \mathbb{Z}/2) \to 0$

... get a long exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & C_n(\mathbb{RP}^n) \arrow[r, "l"] \arrow[d, "g_*"] & C_n(S^n) \arrow[r, "p_*"] \arrow[d, "g_*"] & C_n(\mathbb{RP}^n) \arrow[r] \arrow[d, "g_*"] & 0 \\
0 \arrow[r] & C_n(\mathbb{RP}^n) \arrow[r] & C_n(S^n) \arrow[r] & C_n(\mathbb{RP}^n) \arrow[r] & 0
\end{tikzcd}$$

uses g is odd!

The L.E.S.

$$0 \to \underset{\mathbb{Z}/2}{H_n(\mathbb{RP}^n)} \xrightarrow[\sim]{l_*} \underset{\mathbb{Z}/2}{H_n(S^n)} \xrightarrow[0]{p_*} \underset{\mathbb{Z}/2}{H_n(\mathbb{RP}^n)} \xrightarrow[\sim]{\partial} \underset{\mathbb{Z}/2}{H_{n-1}(\mathbb{RP}^n)} \xrightarrow{l} 0 \xrightarrow{p_*} \underset{\mathbb{Z}/2}{H_{n-1}(\mathbb{RP}^n)} \xrightarrow[\sim]{\partial} \underset{\mathbb{Z}/2}{H_{n-2}(\mathbb{RP}^n)} \to 0$$

$$\to \dots \to \underset{\mathbb{Z}/2}{H_2(\mathbb{RP}^n)} \xrightarrow[\sim]{\partial} \underset{\mathbb{Z}/2}{H_1(\mathbb{RP}^n)} \xrightarrow{l} H_1(S^n) \xrightarrow{p} \underset{\mathbb{Z}/2}{H_1(\mathbb{RP}^n)} \xrightarrow[\sim]{\partial} \underset{\mathbb{Z}/2}{H_0(\mathbb{RP}^n)} \xrightarrow{0} H_0(S^n) \xrightarrow{\sim} H_0(\mathbb{RP}^n) \to 0$$

$$H_0(\mathbb{RP}^n) \xrightarrow{g} H_0(\mathbb{RP}^2)$$

$\square$

The necklace Theorem: n cuts for n gems.

Grading issues.

Math 1300 Geom&Top, Thu April 8 2008, hours II/37–38

on board:

$$\int_D d\omega = \int_{\partial D} \omega$$

Todays goals: "for a smooth manifolds

1. deRham cohomology is equal to singular cohomology & dual to singular homology"

2. Produce leftovers for next time.

First, who cares? Then proof...

who need early results? Draft on board $\int_D dw = \int_{\partial D} w$

Math 1300 Geom&Top, Tue April 8 2006, hours II / 37-38

<u>Thm</u> If M is a smooth manifold and $F=\mathbb{R}$,

$$
\begin{array}{ccc}
(H_p(M))^* & \cong & H^p_{dR}(M) \\
\text{iso 3} \downarrow & \xleftarrow[\text{duality}]{\beta} & \| \text{ iso 2} \\
(H_p^{smooth}(M))^* & \underset{\text{iso 1}}{\cong} & H^p_{smooth}(M)
\end{array}
$$

(Reminder $= H(\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots)$)

$H_p(M) \updownarrow H_p^{sm}(M)$

Who cares? Links homology & diff geom. Allows for "char. classes", computations, cup products, ...

$(\beta[w])([D]) = \int_D w.$

We will prove: iso 1 2 3.

<u>Iso 1</u>

only for V.S.!

$$C_3 \xrightarrow{\partial_3} \underset{D}{C_2} \xrightarrow{\partial_2} C_1$$

$$C_3^* \xleftarrow{d=\partial_3^*} \underset{w}{C_2^*} \xleftarrow{d=\partial_2^*} \underset{\varphi}{C_1^*}$$

$(H_2)^*$ $\overset{\beta}{\uparrow}$ H^2 by $(\beta[w])[D] = w(D)$

$$\operatorname{im}\partial_2^* = (\ker\partial_2)^\perp, \quad \ker\partial_3^* = (\operatorname{im}\partial_3)^\perp$$

$$\ker\partial_3^* / \operatorname{im}\partial_2^* = (\operatorname{im}\partial_3)^\perp / (\ker\partial_2)^\perp = \left(\ker\partial_2 / \operatorname{im}\partial_3\right)^*$$

$$\| $$

$$C_2^* / (\ker\partial_2)^\perp \Big/ C_2^* / (\operatorname{im}\partial_3)^\perp$$

Math 1300 Geom & Top, Thu April 10 2008, hour II/39 last.

M - a smooth manifold, $G = \mathbb{R}$

$$\begin{tikzcd}
(H_p(M))^* \arrow[d, "\text{iso } 3"'] & H^p_{dR}(M) \arrow[dl, "\beta"'] \arrow[d, "\alpha"', shift right] \arrow[d, phantom, "\cong \ \text{iso } 2", shift left=3ex] \\
H_p^{smooth}(M)^* & H^p_{sm}(M) \arrow[l, "\sim"', "\text{iso } 1"]
\end{tikzcd}$$

where α is induced by the obvious $\Omega^p \longrightarrow C^p_{sm}$ (chain map by stokes'!)

---

claim H^*_{dR} has a Mayer-Victoris seq!

$$0 \longrightarrow \Omega^p(U \cup V) \xrightarrow{\begin{pmatrix} i_U^* \\ -i_V^* \end{pmatrix}} \begin{matrix} \Omega^p(U) \\ \oplus \\ \Omega^p(V) \end{matrix} \xrightarrow{i_U^* + i_V^*} \Omega^p(U \cap V) \longrightarrow 0$$

is exact!

$\left(\begin{matrix} f + g = 1 \\ \text{on } U \quad \text{on } V \end{matrix} \qquad \omega \mapsto \underbrace{f\omega}_{\text{extends to } V} + \underbrace{g\omega}_{\text{extends to } U} \right)$

Pf of iso 2

1. Balls iso
2. finite Union of balls
3. Any open set in $\mathbb{R}^n$
4. Any manifold.

Pf of iso 3

Do not turn this page until instructed.

Math 1300 Geometry and Topology

Final Examination

University of Toronto, May 2, 2008

Solve 3 of the 4 problems in Part I and 3 of the 4 problems in Part II of this exam.

Each problem is worth 17 points.

You have three hours to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Advance apology: It will take me around 3 weeks to grade this exam; sorry. If you have a strong reason why you must have your grade ready quicker than that (e.g., you need to graduate), you **MUST** indicate that to Dror **NOW** and he will grade your exam separately and sooner.

Good Luck!

Part I

Solve 3 of the following 4 problems. Each problem is worth 17 points. **Neatness counts! Language counts!**

Problem 1.

1. State (5) and prove (7) the theorem about the local structure of immersions.

5 2. A manifold N is embedded inside a manifold M. Prove that every smooth function on N can be extended to a smooth function on M, at least locally.

Tip. As always in math exams, when proving a theorem you may freely assume anything that preceded it but you may not assume anything that followed it.

Problem 2.

6 1. State the Van Kampen theorem. ⊖ No "$A \cap B$ connected".

11 2. By appropriately gluing a disk to the wedge of two circles (the "∞" space), construct a space X_{34} whose fundamental group is $\langle a, b \colon a^3b^4 = 1\rangle$.

Const: 6
Proof 5

Tip. Of course, you also need to *prove* that X_{34} has the desired property.

Problem 3. Let $p : \mathbb{R}^2_{x,y} = \mathbb{C}_z \longrightarrow \mathbb{C}_w - \{0\} = \mathbb{R}^2_{u,v} - \{0\}$ be given by $w = e^z$ (i.e., by $p(z) = e^z$).

5 1. Prove that there is a unique (1/5) form $\omega \in \Omega^1(\mathbb{R}^2_{u,v} - \{0\})$ such that $p^\star\omega = dy$.

5 2. Find an explicit formula for ω, of the form $\omega = f(u,v)du + g(u,v)dv$.

7 3. Show that ω is closed (3) but not exact (4).

$-\frac{v}{u^2+v^2}$ $\frac{u}{u^2+v^2}$

Problem 4.

−1 connectedness properties misplaced/forgotten.

6 1. State precisely (but don't bother proving) the theorem about existence and uniqueness of lifts of maps $f : Y \to B$, where B is the basis of a covering $p : X \to B$.

11 2. Let $p_1 : X_1 \to B$ and $p_2 : X_2 \to B$ be coverings of a connected and locally connected space B, and assume that $p_{1\star}\pi_1(X_1) = p_{2\star}\pi_1(X_2)$. Prove that X_1 and X_2 are homeomorphic.

−4 no proof that $\tilde{p}_1 \circ \tilde{p}_2 = Id = \tilde{p}_2 \circ \tilde{p}_1$

Part II

Solve 3 of the following 4 problems. Each problem is worth 17 points. **Neatness counts! Language counts!**

Problem 5 "Compute". Embed S^3 inside $\mathbb{C}^2$ as the subset $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$ and consider the map $f : S^3 \to S^3$ given by $(z_1, z_2) \mapsto (z_1^3/|z_1|^2, |z_2|^6/z_2^5)$ (for the purpose of this definition, $\frac{0}{0} = 0$). Compute the degree $\deg f$.

Tip. "Compute", of course, really means "compute and justify your computation".

−7 sign
−8 non-generic pt.

Problem 6 "Reproduce".

4 1. State the exactness axiom for a homology theory.

4 2. State the excision axiom for a homology theory. 7 $\xleftrightarrow{5}$ 8

9 3. Use the exact sequences for a sphere in a disk and for a disk in a sphere, and the excision axiom, to prove that $H_p(S^n) = H_{p-1}(S^{n-1})$ when both p and n are large (that is, don't worry about "basing the induction").

9 8 +homotopy

Problem 7 "Think". The suspension SX of a topological space X is defined to be $X \times [0, 1]$ with $X \times \{0\}$ identified to a point and $X \times \{1\}$ identified to (another) point. Prove that $\tilde{H}_{n+1}(SX) = \tilde{H}_n(X)$ for every n.

Problem 8 "Sketch". The "homotopy axiom" for a homology theory states that if $f \sim g : X \to Y$, then $f_\star = g_\star : H_\star(X) \to H_\star(Y)$. Sketch to the best of your understanding the proof of the homotopy axiom for singular homology.

Tip. A good thumb rule is that you can safely omit details whose completion would qualify as "mechanical exercises".

chain homotopy
prisms

Good Luck!

Do not turn this page until instructed.

Math 1300 Geometry and Topology

Term Test

University of Toronto, November 8, 2007

Solve the 4 problems on the other side of this page.
Each problem is worth 30 points.
You have two hours to write this test.

Notes.

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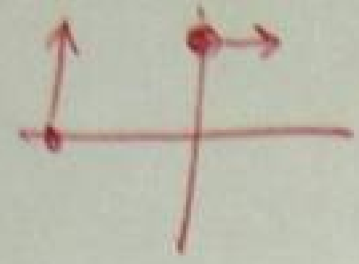
Good Luck!



Solve the following 4 problems. Each problem is worth 30 points. You have two hours. **Neatness counts! Language counts!**

Problem 1 "Compute". Let $\phi : \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$ be given by $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$, let $f : \mathbb{R}^2_{u,v} \to \mathbb{R}$ be given by $f(u,v) = u^2 + v^2$, and let $\xi \in T_{(0,1)}\mathbb{R}^2_{x,y}$ be $\xi = \partial/\partial x$. Compute the following quantities (with at least some justification):

(-6) unequal 3 & 4.

10 1. $\phi_*\xi$. $\partial_x u = 2x$, $\partial_x v = 2y$ $\Rightarrow$ $\phi_*\partial_x = 2x\partial_u + 2y\partial_v = 2\frac{\partial}{\partial v}$

8 2. $\phi^* f$. $= (x^2+y^2)^2 =$

12 { 6 3. $(\phi_*\xi)f$. $= 0$

6 4. $\xi(\phi^* f)$. $= 0$.

Problem 2 "Reproduce". The tangent space $T_0\mathbb{R}^n$ to $\mathbb{R}^n$ at 0 can be defined in the following two ways:

1. $T_0^1\mathbb{R}^n$ is the set of all smooth curves $\gamma : \mathbb{R} \to \mathbb{R}^n$ satisfying $\gamma(0) = 0$, modulo the equivalence relation $\sim$, where $\gamma_1 \sim \gamma_2$ iff $\dot\gamma_1(0) = \dot\gamma_2(0)$, where in general, $\dot\gamma$ denotes the derivative of $\gamma(t)$ with respect to t.
2. $T_0^2\mathbb{R}^n$ is the set of all linear functionals D on the vector space of smooth functions on $\mathbb{R}^n$, which also satisfy Leibnitz' rule, $D(fg) = (Df)g(0) + f(0)(Dg)$.

4 1. $\gamma \mapsto D_\gamma$
4 2. well-def
4 3. into ders
4 4. 1-1
9 5 onto
5 6 quote of Hadamard

Prove that these two definitions are equivalent (i.e., that there is a natural bijection between $T_0^1\mathbb{R}^n$ and $T_0^2\mathbb{R}^n$). If you use a non-trivial lemma from calculus, state it precisely but you don't need to prove it.

Problem 3 "Think". Let $f : M \to M$ be a smooth function from a compact manifold M to itself. Prove that there is a point $y \in M$ so that $f^{-1}(y)$ is finite. (In fact, there are many such points).

Problem 4 "Sketch". Sketch to the best of your understanding the proof of the Whitney embedding theorem, paying close attention to what is important and little attention to what is not. Here, more than anywhere else, **neatness and language count!**

Good Luck!

I	II	III	IV
9	9	9	3

IV, got just one β right: (-4)

III (-2) No construction of a proper function.

II did not normalize $\beta_{1,2}$