Buchweitz - Homological Algebra, Jan 17, 2005

A discussion of the history leading to "modern ~~category~~ homological ~~theory~~ algebra" (see picture)

A discussion of some modern applications / interactions of homological algebra. (see picture)

Definition: A category $Ob\,C$, $C(X,Y)$, compositions, identities (id_X), associativity.

Shortcuts: $X \in C \Leftrightarrow X \in ob\,C$

$$(f: X \to Y) \in C \Leftrightarrow f \in C(X,Y)$$

Each morphism has a unique domain & co-domain.

$$C(X,Y) \leftrightarrow Hom_C(X,Y)$$

$$End_C(X,Y) := C(X,X) \quad \text{an associative monoid.}$$

Examples: Sets, gps, Ab, Lie, Rings (w/ units & unital morphisms)

Rngs (Rings, possibly with no identity)

X - a topological space

Sheaves(X) category of Sheaves.

Small categories $Ob(C)$ is a set.

Examples: Directed graphs $\to$ categories.

pre-ordered ~~posets~~ $\to$ categories

poset $\Leftrightarrow$ a category with at most one morphism between any two objects.

Example X top space:

$\text{Open}\, X := B\{U \subseteq V : U, V \text{ open in } X\}$

Example a category with one object is an associative monoid.

Example G-spaces.

If X is a G-space,

$Ob(B_X G) = X$

$Hom_{B_X G}(x,y) := \{g \in G : gx = y\} \subseteq G$.

Definitions isomorphisms, the opposite category C^{op}, X^{op}, f^{op}

Definition $f: X \to Y$ is an epimorphism if for any Z, $Hom_C(Y,Z) \longrightarrow Hom_C(X,Z)$ is injective.

$f: X \to Y$ is a monomorphism iff f^{op} is an epimorphism. (i.e., f is left-cancellable).

Examples 1. $B(\cdot, \mathbb{Z})$ (the multiplicative monoid of $\mathbb{Z}$)

what are epis, monos, isos?

2. $\mu_2 \longrightarrow S_3$ in gps is a mono, epi but not iso.

two element group. symmetric group on 3 elements

?

3 in <u>CommRings</u> $\mathbb{Z} \to \mathbb{Q}$ is epi & mono but not iso.

? <u>Lemma</u> In <u>grps</u>, $\varphi: H \to G$ is an epi iff $N_G(\varphi(H)) = G$.

<u>Exercise</u> in <u>grps</u>, monomorphisms are as expected

<u>Exercise</u> 1. In <u>CommRings</u> monos and injective morphisms are the same.

2. A ring homomorphism $\varphi: S \to R$ is an epi <u>CommRings</u> iff

$$R \otimes_S R \longrightarrow R \text{ is a bijection.}$$

Q: In which category is iso $\Leftrightarrow$ epi & mono?

Q: In which categories every morphism is a composition mono $\circ$ epi?

<u>Def</u> A magma is a set with a binary operation

<u>def</u> idempotent, 0, units

---

small categories $\Leftrightarrow$ Associative magmas with 0, with a distinguishing set of idempotents. (DSOI)

__Prop__ An assoc^e magma with $\circ$ contains at most one ~~set~~ DSOI.

__Prop__ ("Set theoretic version of Gabriel's thm")

$$C: \text{small category} \xmapsto{Mg} M := \coprod_{x,y \in C} C(x,y) \cup \{0\}$$

↑ "the magma of C"

is an equivalence of "categories" with "magmas with 0, associative & with DSOI"

__Exercise__ What are "homomorphisms of categories"

Jan 24, 2005

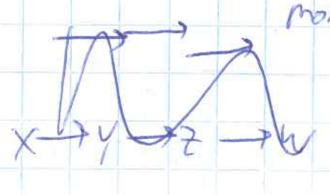
No class on Feb 7!

__Opposite category__ C cat $\to C^{op}$ "opposite category"

Monoid: $(M, \cdot) \longmapsto (M^{op}, \cdot^{op})$

$$B(M, \cdot)^{op} = B((M^{op}, \cdot^{op}))$$

↑ one object category defined by the monoid



In the case of a ring, $(R, +, \cdot)$

$R^{op} := (R, +, \cdot^{op})$

For groups $G^{op} \cong G$ by $x \mapsto x^{-1}$

Exercise $T_{\mathbb{Z}}(x_1 \ldots x_n)$ is isomorphic to its opposite.

a the tensor algebra

R ring $\rightarrow$ Mod-R category of right R-module

Mod-$(R^{op}) \cong R$-mod

$\neq (\text{Mod-}R)^{op}$

its any module category.

For groups, epimorphism and surjective morphisms <u>are</u> the same thing.

From M. Auslander (Queen Marry Lecture Notes) in collected works.

f is epimorphism $f: X \to Y$ $\Longleftrightarrow$ $\mathrm{Hom}(f, Z): \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ is injective

$gf = hf \Rightarrow g = h$

if g, h are epi, then so is gh

if gh is epi, so is g.

If g has a section: $\exists h\ gh = id$

$\Rightarrow$ g is epi

(g is a "split epimorphism")

<u>Example</u> in <u>Sets</u> every epi is split

For a mono, a "co-section" is called a retract

In <u>Vect</u>, every epi/mono is split.

In <u>Grps</u>, <u>Ab</u>, not every epi is split.

$$\begin{array}{ccccc} & \nearrow & f \text{ is split mono} & \Rightarrow & f \text{ is mono} \\ f \text{ is iso} & & & & \\ & \searrow & f \text{ is split epi} & \rightarrow & f \text{ is epi} \end{array}$$

split mono and split epi together $\Rightarrow$ f is iso

<u>Exercise</u> if f is mono and a split epi, then f is iso.

~~if~~ ~~$fg = id$~~ ~~AND~~ $fg = id$

<u>Def'n</u> f is a <u>strong</u> epimorphism if in every diagram (with k a mono)

$$\begin{tikzcd}
 & Y \arrow[dr, "a"] \arrow[dd, "\exists!\,\alpha"'] & \\
X \arrow[ur, two heads, "f"] \arrow[dr, "b"'] & & V \\
 & U \arrow[ur, tail, "k"'] &
\end{tikzcd}$$

$\exists!\ \alpha : Y \to U$ s.t. $k\alpha = b$

it follows $\alpha f = b$, α is unique.

A strong mono is the same, though starting with k.

---

g, h strong epis $\Rightarrow$ gh is a strong epi

gh strong epi $\Rightarrow$ g is a strong epi

any iso is a strong epi & mono.

~~any split~~ (split $\Rightarrow$ strong)

If $f: A \to B$ is epi & mono then TFAE:

1. f is iso.
2. f is strong epi
3. f is strong mono.

Def $\mathcal{C}$ a category, $f: X \to Y$ morphism.

(a) A coimage of f is a factorization $X \xrightarrow{g} Z \xrightarrow{k} Y$ s.t. g is a strong epi & k mono epi

~~(b)~~ ~~epi~~ and then $Z := \mathrm{coim}\, f$

(b) An image of f is a factorization

$$X \xrightarrow{g'} U \xrightarrow{k'} Y$$

g' epi, k' strong mono

$U := \mathrm{im}\, f.$

<u>claim</u> $\mathrm{coim} f$, $\mathrm{im} f$ are unique

Let $f: X \to Y$ be a morphism that admits a ~~[illegible]~~ co-image and assume

$f = ab$ is a factorization

w/ a mono & b epi

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & \mathrm{coim} \arrow[r] \arrow[d, "\alpha"', "\exists !"] & Y \arrow[d, equal] \\ X \arrow[r, "b"] & A \arrow[r, hook, "a"] & Y \end{tikzcd}$$

α is an epi and a mono.

if b is a strong epi, α is a ~~[illegible]~~ unique.

i.e. — $\mathrm{coim}(f)$ is unique up to unique iso.

Assume f admits image & co-image

$\exists !\, \alpha: \mathrm{coim} f \to \mathrm{im} f$

α is both mono & epi

~~If every morphism~~

IUG

$\Rightarrow$ If (mono+epi $\Rightarrow$ iso) then coim $f \cong$ im f canonically.

<u>Lemma</u> For f TFAE:

(a) $f = hg$ where g is a strong epi & h is a strong mono

(b) f admits an image & a coimage, and the canonical morphism between them is an iso.

<u>Defn</u> The morphism f has an "analysis" if $f = hg$ with g a strong epi & h a strong mono.

<u>Prop</u> If f has analysis,

1. every factorization $f = h'g'$ with h' mono & g' epi is an analysis.

$$\begin{tikzcd}
X \arrow[r] & \mathrm{coim} \arrow[r] \arrow[d, "\exists!\alpha"] \arrow[dd, bend right, "\mathrm{iso}"'] & Y \\
X \arrow[r, "g'"] & Z \arrow[r, "h'"] \arrow[d, "\exists!\beta"] & Y \\
X \arrow[r] & \mathrm{im} \arrow[r] & Y
\end{tikzcd}$$

A category where every morphism has an analysis is "balanced".

Prop For $\underline{C}$, TFAE:

1. $\underline{C}$ is balanced
2. Each morphism can be written uniquely up to unique iso as an epi followed by a mono.
3. Every epi is strong, every mono is strong and every morphism can be factored as mono∘epi
4. Every morphism has a co-image & all morphisms that are epi & mono are iso
5. Dual of 4.
6. Every morphism has image & co-image & the canonical map between them is an iso.

Balanced categories Sets, Gps, Ab, K-vect, Mod-R

But Rings & comm-rings are not.

Example $\mathbb{Z} \hookrightarrow \mathbb{Q}$

is epi & mono but not iso, has image & co-image.

co-image $\mathbb{Z} = \underset{\text{co-im}}{\mathbb{Z}} \hookrightarrow \mathbb{Q}$

image $\mathbb{Z} \hookrightarrow \underset{\text{im}}{\mathbb{Q}} \cancel{=} \mathbb{Q}$

In general: $R \xrightarrow{f} S$ ring morphism:

$R \longrightarrow R/\ker f \hookrightarrow S$ coimage

for commrings only! $\left\{ \begin{array}{l} R \longrightarrow T \longrightarrow S \quad \text{image} \\ \text{where } T = \{ s \in S \text{ s.t. } s \otimes 1 = 1 \otimes s \text{ in } S \otimes_R S \} \end{array} \right.$

So in comm-rings Every morphism has image & coimage but they are in general not same.

Q: in rings, same holds with $S \otimes_R S$ replaced by $S \ast_R S$?

~~(Mor)~~

<u>Def'n</u> A functor $F: \underline{C} \longrightarrow \underline{D}$

<u>Example.</u> $\underline{Grs} \longrightarrow \underline{Ab}$ via
$G \longmapsto G/[G,G]$

<u>Example</u> Δ: the simplicial category,
Obj: finite linearily ordered sets
Mor: non-decreasing maps.

~~Simpl~~ If $\underline{C}$ is a category, a simplicial ~~o~~object over $\underline{C}$ is a functor $F: \Delta^{op} \longrightarrow \underline{C}$

A functor $G: \Delta \rightarrow \underline{C}$ is a "co-simplicial" object.

<u>Example</u> $V \longmapsto V^*$ on Vect is a "contravariant" functor.

$X \in \underline{C}$, $h^X: Hom(X,-): \underline{C} \longrightarrow \underline{sets}$

$h_X: Hom(X,-): \underline{C}^{op} \longrightarrow \underline{sets}.$

Defn $\underline{C}$ category.

A presheaf on $\underline{C}$ is a contravariant functor to sets, i.e. $F : \underline{C}^{op} \to \underline{sets}$

F is "representable" if $\exists X \in \underline{C}$ s.t. $h_X \cong F$ (isomorphism of functors is to be defined).

Morphism of functors ("Natural Trans")

$$F, G : \underline{C} \longrightarrow \underline{D}$$

a morphism $\alpha : F \to G$ is $\alpha_X : F(X) \to G(X)$ in $\underline{D}$ for every $X \in \underline{C}$

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr, dashed] & F(X) \arrow[d, "\alpha_X"] \\ & G(X) \end{tikzcd}$$

s.t. if $f : X \to Y$ in $\underline{C}$, the ~~square~~ square:

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\alpha_Y"] & GY \end{tikzcd}$$

commutes

Ex fix $n \in \mathbb{Z}, n > 0$ — det is a nat. trans.

$$\underline{Rings} \xrightarrow{Gl_n} \underline{Gps} \qquad \underline{Rings} \longrightarrow \underline{R^*\ \text{units}\ Gps}$$

$$R \longmapsto Gl_n(R) \qquad R \longrightarrow \text{units in } R : R^*$$

Example There's a natural trans.

$$\mathrm{Id}_{Grps} \longrightarrow \text{Abelianization}$$

Example On Vect,

$$\begin{tikzcd} k\text{-Vect} \arrow[r, bend left, "id"] \arrow[r, bend right, "V \mapsto (V^*)^*"'] \arrow[r, phantom, "V\ \text{norm}"] & k\text{-Vect} \end{tikzcd}$$

Lemma 0 (Functors from $\underline{C}$ to $\underline{D}$) form a category, mod. set theory.

Lemma 1 A morphism $\alpha: F \to G$ of functors is an iso iff $\alpha_X : FX \to GX$ is an iso in $\underline{D}$ for each $X \in \underline{C}$

Lemma 2 $F: C \to D$ & $G: D \to E$ are functors, so is $G \circ F$

$F: \underline{C} \to \underline{D}$ is an equivalence of categories if $\exists G: \underline{D} \to \underline{C}$ s.t.

$$FG \simeq id_D$$
$$G \circ F \simeq id_C$$

__Example__ __F.D. k-vect__ is equiv. to C_k,

$\text{obj } C_k = \{0, 1, \dots\}$

$\text{Hom}(n, m) = M_{m \times n}(k)$

__Example__ Δ is equiv. to the category

$[0, n]_{\mathbb{N}}$ $n \geq 0$ (intervals of natural numbers)

A category is "skeletaly small" or "essentially small" if it is equiv. to a small category.

A "skelleton" in $\underline{C}$ is a $\overbrace{\text{subcategory}}^{\text{full}}$ $\underline{S} \subset \underline{C}$ s.t. $X \cong Y \Rightarrow X = Y$ in $\underline{S}$.

in the "subset of objects" sense.

__Jan 31, 2005__

Reminder: $f: X \to Y$ in $\underline{C}$

f mono: $\text{Hom}_C(Z, X) \to \text{Hom}_C(Z, Y)$ injective $\forall Z$

f epi : $\text{Hom}_C(Y, Z) \to \text{Hom}_C(X, Z)$ is surj. $\forall Z$

What about:

surjectivity of $\mathrm{Hom}_C(Z,X) \to \mathrm{Hom}(Z,Y)$

$$\begin{tikzcd} Y & X \arrow[l] \\ & Z \arrow[ul] \arrow[u, dashed, "?"'] \end{tikzcd}$$

injectivity of $\mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$?

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \\ Y \arrow[ur, dashed, "?"'] & \end{tikzcd}$$

<u>Def'n</u> The composition of functors.

$\mathrm{Hom}(\underline{C},\underline{D})$: The category of all functors with "morphisms of functors" as morphisms.

$$\begin{tikzcd} C \arrow[r, "F"] & D \arrow[r, "G"] & \underline{E} \\ C \arrow[r, "F'"] & D \arrow[r, "G'"] & \underline{E} \\ C \arrow[r, "F''"] & D \arrow[r, "G''"] & \underline{E} \arrow[from=1-1, to=2-1, phantom, ""] \end{tikzcd}$$

with $\downarrow\alpha$, $\downarrow\beta$ between the first and second rows, $\downarrow\alpha'$, $\downarrow\beta'$ between the second and third.

The composition $\alpha'' \circ \alpha'$ makes sense.

$$\beta * \alpha_X : GFX \xrightarrow{G\alpha_X} GF'X \xrightarrow{\beta_{F'X}} G'F'X$$

$\beta * \alpha$ is a morphism of functors from GF to $G'F'$

Claim

$$\begin{tikzcd}
GFX \arrow[r, "\beta_{FX}"] \arrow[d, "G\alpha_X"'] \arrow[dr, "\beta*\alpha_X"] & G'FX \arrow[d, "G'\alpha_X"] \\
GF'X \arrow[r, "\beta_{F'X}"] & G'F'X
\end{tikzcd}$$

Commutes.

without x's:

$$(G'\alpha)(\beta F) = (\beta F')(G\alpha) =: \beta * \alpha$$

Claim $(\beta'\beta) * (\alpha'\alpha) = (\beta' * \alpha')(\beta * \alpha)$

"the Hilton-Eckmann relation".

Aside Assume $*$, $\cdot$ satisfy (on a set S)

(i) Hilton-Eckman

(ii) have the same 2-sided identities.

Then $*$ & $\cdot$ are commutative.

[Application $\pi_i(X)$, $i \geq 2$ is Abelian]

Pf $\beta' = \alpha = \alpha$

$(\beta e) * (e\alpha') = (\beta * e)(e * \alpha') = \beta \cdot \alpha'$

$(\beta'\beta) * (\alpha'\alpha) = (e\beta) * (\alpha' e) = \beta * \alpha'$

$(\beta' * \alpha) \cdot (\beta * \alpha) = (e * \alpha')(\beta * e) = \alpha' \cdot \beta$

Def'n If $\underline{C}$ is a category, then the centre of $\underline{C}$ is

$$Z(C) := \mathrm{End}(\mathrm{id}_C)$$

i.e.,

$$\begin{tikzcd} X \arrow[r, "z_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "z_Y"'] & Y \end{tikzcd}$$

commutes.

If $Z(C)$ is a set, then it is a Abelian monoid.

Lemma If $C = \text{Mod-}R$, R a ring, then $Z(C) \cong Z(R)$ via

$$\begin{aligned} &Z(C) \\ &\ \ \rotatebox{90}{$\in$} \\ &z \longmapsto (z_R : R \to R)(1) \end{aligned}$$

Pf ~~Obvious~~ Use the fact that "R generates Mod-R".

Def'n A presheaf on $\underline{C}$ is a contravariant functor $\underline{C} \longrightarrow \underline{\text{sets}}$, $F : \underline{C}^{op} \longrightarrow \underline{\text{sets}}$

Example X top space,

$\underline{C} := \mathrm{open}(X) = B(\mathrm{open}(X), \subseteq)$

F a presheaf is

$$U \longmapsto F(U)$$

$$U \subset V \longmapsto F(V) \xrightarrow{\rho_{UV}} F(U) \quad \text{"restriction"}$$

Representable Functors:

$$F : \underline{C}^{op} \longrightarrow \underline{sets}$$

is representable if

$$F \cong \mathrm{Hom}(?, X) \quad \text{for some } X \in \underline{C}.$$

$$G : \underline{C} \longrightarrow \underline{sets}$$

is represantable if

$$G \cong \mathrm{Hom}(Y, -) \quad \text{for some } Y \in \underline{C}$$

Examples $\underline{gps} \longrightarrow \underline{sets}$ forgetfull.

represantable by $\mathbb{Z}$:

$$\mathrm{Hom}_{\underline{gps}}(\mathbb{Z}, G) \cong G$$

because $\mathbb{Z}$ is the free group on one generator.

$\underline{Ab} \longrightarrow \underline{sets}$ forgetful

$\text{Mod-}R \longrightarrow \underline{sets}$
(R has identity)

$\underline{Rings} \longrightarrow \underline{sets}$

$\underline{\text{Comm-rings}} \longrightarrow \underline{sets}$

$\text{Lie Alg}_K \longrightarrow \underline{sets}$

$\underline{sets} \xrightarrow{id} \underline{sets}$

representing object

$\mathbb{Z}$

R

$\mathbb{Z}[X]$

~~$\mathbb{Z}[x]$~~ $\mathbb{Z}[x]$

K as a 1-d Abelian

$\{e\}$

~~End End~~

$\text{End}(\underline{sets}) \longrightarrow \underline{sets}$

obj (X, f) $f: X \to X$

mor

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

$X \longmapsto X$

$(\mathbb{N}, +1)$

Finite groups $\longmapsto$ sets

"profinite representability"

not representable.

but $\{\mu_n : n \geq 2\}$ is a "representing family".

cyclic group of order n

$\mathrm{Hom}_{\mathcal{C}}(-,-): \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \underline{\text{Sets}}$

<u>Def'n</u> A k-linear structure on $\mathcal{C}$ is a lift of Hom: (k a commutative ring)

$$\begin{tikzcd} & \underline{\text{Mod-}k} \arrow[d, "\text{forget}"] \\ \mathcal{C}^{op} \times \mathcal{C} \arrow[ur, dashed, "F"] \arrow[r, "\mathrm{Hom}"] & \underline{\text{Sets}} \end{tikzcd}$$

(on each $\mathrm{Hom}(X,Y)$ we have a k-module structure s.t. compositions are k-bilinear)

If $k = \mathbb{Z}$ we say "pre-additive".

<u>Def'n</u> $\mathcal{C}' \subseteq \mathcal{C}$ is a full category if $\mathrm{obj}(\mathcal{C}') \subseteq \mathrm{obj}(\mathcal{C})$ and Hom sets are same when makes sense.

$F: \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{D}}$ is full (or faithful or fully faithful)

if $F: \mathrm{Hom}_{\underline{\mathcal{C}}}(X,Y) \longrightarrow \mathrm{Hom}_{\underline{\mathcal{D}}}(FX, FY)$

is surjective (or injective, or bijective)

<u>Def'n</u> $F: \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is <u>dense</u> if $\forall Y \in \mathcal{D}\ \exists X \in \mathcal{C}$ s.t. $F(X) \cong Y$.

Prop F is an equiv. ($F: \underline{C} \to \underline{D}$) of categories iff it is dense and fully faithful.

Pf $\Rightarrow$ trivial

$\Leftarrow$ Use AC to chose $G: \underline{D} \to \underline{C}$

Given $\underline{C}$, $\hat{\underline{C}} = \mathrm{Hom}(C^{op}, \underline{Sets})$ is the category of presheafs.

$\underline{C} \longrightarrow \hat{\underline{C}}$ via (Yoneda functor)

$$X \longmapsto h^X = \mathrm{Hom}_C(-, X)$$

Yoneda's lemma (1) $h^? : C \to \hat{C}$ is fully faithful.

(2) For each $X \in C$, $F \in \hat{C}$,

$$\mathrm{Hom}_{\hat{C}}(h^X, F) \cong F(X)$$

via $\omega_X : \varphi \longmapsto \varphi_X(id_X)$

(3) (Eilenberg–Kelly) Given $\underline{C}, \underline{D}$, $F: \underline{C} \to \underline{D}$, $K \in C$, $M \in D$ there is a natural bijection

$$\left\{ \begin{array}{l} \text{morphisms of functors} \\ \text{on } C \text{ from} \\ \mathrm{Hom}_C(K, ?) \to \mathrm{Hom}_D(M, F(?)) \end{array} \right\} \underset{\beta}{\overset{\gamma}{\rightleftarrows}} \mathrm{Hom}_D(M, FK)$$

by

$p \in \ell k s \mapsto \varphi(p) = p(id_K)$

$\theta \in Hom_D(M, FK) \longmapsto$

$$\begin{array}{l} \beta(\theta) : Hom_C(K,-) \xrightarrow{F} Hom_D(FK, F-) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \theta \\ \qquad\qquad\qquad\qquad\qquad Hom_D(M, F(-)) \end{array}$$

---

(3) $\Rightarrow$ (2) take $D = \underline{sets}$, $M = \{*\}$

(2) $\Rightarrow$ (1) $Hom_C(X,Y) \underset{?}{\xrightarrow{\sim}} Hom_{\hat{C}}(h^X, h^Y)$

$\underset{(2)}{\text{by}}$ $h^Y(X) \underset{?}{=} Hom(X,Y)$

<u>pf of (2)</u> need to show that ev_X is bijective.

<u>injective</u>: $\varphi, \psi : h^X \to F$ $\varphi \neq \psi$ means $\exists Y \in C$,

$\varphi_Y \neq \psi_Y : \underbrace{h^X(Y)}_{Hom(Y,X)} \longrightarrow F(Y)$

$\Rightarrow \exists g : Y \to X$ s.t. $\varphi_Y(g) \neq \psi_Y(g)$

$\varphi_Y(g) = F(g)\varphi_X(id_X)$

Limits and co-limits

I - small category

given $\underline{C}$, $X \in C$ is final if

$Hom_{\underline{C}}(Y, X) = $ singleton

for each $Y \in C$

(by Yoneda, $F(Y) = \{\cdot\}$

is representable)

Examples	$\underline{C}$	final	initial
	<u>sets</u>	$\{*\}$	$\emptyset$
(a "zero object" final = initial)	<u>grps</u>	$\{1\}$	$\{1\}$
	<u>rings</u>	$\{0=1\}$	$\mathbb{Z}$
	<u>categories</u>	$*\circlearrowleft$	$\emptyset$

---

let $F: I \to C$ be a functor,

$$\ell_X : I \longrightarrow \circlearrowleft \longrightarrow (X, id_X)$$

<u>Def'n</u> (of the inverse limit over F)

$$\varprojlim_F : C^{op} \longrightarrow \underline{sets}$$

via

$$X \longmapsto Hom_{Hom(I,C)}(\ell_X, F)$$

If $\varprojlim_F$ is representable, we write

$\varprojlim_F$ or $\varprojlim_I F$ for the representing objects

Exercise If I is a discrete category, $\varprojlim_F$ is ?

when is it representable?

what if C is replaced by C^{op}?

If $F: I \to \underline{C}$ is any functor,

$\varinjlim_F$ defined via

(direct limit, co-limit)

$$\varinjlim_F (X) := Hom_{C^{op}}(F, c_X)$$

same exercise.

Feb 14, 2005

Limits & colimits

Recall: I: small "index" category, C any category, $F: I \to C$ functor

$\forall X \in ob(C)$ $\quad C_X : I \to C$ "the constant functor"

$$\begin{array}{ccc} C_X : I & \to & C \\ a & \longmapsto & X \\ a \downarrow & & \downarrow id_X \\ b & \longmapsto & X \end{array}$$

$$\varprojlim_I F : C^{op} \longrightarrow sets$$
$$X \longmapsto Hom_{Hom(I,C)}(C_X \to F)$$

$\varprojlim_I F$ <u>exists</u> if this functor is representable; $\exists Z \in C$ s.t.

$$\alpha : Hom(-, Z) \xrightarrow{\simeq} \varprojlim_I F$$ as presheafs

Yoneda's Lemma:

Alt. perspective: Z is $\varprojlim$ if

$$\begin{tikzcd}
 & & F(a) \arrow[dd, "F(f)"] \\
X \arrow[urr] \arrow[r, "\exists!"] \arrow[drr] & Z \arrow[ur] \arrow[dr] & \\
 & & F(b)
\end{tikzcd}$$

$f : a \to b$ in I

Examples: I is discrete (i.e., a set w/ only identities as morphisms)

① $Fa = X_a$

$$(\varprojlim_I F)(Y) = \{(f_a : Y \to X_a) : a \in I\}$$
$$= \prod_{a \in I} \mathrm{Hom}(Y, X_a)$$

If $\varprojlim F$ exists, one writes a representing objects as $\prod_{a \in I} X_a = Z$

$p_a := \mathcal{Z}(a) : \prod X_a \to X_a$ "the projections"

In <u>sets</u> all products exist (in ZFC)

<u>Gps</u> –||–

<u>Ab</u> –||–

<u>Finite gps</u> : finite products exist

$(M, \cdot)$ a monoid $\quad BM : \bullet \circlearrowleft M$

products exist only over a singleton, or if M is a singleton.

② $I = \{\bullet \longrightarrow *\}$ $(= B(0<1))$

$F : X \xrightarrow{f} Y \quad \varprojlim F = X$

$I = (\bullet \rightrightarrows *)$ Exists in Grps!

The limit is the "equalizer":

in sets,

$$\varprojlim\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) \cong \{x \in X : f(x) = g(x)\}$$

If C has a zero object:

$$\begin{tikzcd} X \arrow[rr, "\text{called the zero map}"] \arrow[dr, "\exists!"'] & & Y \\ & 0 \arrow[ur, "\exists!"'] & \end{tikzcd}$$

Equalizer of $X \underset{0}{\overset{f}{\rightrightarrows}} Y$ is "$\ker f$".

$$I = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

"the pull back" "fibered product"

~~Lemma: $f : X \to Y$, $g : X \to Y$ be morphisms in C, then $eq(f, g) \cong$ fibre product of~~ $\cancel{\begin{tikzcd} & X \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}}$,

~~Provided all presheaves on C.~~

~~Provided~~

Lemma {Equalizers} = {free products}

Provided the product of any two objects exists.

Pf

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \quad \leftrightarrow \quad \begin{tikzcd} X \times_{X\times Y} X \arrow[r] \arrow[d] & X \arrow[d, "I\times g"] \\ X \arrow[r, "I\times f"'] & Y \end{tikzcd}$$

Likewise

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \qquad X \times_Z Y = Eq\left(X\times Y \overset{f\circ\pi_1}{\underset{g\circ\pi_2}{\rightrightarrows}} Z\right)$$

Thm & defn A category C is "closed under all (finite) limits" (Obvious def) iff all products & all pullbacks of pairs exist, iff all product & all equalizers exist.

aka "complete categories".

$$\lim_{n\in B(\mathbb{N},\cdot)} \mathrm{Hom}_{grps}(\mathbb{Z}/n\mathbb{Z} \to G)$$

$$= \mathrm{Hom}_{grps}\Big(\underbrace{\lim_{n\in B(\mathbb{N},\cdot)} \mathbb{Z}/n\mathbb{Z}}_{\text{The profinite completion of } \mathbb{Z}}, G\Big)$$

__Def'n__ If C is any category, the the ~~pro~~ completion of C is the full subcategory of $\hat{C} = \mathrm{Hom}(C^{op}, sets)$ of the form $\varprojlim_{I} F$ for $F : I \to C$ a functor

This defines the "pro-objects" of C

GPS is complete so a profinite group is (iff) the inverse limit over all ~~G~~ (surjective) $G \longrightarrow H$ homomorphisms into finite groups.

__Thm__ $$\varprojlim_{n \in (\mathbb{N}, |)} \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}} = \prod_{p \text{ prime}} \hat{\mathbb{Z}}_p$$

Dual statements Formulate everything for ~~direct limit~~ co-limits!

co product: $X_\alpha \coprod X_{\alpha'}$ the disjoint limit.

$X \coprod_Z Y$ = disjoint union, modulo the image of Z.

$$\begin{tikzcd} X \arrow[r] & X \coprod_Z Y \\ Z \arrow[u] \arrow[r] & Y \arrow[u] \end{tikzcd}$$

Examples

in $\underline{\text{Gps}}$, $G \amalg H = G * H$

in $\underline{\text{Ab}}$ $G \amalg H = G \times H$ $(= G \prod H)$

in $\underline{\text{rings}}$ $R \amalg S = R * S$ (same as $G*H$ but allowing formal linear combs & mod linearity)

in $\underline{\text{commrings}}$ $R \amalg S = R \otimes_{\mathbb{Z}} S$

Lemma Assume products & co-products of a family $(X_a)_{a \in I}$ exist in $\underline{C}$, and $\underline{C}$ has a zero object. Then there exist a natural "comparison morphism"

$$\sigma : \coprod_a X_a \longrightarrow \prod X_a$$

$$\begin{tikzcd}
X_a \arrow[dr, "i_a"] \arrow[rrr, bend left] & & & X_a \\
& \coprod X_a \arrow[r, "\sigma"] & \prod_a X_a \arrow[ur, "p_a"] \arrow[dr, "p_b"] & \\
X_b \arrow[ur, "i_b"] \arrow[rrr, bend right] & & & X_b
\end{tikzcd}$$

Homework describe $PSL(2,\mathbb{Z}) \xrightarrow{\sigma} \mathbb{Z}_6$

$\mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_3$

Def'n $\underline{A}$ is additive if
(a) it is a pre-additive
(b) there is a 0 object.
(c) Product & co-products of any 2 objects exist.
(d) σ is an isomorphism for any X, Y.

Def'n A is Abelian if it is additive and any morphism has an analysis.

Feb 21, 2005

A: pre-additive category

Lemma for $Z \in \mathrm{Obj}\, A$ TFAE
(1) Z is initial (2) Z is terminal
(3) Z is a zero object.
Moreover, $\mathrm{End}_A(Z) = \{0\}$ (4)

Pf (1) $\Rightarrow$ (4): $\exists! \; Z \to Z$; it must be both 0 and id.
(1) $\Rightarrow$ (3) Z initial $\Rightarrow 0 = id_Z$; take any object X & take $0: X \to Z$. If $f: X \to Z$ is another morphism, $f = id \circ f = 0 \circ f = 0$.
So $0: X \to Z$ is unique.

Direct sums: Given $X, Y \in A$ (pre additive) a direct-sum of X and Y consists of

$$X \underset{i_1}{\overset{p_1}{\rightleftarrows}} Z \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$$

s.t.
a. $p_i i_i = id$
b. $\sum i_i p_i = id.$

} it follows that $p_1 i_2 = 0$, $p_2 i_1 = 0$

Prop: A pre-additive, $X, Y \in \mathrm{Obj}\, A$, TFAE:
(1) X, Y admit a direct sum
(2) X, Y admit a product
(3) X, Y admit a coproduct.

Moreover,
$$X \amalg Y \longrightarrow X \times Y$$
is an isomorphism (& both are direct sums)

Proposition: For morphisms $f, f' : X \to Y$,

$$f + f' = \Delta^Y \circ (f \oplus f') \circ \Delta_X.$$

$$X \xrightarrow{\Delta_X} X \times X \cong X \oplus X \xrightarrow{f \oplus f'} Y \oplus Y \cong Y \amalg Y \xrightarrow{\Delta^Y} Y$$

Cor: If A is a pre additive category in which any pair of objects has a product (or coproducts) for all then the additiv structure is unique.

Assume A, B are pre-additive with direct sums, & $F: A \to B$ is a functor.

<u>TFAE</u> (1) For any $X, Y \in A$,

$$F: Hom_A(X,Y) \to Hom_B(FX, FY)$$

is a group homomorphism

(2) F preserves products

(3) F preserves coproducts / direct sums.

Gabriel's realization of small pre-additive categories

A: small pre-additive category. The "Gabriel Ring" is

$$R(A) := \bigoplus_{X,Y \in A} Hom_A(X,Y)$$ componentwise addition,

$$f * g = \begin{cases} f \circ g & \text{if } dom(f) = codom(g) \\ 0 & \text{otherwise.} \end{cases}$$

<u>Properties</u>

(1) $R(A)$ has a mult. unit iff $\operatorname{obj} A$ is finite

(2) There is a family of orthogonal idempotents,

$\{e_x := id_x\}_{x \in \mathrm{obj} A}$ s.t.

$e_x e_x = e_x$

$e_x e_y = 0 \quad x \neq y$

$\forall a \in R(A)$,

leftsupport$(a) := \{e_x : e_x a \neq 0\}$

rightsupport$(a) := \{e_x : a e_x \neq 0\}$

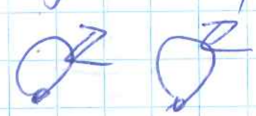
are finite.

Mitchell: "rings with several objects"

$R(A)$: "tic-tac-toe algebra"

Example: $R = \mathbb{Z} \times \mathbb{Z}$ is the Gabriel ring of both

and

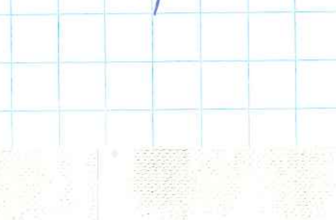


C: a small category

$\mathrm{Hom}(C, k\text{-mod})$ "a quiver rep."

<u>HW</u> Classify all isomorphism classes of quiver reps of the types

with dims as stated



Def'n $\mathcal{C} \underset{L}{\overset{R}{\rightleftarrows}} \mathcal{D}$ (diagram of categories & functors)

R & L are called "adjoint functors" if

$$\mathrm{Hom}_{\mathcal{D}}(Y, RX) \cong \mathrm{Hom}_{\mathcal{C}}(LY, X)$$

$\forall X \in \mathcal{C}$ & $Y \in \mathcal{D}$

via an ~~iso~~ isomorphism of functors on $\mathcal{D}^{op} \times \mathcal{C}$

Examples

R = forgetfull: $\underline{Ab} \longrightarrow$ ~~Abe~~ $\underline{sets}$

L: Free group generated by ?

R = forgetful: $\underline{\text{Rings}} \longrightarrow \underline{\text{sets}} \ni S$

L: Tensor algebra on S

R: forgetfull k-algebras $\xrightarrow{\text{forget}}$ k-Lie algs

L: The Universal Enveloping Algebra.

R: $\underline{Ab} \xrightarrow{\text{inclusion}}$ GPS

L: Abelianization

Assume $f: A \to B$ a ~~ring~~ ring homomorphism

$$f_*: B\text{-mod} \to A\text{-mod}$$

("restriction of scalars along f")

Left adjoint: $f^*(M) = B \otimes_A M$ (easy)
right adjoint $f^!(M') = \mathrm{Hom}_A(B, M')$

$N \in B\text{-mod}$ $M' \in A\text{-mod}$

$$\mathrm{Hom}_A(f_* N, M') \cong \mathrm{Hom}_B(N, f^! M')$$

Exercise $A = \mathbb{Z}$, G = finite group
$B = \mathbb{Z}G$, $f: A \to B$ ~~via~~ $1 \mapsto 1$

Show that each of $f^*, f^!$ have left and right adjoints, and you can keep going.

What property of f makes this possible?

Feb 28, 2005 Today's plan

Equiv. of categories, Fourier-Mukai trans. Non Comm. Alg. Geom ↑ (inspired by thesis of Wendy Lowen)

1. structure of adjoint functors
2. Ubiquity of tensor products / left adjoints
3. "standard resolutions" / cotriples / Godement construction.

Recall $\underline{C} \underset{L}{\overset{R}{\rightleftarrows}} \underline{D}$ is an adjoint pair if

$$\phi_{-,=} : \mathrm{Hom}_C(L(-), =) \cong \mathrm{Hom}_D(-, R(=))$$

as functors $D^{op} \times C \longrightarrow$ Sets.

<u>Properties</u> R commutes with all limits that exist in C; L commutes with all colimits that exist in D.

I.e., assume $F : I \to C$ functor s.t. $\varprojlim_I F$ exists in C. Then

$$\varprojlim_I R \circ F = R \circ \varprojlim_I F$$

(so R preserves products, fibered products, equalizers etc., final objects, etc.)

Assume C is complete (all limits exist) enough: I-complete, all functors $I \to C$ admit a limit then $C_? : C \longrightarrow \mathrm{Hom}(I, C)$ admits $\varprojlim$ as its right adjoint.

<u>Pf</u> $\mathrm{Hom}_{\mathrm{Hom}(I,C)}(Cx, F) \cong \mathrm{Hom}(x, \varprojlim_I F)$

unit and counit of an adjunction

$C \underset{L}{\overset{R}{\rightleftarrows}} D$ adjoint pair. Then

$Y \in D,\ X \in C \Rightarrow$

$$\phi : \mathrm{Hom}_C(LY, X) \cong \mathrm{Hom}_D(Y, RX)$$

$$id_{LY} \in \mathrm{Hom}_C(LY, LY) \underset{\phi}{\cong} \mathrm{Hom}_D(Y, RLY) \ni u$$

$$u : Y \longrightarrow RLY$$

Naturality of $\phi \Rightarrow u : id_D \longrightarrow RL$ is a morphism of functors.

Likewise, take $Y \in RX$ then $\phi^{-1}(id_{RX}) : \eta_X :$

$$: LRX \longrightarrow X$$

& $\eta : LR \longrightarrow id_C$ is a morphism of functors.

Now

$LY \xrightarrow{f} X \longrightarrow RLY \xrightarrow{Rf} RX \longrightarrow Y \xrightarrow{u_Y} RLY \xrightarrow{Rf} RX$

$$\begin{array}{c} Y \xrightarrow{\phi(f)} RX \\ \| \\ Y \xrightarrow{u_Y} RLY \xrightarrow{Rf} RX \end{array}$$

<u>claim</u> $\phi(f) = R(f) \circ u_Y$

likewise if $Y \xrightarrow{g} RX$ then $\phi^{-1}(g) = \eta_X \circ L(g)$

(Aside $C^{op} \underset{R^{op}}{\overset{L^{op}}{\rightleftarrows}} D^{op}$ L^{op} is right adjoint to R^{op})

Moreover:

$$\begin{tikzcd} RX \arrow[r, "u_{RX}"] \arrow[rr, bend right, "id_{RX}"'] & RLRX \arrow[r, "R\eta_X"] & RX \end{tikzcd}$$

split mono

split epi

$$\begin{tikzcd} LY \arrow[r, "Lu_Y"] \arrow[rr, bend right, "id_{LY}"'] & L(RLY) \arrow[r, "\eta_{LY}"] & LY \end{tikzcd}$$

Example: $f: A \to B$ ring homomorphism,

$R = f_*: \text{Mod-}B \to \text{Mod}A$

$L = f^*: \text{Mod}A \to \text{Mod}B$ via $M_A \to M \otimes_A B$

$u: id_{\text{Mod-}A} \to RL$

$u_M: M \to RL(M) = (M \otimes_A B)_A$

$m \longmapsto m \otimes_A 1$

$\eta: LR \to id_{\text{Mod-}B}$

$N \in \text{Mod-}B$ $\eta_N: LR(N) = N \otimes_A B \to N$

via $n \otimes b \mapsto nb$.

Existance and construction of adjoints.

$R: C \to D$ a functor.
given $Y \in D$. consider the category

$Y \downarrow R$: objects = pairs

$(Y \longrightarrow RX, X)$ $X \in \mathrm{Ob}\, C$

Morphisms

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, equal] & RX \arrow[d, "g"] & X \arrow[d, "g"] \\ Y \arrow[r, "f'"] & RX' & X' \end{tikzcd}$$

$\mathrm{Hom}_{Y \downarrow R}$

<u>Thm</u> (Freyd) A functor $R: C \to D$ admits a left adjoint iff

1. R commutes with all limits that exist in C
2. For each $Y \in D$, $Y \downarrow R$ has an initial object.

or for each $Y \in D$,

$\varprojlim_{Y \downarrow R} \left((Y \to RX) \mapsto X \right)$ exists.

$\nwarrow$ not always small.

Tensor Products

A, B rings (or k-algebras over some comm. ring k)

Def ${}_A M_B$ $(A-B)$-bimodule

$\Longleftrightarrow$ A ring homomorphism $A \to End_B(M)$

$\Longleftrightarrow$ ring homomorphisms $B \to (End_{A^{op}}(M))^{op}$

N_B is any B-module, Then

$Hom_B({}_A M_B, N_B)$ is a right A-module

by $(\varphi \cdot a)m := \varphi(am)$

~~We~~ define for any A-module L $Mod\text{-}B \longrightarrow \underline{Sets}$ via

$$N \longmapsto Hom_A(L, Hom_B(M, N))$$

is it representable?

More generally, does

$$Hom_B(M, ?) : Mod\text{-}B \longrightarrow Mod\text{-}A$$

have a left adjoint?

<u>Thm</u> Yes, given by $(? \otimes_A M)_B$

<u>Lemma</u> Given any A-module L, it can be obtained as $L = coker(\underset{free}{A^{(J)}} \longrightarrow \underset{modules.}{A^{(I)}})$

From the lemma, it is enough to define

$$(?\otimes_A M) \text{ on } ? = M. \quad \text{s.t.}$$

$$A \otimes_A M \cong M.$$

In general, if L is an arbitrary A-module, choose a presentation

$$A^{(J)} \xrightarrow{(a_{ij})} A^{(I)} \longrightarrow L \longrightarrow 0$$

convert to

$$M^{(J)} \xrightarrow{(a_{ij}|_M)_{ij}} M^{(I)} \longrightarrow L \otimes_A M \longrightarrow 0$$

If A is an Abelian group,

$$A \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong A/nA$$

(true for $A = \mathbb{Z}$, and both sides commute with co-limits)

<u>Mar 7, 2005</u> Extensions & Abelian categories

Q How many different groups of order 48 have S_4 as a quotient?

$$1 \longrightarrow \mu_2 \longrightarrow \overset{48}{G} \longrightarrow\!\!\!\!\rightarrow \overset{24}{S_4} \longrightarrow 1$$

observations

$1 \to \mu_2 \to G \xrightarrow{p} S_4 \to 1$

1. μ_2 lies in the center of G

in general,

$1 \to H \xrightarrow{i} G \xrightarrow{\pi} K \to 1$

1. G acts on H. $\varphi: G \to \mathrm{Aut}(H)$ via $g \mapsto (h \mapsto ghg^{-1} = {}^g h)$
2. $\exists$ set-theoretic section $\sigma: K \to G$ (σ is "normalized" if $\sigma(1) = 1$)

Such σ yields a bijection of sets

$$H \times K \xrightarrow{F} G$$
$$(h,k) \longmapsto i(h)\cdot\sigma(k)$$

So we just have to find a multiplicative structure $*$ on $H \times K$.

$$F((h,k)*(h',k')) = i(h)\left[\sigma(k)i(h')\sigma(k)^{-1}\right]\underbrace{\sigma(k)\sigma(k')}_{u(k,k')\sigma(kk')}$$

So

$$(h,k)*(h',k') = (\cancel{\ \ }$$
$$= (h \cdot {}^{\sigma(k)}h' \cdot u(k,k'), kk')$$

if H is central, this is

$$= (hh' \cdot u(k,k'), kk')$$

So we need $u: K \times K \to H$

$$\left(u(k,k') = i^{-1}\left(\sigma(k)\sigma(k')\sigma(kk')^{-1}\right)\right)$$

G associative $\Longleftrightarrow$

$$u(k,k')\,u(kk',k'') = \cancel{u(k,k'k'')\,u(k',k'')}$$
$$u(k',k'')\,u(k,k'k'')$$

"The co-cycle condition" (defines $Z^2(K,H)$)

There's also a normalization cond: $u(1,k)=u(k,1)=1$

The dependence on σ: (if τ is another section)

$$u(k,k') = \sigma(k)\sigma(k')\sigma(kk')^{-1}$$
$$v(k,k') = \tau(k)\tau(k')\tau(kk')^{-1}$$

~~So~~

$$\cancel{uv^{-1} = \sigma\tau^{-1}(k)\,\sigma\tau^{-1}(k')\,(\sigma\tau^{-1}(kk')^{-1})}$$

~~Where~~ $\cancel{\sigma\tau^{-1} = \sigma(k)\tau^{-1}(k)}$

~~is a map~~ $\cancel{\sigma\tau^{-1} : K \to H}$

Let $\varphi(k) = \sigma(k)\tau^{-1}(k) : K \longrightarrow H$

So $$u(k,k') = \underbrace{\varphi(k)\varphi(k')\varphi(kk')^{-1}}_{\text{A co-boundary (defines } B^2(K,H))} v(k,k')$$

So extensions are classified by

$$H^2(K,H) := Z^2(K,H)\,/\,B^2(K,H)$$

(For extensions $1 \to H \to G \to K \to 1$ with H central in G where
$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d, "\cong"] & K \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & H \arrow[r] & G' \arrow[r] & K \arrow[r] & 1 \end{tikzcd}$$
are considered as equiv.)

Examples $H^2(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$

$$G = \mathbb{Z}/4 \text{ or } (\mathbb{Z}/2)^2$$

Ex $H^2(\mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2$

$$G = \mathbb{Z} \times (\mathbb{Z}/2) \text{ or } G = \mathbb{Z}$$

Example

$$(0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0) \in H^2(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$$

Example $H^2(\underbrace{\mathbb{Z}/2 \times \mathbb{Z}/2}_{V}, \mathbb{Z}/2)$ classifies

"the klein 4-group".

$$1 \to \mathbb{Z}/2 \to G \to V \to 1$$

$G = (\mathbb{Z}/2)^3$, $\mathbb{Z}/4 \times \mathbb{Z}/2$ in two ways, $\{\pm 1, \pm i, \pm j, \pm k\}$, D_8

+ at least one more.

Defn A category $\mathcal{A}$ is Abelian if it is additive &

1. Every morphism has a kernel & coker
2. Every morphism has an analysis

$$\begin{tikzcd}
\ker f \arrow[d, "i"] & \ker(\mathrm{cok} f) = \mathrm{im} f \arrow[d, "\varphi?"] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "q"] \\
\mathrm{coim} f = \mathrm{cok}(\ker(f)) \arrow[uur, dashed, "\exists!"] & \mathrm{coker} f
\end{tikzcd}$$

φ is an iso.

$2+2+2+3+3$

Def'n A 3-term seq. $0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0$ is exact if $a = \ker b$ & $b = \operatorname{cok} a$

$$
\begin{tikzcd}
0 \arrow[dr] & & & & & 0 \\
& \ker f \arrow[dr] & & & \operatorname{cok} f \arrow[ur] & \\
& & X \arrow[r, "f"] \arrow[d] & Y \arrow[ur] & & \\
& & \operatorname{coim} f \cong \operatorname{im} f \arrow[ur] \arrow[dr] & & & \\
& 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

moral ~~stud~~ studying morphisms in an Abelian category is same as studying short exact sequences; these are pairs (object, sub) or (obj, quo)

Example A = Finitely gen. Abelian groups
B = all ~~a~~ Abelian groups

simples: $\mathbb{Z}/p$

Def (The Ringel-Hall Alg of an ~~d~~ Abelian cat A)

$$\left\{ \sum n_i [X_i] : \begin{array}{l} [X_i] \text{ is an isomorphism} \\ \text{class on an indecomposable} \\ \text{object in } A \end{array} \right\}$$

$$[X] \cdot [Y] = \sum_{[Z]} C^Z_{X,Y} [Z] \qquad C^Z_{X,Y} = \# \text{ of exact seq}$$

$0 \to X \to Z \to Y \to 0$
up to equiv as above

Example Q a finite quiver,
A f.d. rep. of Q in a finite field $\mathbb{F}_q$
obvious simple obj: all v.s. 0 except one which is 1-d.

Ringel Hall algebra makes sense?
Thm (Ringel) This RH algebra is the positive part of a quantum group. (Inventiones 1990)

More on extensions:
$\mathrm{Ext}^1_A(X,Y) = \left(\text{equiv classes of ext. } 0 \to Y \to Z \to X \to 0\right)$

Thm $\mathrm{Ext}^1_A(X,Y)$ is naturally an Abelian group, such that
$\mathrm{Ext}^1_A(?,??) : A^{op} \times A \longrightarrow Ab$
is an additive bi-functor.

Defn $0 \to Y \xrightarrow{a_0} Z_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} Z_n \xrightarrow{a_n} X \to 0$
is an "n-extension" if it is exact.
There is a concatanation map: (The Yoneda product)
$$\mathrm{Ext}^m_A(Z,Y) \times \mathrm{Ext}^n_A(X,Z) \longrightarrow \mathrm{Ext}^{m+n}_A(X,Y)$$

Def'n

$$0 \to X \to U_0 \to \cdots \to U_n \to Y \to 0$$

$$0 \to X \to V_0 \cdots V_n \to Y \to 0$$

are equivalent modulo the equiv. rel. generated by

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, equal] & U_0 \arrow[r] \arrow[d, "\alpha_0"] & \cdots \arrow[r] & U_n \arrow[r] \arrow[d, "\alpha_n"] & Y \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & V_0 \arrow[r] & \cdots & V_n \arrow[r] & Y \arrow[r] & 0
\end{tikzcd}$$

So two n-extensions are equiv. if $\exists$ morphisms

$$E \leftarrow E_0 \rightarrow E_1 \leftarrow E_2 \rightarrow E_1 \cdots \rightarrow F$$

Buchweitz' course

March 14, 2005

study extension

$$1 \to \mathbb{Z}_2 \to G \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

Claim They are

$$[\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2] + 3[\mathbb{Z}_4 \times \mathbb{Z}_2] + 3[D_8] + [Q_8]$$

Q_8 is the quaternion group $\pm 1, \pm i, \pm j, \pm k$.

furthermore, $H^2(\quad) = (\mathbb{Z}_2)^3$

Double covers of S_4:

$$G \xrightarrow{b} S_4 \quad |\ker b| = 2$$

Example

$$1 \to \mathbb{Z}_2 \to \underset{\{\pm 1, \pm i\}}{\mathbb{Z}_4} \xrightarrow{(\ )^2} \mathbb{Z}_2 \to 1$$

$$\uparrow \text{sign}$$

has 7 elements of order 2

$$1 \to \mathbb{Z}_2 \to \underset{\overset{\shortparallel}{G}}{\mathbb{Z}_4 \times_{\mathbb{Z}_2} S_4} \longrightarrow S_4 \to 1$$

"$\sqrt{\text{sign}}$" $= G = \{\sqrt{\text{sign}\,\sigma} \cdot \sigma \mid \sqrt{\text{sign}\,\sigma} \in \mu_4 = \{\pm 1, \pm i\}\}$

Example $\quad 1 \to \{\pm 1\} \to SU(2) = \mathbb{H}_1 \to SO(3) \to 1$

$$\uparrow \qquad\qquad \uparrow$$

has one element of order 2 $\quad 1 \to \{\pm 1\} \to 2O \longrightarrow S_4 \to 1$

the binary octahedral ... octahedral.

Addition of extensions:

$E_1: \quad 1 \to H_1 \to G_1 \to K_1 \to 1$

$E_2: \quad 1 \to H_2 \to G_2 \to K_2 \to 1$

$$\begin{tikzcd} E_1 \times E_2 : & 1 \arrow[r] & H_1 \times H_2 \arrow[r] & G_1 \times G_2 \arrow[r] & K_1 \times K_2 \arrow[r] & 1 \\ & & & \Delta^* \arrow[r] & K \arrow[u, "\Delta"', "\text{diag}"] & \end{tikzcd}$$

if $K_1 = K_2 = K$

can consider $\Delta^*(E_1 \times E_2)$, an extension of K by $H_1 \times H_2$

if $H_1 = H_2$, push forward using the product law ∇_H of $H = H_1 = H_2$.

<u>Definition</u> If $E_i : 1 \to H \to G_i \to K \to 1$

The "Baer sum" of extensions E_1 & E_2 is

$$E_1 \times_{Baer} E_2 = \cancel{\nabla_{H*} \Delta_K^* (E_1 * 1}$$

$$= \nabla_{H*} \Delta_K^* (E_1 \times E_2),$$

an extension of K by H.

<u>Exercise</u> What is ~~QQ~~ $2O \times_{Baer} \mathrm{sign}^*$? Is it new?

Q: ~~QQ~~ $2O \neq GL_2(\mathbb{F}_3)$

Extensions in Abelian categories Fix $\underline{A}$

An extension of X by Y is a short exact sequence

$$E: \ 0 \to Y \xrightarrow{a} Z \xrightarrow{b} X \to 0$$

~~Any (they ar~~ (Two such are equiv iff)

$$\begin{tikzcd} E & 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \\ E' & 0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u] & X \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

$$\mathrm{Ext}^1_{\underline{A}}(X,Y) := \{0 \to Y \to Z \to X \to 0\}/\cong$$

Lemma If $f: X' \to X$ is a morphism, then for any extention E,

$$f^*E: \ 0 \to Y \xrightarrow{a'} Z \times_X X' \xrightarrow{pr_2} X' \to 0$$

is an extension of X' by Y, & this op is well defined mod. equiv. of extensions. $\square$

Likewise, Given $g: Y \to Y'$

$$\begin{tikzcd} g_*E: & 0 \arrow[r] & Y' \arrow[r] & \frac{Z \oplus Y'}{Y} \text{ "}Z'\text{"} \arrow[r] & X \arrow[r] & 0 \\ & 0 \arrow[r] & Y \arrow[r, "a"] \arrow[u, "g"] & Z \arrow[r, "b"] \arrow[u] & X \arrow[r] & 0 \end{tikzcd}$$

$$Z' = \mathrm{coker}\left(Y \xrightarrow{(a,g)} Z \oplus Y'\right)$$

well defined mod equiv, commutes with f^*.

So $Ext^1_{\underline{A}}(?, ??)$ is a bifunctor on $\underline{A}^{op} \times \underline{A}$.

Assume that $\underline{A}$ is a k-category. Then k acts on $Ext^1_{\underline{A}}(X, Y)$ and all possible such actions are the same (pull back of X via mult. by k, pushforward of Y by mult by k)

In particular, there's always $(-E)$.

Given $E: 0 \to Y \xrightarrow{a} Z \xrightarrow{b} X \to 0$

$(-E)$ is $0 \to Y \xrightarrow{-a} Z \xrightarrow{b} X \to 0$

or $0 \to Y \xrightarrow{a} Z \xrightarrow{-b} X \to 0$

~~$E \oplus$~~ ~~$E +_{Baer} E'$~~ $E +_{Baer} E'$ is defined as before to be

$$\nabla_{Y*} \Delta_X^* (E \oplus E')$$

So $Ext^1_{\underline{A}}$ take values in ~~A~~ Mod-k; it is also bi-additive.

Given any exact seq.

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$$

Get an exact

$$0 \to Hom(X,A) \to Hom(X,B) \to Hom(X,C)$$
$$\xrightarrow{\delta} Ext^1_\Lambda(X,A) \xrightarrow{a_*} Ext^1_\Lambda(X,B)$$
$$\xrightarrow{b_*} Ext^1_\Lambda(X,C)$$

δ is $(\ )^*$ i.e.,

$$E: 0 \to A \to B \to C \to 0$$
$$\uparrow f$$
$$f^*E: 0 \to A \to f^*B \to X \to 0$$

$\delta f = \cancel{}\ f^*E$

The 0 of $Ext^1(X,Y)$ is

$$0 \to Y \to X \oplus Y \to X \to 0$$

March 28, 2005

An n-extension of X by Y is an exact sequence

$$0 \to Y \xrightarrow{\alpha_0} Z_1 \xrightarrow{\alpha_1} \cdots \to Z_n \xrightarrow{\alpha_n} X \to 0$$

Addition (external addition)

$$0 \to Y \to \cdots \to X \to 0$$
$$0 \to Y' \to \cdots \to Y' \to 0$$
$$\downarrow$$
$$0 \to Y \oplus Y' \to \cdots \to X \oplus X' \to 0$$

pullbacks:

$$\begin{tikzcd} E: & 0 \arrow[r] & Y \arrow[r] & \cdots \arrow[r] & Z_{n-1} \arrow[r] & Z_n \arrow[r] & X \arrow[r] & 0 \\ f^*E & 0 \arrow[r] & \cdots \arrow[r] & & Z_{n-1} \arrow[r] & Z_n \times_X X' & X' \arrow[u, "f"] \arrow[r] & 0 \end{tikzcd}$$

pushout

$$\begin{tikzcd} g_*E: & 0 \arrow[r] & Y' \arrow[r] & Y' \oplus_Y Z_1 \arrow[r] & \cdots \\ E: & 0 \arrow[r] & Y \arrow[u, "g"] \arrow[r] & Z_1 \arrow[u] \arrow[r] & Z_2 \arrow[r] & \cdots \arrow[r] & Z_n \arrow[r] & 0 \end{tikzcd}$$

$Y' \oplus_Y Z_1 = \operatorname{cok}(Y \xrightarrow{(g,\alpha_0)} Y' \oplus Z_1)$

(pullbacks commute with pushouts (up to iso))

addition: ("the "Baer" sum")

$$E + E' = \Delta^Y_* \Delta_X^* (E \oplus E')$$

where $\Delta_X : X \to X \oplus X$ & $\Delta^Y : Y \oplus Y \to Y$

(one needs to verify commutativity & associativity)

Equivalence:

$$\begin{tikzcd} E & 0 \arrow[r] & Y \arrow[r, "\alpha_0"] & \cdots \arrow[r, "\alpha_n"] & X \arrow[r] & 0 \\ E' \arrow[u] & 0 \arrow[r] & Y' \arrow[u, equal] \arrow[r, "\beta_0"'] & \cdots \arrow[r, "\beta_n"'] & X' \arrow[r] & 0 \end{tikzcd}$$

Example

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_4 \arrow[r, "\cdot 2"] & \mathbb{Z}_4 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z}_2 \arrow[u, equal] \arrow[r] & \mathbb{Z}_8 \arrow[u] \arrow[r, "\cdot 2"'] & \mathbb{Z}_8 \arrow[u] \arrow[r] & \mathbb{Z}_2 \arrow[u, equal] \arrow[r] & 0 \end{tikzcd}$$

(not symmetric, one has to complete...)

Theorem (Yoneda)

Set $\mathrm{Ext}^n_{\underline{A}}(X,Y) = n\text{-extensions} / \text{equivalence}$

is an Abelian group using Baer sums,

is functorial in each argument,

(0 is

$$0 \to Y \xrightarrow{\alpha_0} Z_1 \to \cdots Z_n \xrightarrow{\alpha_n} X \to 0$$

with α_0 a split mono or α_n a split epi)

is half-exact in each argument

$\left(\begin{array}{ll} 0 \to A \to B \to C \to 0 & \text{exact goes to} \\ F(A) \to F(B) \to F(C) & \text{exact} \end{array} \right)$

& one has a long exact sequence:

given $0 \to A \to B \to C \to 0$ becomes

$$0 \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C) \to$$
$$\to \mathrm{Ext}^1(X,A) \to \cdots$$
$$\to \mathrm{Ext}^2(X,A)$$
$$\vdots$$

Likewise,

$$0 \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(B,Y) \to \mathrm{Hom}(A,Y) \to$$
$$\to \mathrm{Ext}^1(C,Y) \to \cdots$$
$$\to \mathrm{Ext}^2(C,Y) \to \cdots$$

Def'n the global dim of $\underline{A}$:

$$\mathrm{gldim}\,\underline{A} := \min\{j : \mathrm{Ext}_{\underline{A}}^{j+1}(X,Y) = 0 \ \forall X,Y\}.$$

Example $\mathrm{gldim}\ \underline{\mathrm{Vect}} = 0.$

(If $\mathrm{gldim}\,\underline{A} = 0$ $\underline{A}$ is called "semi-simple")

Example Let k be a field, $n \geq 1$

$\underline{A}$ = Modules over $M_{n\times n}(k)$

then $\mathrm{gldim}\,\underline{A} = 0$ (hard, conseq. of Morita equivalence)

a says: Mod $M_{n\times n}(k)$ ~~is~~ is equiv Mod ~~k~~ k

Weyl: $A_n := \dfrac{\mathbb{C}\langle p_1 \ldots p_n ; q_1, \ldots, q_n \rangle}{[p_i p_j] = 0,\ [q_i, q_j] = 0,\ [p_i, q_j] = \delta_{ij}}$

$\mathrm{gldim}\ A_n = n$

$H_n :=$ same but $[p_i, q_j] = \delta_{ij} \cdot z$

with central z.

then $\mathrm{gldim}\, H_n = 2n+1$?

Thm $\mathrm{gldim}\ k[x_1, \ldots x_n] = n$

"The Hilbert syzygy theorem"

(also for ~~formal~~ formal power series,
~~and~~ and for convergent power series)

Ex 3 $q : V \to k$ quadratic form

$\text{gldim Cliffor Alg}(q) = 0$.

Ex 4 G a finite group, k a field of char 0, kG the groupalg. of G

$\text{gldim Mod-}kG = 0$
(Maschke's Theorem)

Categories of gldim $= 1$ ("hereditary categories")

Theorem if R is a PID then
$\text{gldim Mod-}R = 1$
unless R is a field!
(this is the structure theorem for modules over a PID)

Ex $Q = (Q_0, Q_1)$ a quiver
Q_0: verts, Q_1: arrow

$\text{gldim } \overset{\text{Rep}}{\cancel{\text{Ref}}}(Q, k) \leq 1$

(so for tensor algebras, gldim ≤ 1 as well)

Claim if $\underline{B} \subset \underline{A}$ is a subcategory
then $\mathrm{gldim}\, \underline{B} \leq \mathrm{gldim}\, \underline{A}$

satisfying: full, Abelian,
extension closed:
$X, Y \in B \Rightarrow$ in any
$0 \to Y \to Z \to X \to 0$
$Z \in B$

So in the above examples, F.D. of F.G. also have gldim ≤ 1.

$\underline{V}, \underline{W}$ to reps of a quiver Q
an extention

$$0 \longrightarrow \underline{W} \xrightarrow{a} \underline{X} \xrightarrow{b} \underline{V} \longrightarrow 0$$

:

Prop There is an exact seq of vect. spaces:

$$0 \to \mathrm{Hom}_{Rep(Q,k)}(\underline{V}, \underline{W}) \to \bigoplus \mathrm{Hom}_k(V_i, W_i) \xrightarrow{d_W^V} \bigoplus_{i \xrightarrow{\alpha} j} \mathrm{Hom}_k(V_i, W_j) \longrightarrow \mathrm{Ext}^1_{Rep(Q,k)}(\underline{V}, \underline{W}) \to 0$$

So $\dim \mathrm{Ext}^1_{Rep(Q)}(\underline{V}, \underline{W}) =$

$$= \sum_{\substack{i \xrightarrow{\alpha} j \\ \in Q}} \dim V_i \cdot \dim W_j - \sum_i \dim V_i \dim W_i + \dim \mathrm{Hom}_{Rep(Q)}(\underline{V}, \underline{W})$$

So

$$\dim \mathrm{Hom}_{Rep}(\underline{V}, \underline{W}) - \dim \mathrm{Ext}^1(\underline{V}, \underline{W}) =$$

$$= \sum_i \dim V_i \dim W_i - \sum_{i \xrightarrow{a} j} \dim V_i \dim W_j$$

PID, structure theorem

Any submodule of a free R-module is free

So if M is a f.g. module, we have

$$0 \to R^n \xrightarrow{A} R^m \to M \to 0$$

<u>Lemma</u> if R is any ring, then

$$\mathrm{Ext}^n_{Mod_R}(R, ?) = 0.$$

(trivial)

So if M is as above,

$$\mathrm{LES} \quad 0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(R^m, N) \to \mathrm{Hom}(R^n, N)$$
$$\to \mathrm{Ext}^1(M,N) \to \cancel{\mathrm{Ext}^1(R^m, N)}.$$

So $\mathrm{Ext}^1(M,N) = N^m / A^T N^n$

& $\mathrm{Ext}^{>1}(M,N) = 0$.

The Hilbert Syzygy Theorem:

If P is a direct summand of R^{I} then

$$\mathrm{Ext}_R^{>0}(P, ?) = 0.$$

Such a P is called projective.

Now given a general module over a general ring. Form

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & R^{(I_2)} \arrow[rr] \arrow[dr] & & R^{(I_1)} \arrow[rr] \arrow[dr] & & R^{(I_0)} \arrow[r] & M \arrow[r] & 0 \\
& 0 \arrow[r] & K_2 \arrow[ur] \arrow[r] & 0 \quad 0 \arrow[r] & K_1 \arrow[ur] \arrow[r] & 0 & &
\end{tikzcd}$$

The syzygy modules. ⟶

The result is an exact sequence

$$R^{I_2} \to R^{I_1} \to R^{I_0} \to M \to 0.$$

If one of the syzygy modules say K_n, is projective, it follows that

$$\mathrm{Ext}^{>n}(M, N) = 0.$$

Hilbert's syzygy thm says that over a polynomial ring, ~~s~~ n K_n will be projective.

Def gldim $R \leq n$ iff each chain of syzygies terminates with a projective after at most n steps

~~dd~~ April 4, 2005

<u>Projectives and Injectives</u> $\underline{A}$: Abelian category

informal def — Projective object $P \in \underline{A} \iff Ext^i_{\underline{A}}(P, ?) = 0 \ \forall i > 0$.

Formal — <u>Def</u> an object $P \in \underline{A}$ is projective if for any epimorphism $X \xrightarrow{p} Y$ and any ~~m~~ morphism $P \xrightarrow{f} Y$ there exists a morphism $g : P \to X$ s.t. $f = pg$

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed, "\exists g"'] \\ X \arrow[r, two heads, "p"'] & Y \end{tikzcd}$$

<u>TFAE</u> P an object of $\underline{A}$

1. P is projective
2. $Hom_{\underline{A}}(P, ?)$ is right exact.
3. Any epimorphism onto P splits
4. $Ext^1_{\underline{A}}(P, ?) = 0$
5. $Ext^{\geq 1}_{\underline{A}}(P, ?) = 0$

<u>Pf</u> $5 \to 4$ ✓ $4 \to 3$ ✓ $3 \to 5$ ✓

$1 \to 2$ ✓ $2 \to 3$ ✓

<u>$3 \to 1$:</u> given

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & Y \arrow[r, "p"] & Z \arrow[r] & 0 \\ 0 \arrow[r] & X \arrow[r] \arrow[u] & X \times_Z P \arrow[r, "p'"] \arrow[u] & P \arrow[r] \arrow[u, "f"] \arrow[l, bend left, "\text{splits}"] & 0 \end{tikzcd}$$

make

Warning: $\mathrm{Hom}_A(P, ?)$ does not necessarily commute with infinite direct sums. If it does, P is called "compact".

Lemma If A is a ring, then $P \in \mathrm{Mod}\,A$ is projective iff P is a direct summand of a free module.

Example $R = \mathbb{C}[X, Y, Z] / x^2+y^2+z^2=1$

$$\mathrm{Der}_{\mathbb{C}}(R) = \left\{ D : R \to R : \begin{array}{l} D \text{ is } \mathbb{C}\text{-linear} \\ D(ab) = D(a)b + aD(b) \end{array} \right\}$$

is projective but not free.

Example R Dedekind ...

Thm (Serre) If X is a diff'able manifold, $R = C^\infty(X)$,

$$\begin{pmatrix} \text{iso. classes of} \\ \text{vector bundles over} \\ X \text{ of finite rank} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} \text{iso. classes of} \\ \text{f.g. projective} \\ C^\infty(X)\text{-modules} \end{pmatrix}$$

Def $\underline{A}$ has "enough projectives" iff for every $X \in \underline{A}$ $\exists P \in \underline{A}$ projective along with a surjection $P \to X$.

Thm Mod-A has enough projectives

However, "Most" Abelian categories do not have enough projectives.

If there are enough projectives, it's "easy" to calculate ext, as below

Every object admits a projective resolution

$$\begin{tikzcd}
\arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] \arrow[dr] & P_0 \arrow[r, "\partial_0"] & X \arrow[r] & 0 \\
& & K_0 \arrow[u] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

So Take $\mathrm{Hom}_{\underline{A}}(-, Y)$ & apply to the resolution:

$$\mathrm{Hom}_{\underline{A}}(P_0, Y) \xrightarrow{\partial_1^* = \mathrm{Hom}(\partial_1, Y)} \mathrm{Hom}_{\underline{A}}(P_1, Y) \to \cdots$$

Thm

$$\mathrm{Ext}^n_{\underline{A}} \cong \frac{\ker \partial^*_{n+1}}{\mathrm{Im}\, \partial^*_n}$$

Pf $0 \to K \to P \to C \to 0$ with P projective

Apply $\mathrm{Hom}(-,Y)$:

$$\mathrm{Hom}(C,Y) \to \mathrm{Hom}(P,Y) \to \mathrm{Hom}(K,Y) \to \mathrm{Ext}^1(C,Y) \to \mathrm{Ext}^1(P,Y) = 0$$

So $\mathrm{Ext}^1(C,Y) = \mathrm{Hom}(K,Y)/\mathrm{im}\,\mathrm{Hom}(P,Y)$

So ...

Injectives $I \in \underline{A}$ is "injective" if $I^{op} \in \underline{A}^{op}$ is projective. I.e.,

TFAE:

1.
$$\begin{tikzcd} & I & \\ 0 \arrow[r] & X \arrow[u, "\forall f"] \arrow[r] & Y \arrow[ul, dashed, "\exists"'] \end{tikzcd}$$

2. $\mathrm{Hom}(-, I)$ is right exact.
3. A monomorphism into I splits
4. $\mathrm{Ext}_{\underline{A}}(?, I) = 0$
5. $\mathrm{Ext}^i_{\underline{A}}(?, I) = 0 \quad \forall i \geq 1$

Example $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

Note Any injective module is divisible.

X.R.V

Def an A-module M is divisible if for every left non-zero-divisor $a \in A$ and any $m \in M$ there is a solution x to

$$m = x \cdot a.$$

Pf of note

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "a\cdot"] \arrow[d, "m"] & A \arrow[dl, "\exists x"] \\ & M & \end{tikzcd}$$

is an injection

Thm An Abelian group is injective iff it is divisible.

Pf Use Zorn's lemma to extend one element at a time.

Prop An A module I is injective iff for any ideal J in A, any morphism $J \to I$ extends to A.

Pf $\Rightarrow$ obvious.

$\Leftarrow$ Zorn's lemma in the same way.

Claim A quotient ~~is an~~ of ~~an injective~~ a divisible group is divisible.

Corollary: $\mathbb{Q}/\mathbb{Z}$ is injective

<u>Cor</u> <u>Ab</u> has "enough" injectives – every group injects in an injective one.

<u>Pf</u> Let $G \in \underline{Ab}$, write

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[dr] & \mathbb{Z}^{(I)} \arrow[r] \arrow[d, hook] \arrow[drr, dashed] & G \arrow[r] \arrow[dd] & 0 \\ & & \mathbb{Q}^{(I)} \arrow[drr] & & \\ & & & & \mathbb{Q}^{(I)}/U \arrow[r] & 0 \end{tikzcd}$$

exact

$\mathbb{Q}^{(I)}$ divisible

$(\mathbb{Q}/\mathbb{Z})^{(I)}$

$\mathbb{Q}^{(I)}/U$ hence divisible

So G is embedded in a divisible group.

<u>Aside</u> $0 \to G \to \dfrac{\mathbb{Q}^{(I)}}{U} \to (\mathbb{Q}/\mathbb{Z})^{I} \to 0$

is exact, thus it is an injective resolution.

<u>Cor</u> $\operatorname{gldim} \underline{Ab} = 1$

<u>Exercise</u> TFAE for an Abelian category $\underline{A}$:

1. $\operatorname{gldim} A = 0$
2. Every object is proj.
3. Every object is inj.

Thm Every category of the form Mod$A$ has enough injectives.

Prop Assume $\underline{A} \underset{L}{\overset{R}{\rightleftarrows}} \underline{B}$ is an adjoint pair of functors between Abelian categories (a) if R is exact, L preserves projectives

(b) if L is exact, R preserves injectives.

Pf By duality, it is enough to prove (a). Assume $P \in \underline{B}$ is projective, LP is projective iff $\mathrm{Hom}_{\underline{A}}(LP, -)$ is exact by adjunction, iff

$$\mathrm{Hom}_B(P, R(?))$$ is exact

but R is exact and so is $\mathrm{Hom}_B(P, -)$.

Pf of Thm L = forgetful from A modules to $\underline{Ab}$ (= Mod $\mathbb{Z}$),

$L = \chi_*$, $\chi : \mathbb{Z} \to A$.

L is exact; $\chi^! = \mathrm{Hom}_{\mathbb{Z}}(A, ?)$ is its right adjoint. So by prop

$\chi^!$ (injective) is injective ...

Example. A commutative ring R is called Gorenstein if $\operatorname{inj\,dim}_R R < \infty$, i.e., $\exists$ exact

$$0 \to R \to I_0 \to I_1 \to \cdots \to I_d \to 0$$

with each I_i injective.

Remark $\operatorname{gldim} A \le n \iff$ inj & proj dim of any object $\le n$.

Example

	$k[x,y]/x^2,xy,y^2$	$k[x,y]/x^2,y^2$
gldim	∞	∞
Gorenstein	No	Yes

Example. For any field k & finite group G, kG is Gorenstein.

Morita Equivalence:

General question What are equivalences of Abelian categories like?

If both are module categories, the answer is Morita's.

Given A, B rings. When are Mod$A$ & Mod$B$ equivalent?

Answer Mod$A$ is equiv. to Mod B iff

$\exists$ an A-module P s.t.

1. P is projective
2. P is F.G.
3. P is a generator (A is a direct summand of $P^{(I)}$ for some I)
4. $B \cong \mathrm{End}_A(P)$

The equivalence is then given by

$$\mathrm{Hom}_A(P, -) : \mathrm{Mod}\, A \longrightarrow \mathrm{Mod}\, B$$

Steps ① Assume $F : \mathrm{Mod}\, A \xrightarrow{\sim} \mathrm{Mod}\, B$

Then F commutes with all limits & colimits and it has a bi-adjoint that is, $\exists G:$ Mod-$B \to$ Mod$A$ which is both a right & a left adjoints.

(2) So $F(-) \cong (-) \otimes_A F(A)$ (by Eilenberg-Watts.)

(2') $G \cong (-) \otimes_B G(B)$

So $\mathrm{Hom}_A(G(B), -)$ is the right adjoint to $(\ ) \otimes_B G(B)$, so

$$F = \mathrm{Hom}_A(G(B), -).$$

<u>Claim</u> $G(B) = P$

(3) F equiv $\Rightarrow F$ exact, so $G(B)$ is projective as A-module

(4) F commutes with all direct sums $\xrightarrow[\text{Needs proof.}]{\&} G(B)$ is F.G.

April 11, 2005.

If $(\underline{C}, *)$ is a category with a zero object $*$, then a sequence of morphisms

$$\to X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} X_{-1}$$

is a complex if $f^2 = *$.

The complex is bndd ~~below~~ above if

$$\cdots \to * \to * \to A_i \to A_{i-1} \cdots$$

bndd ~~above~~ below and bndd are likewise.

Complexes form a category.

$\mathcal{C}^-$ bndd in future

$\mathcal{C}^+$ bndd in past.

$\mathcal{C}^b$: bndd complexes

In an Abelian category, homology makes sense.

Shifts: $X[1]^n := X^{n+1}$

Homology is a functor.

cycles & boundaries

Homotopies

The Hom complex

$$h \in \mathrm{Hom}^i(Y^\circ, X^\circ) = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_A(Y^j, X^{j+i})$$

with $(Dh)_j = d_X h^j - (-1)^i h^{j+1} d_Y$

So $\ker D^0 =$ morphisms of complexes
$\operatorname{im} D^{-1} =$ homotopies acting on Hom^0.

So $H^0(\mathrm{Hom}(Y,X)) = \dfrac{\text{morphisms of complexes}}{\text{homotopy}}$

The homotopy category

$K^*(\underline{A})$: objects are $*$-complexes
Morphisms: homotopy classes of morphisms of cmplxs.

<u>Note</u> $K(A)$ is additive but not Abelian

<u>Def</u> $F: Y \to X$ of complexes is a quasi-isomorphism if it induces an isomorphism of homologies.

<u>Example</u>

$$\begin{tikzcd}
 & \cdots 0 \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, "2"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{Z}_2 \arrow[r] \arrow[u, "1/2"'] & 0 \arrow[u] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 0 &
\end{tikzcd}$$

quasi isom

quasi isom

No backward maps!

$0 \to \mathbb{Z} \xrightarrow{4} \mathbb{Z} \to$

$\mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{2}$

<u>Example</u>

$$0 \to \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_4 \to 0$$

$H^0 = 0$ $H^{-1} = \mathbb{Z}/2$ but not quasi-iso (not even by a chain of quasi-iso) to

$$0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

Given $X \in K(A)$ define

Quasim/X : objects: $Y \xrightarrow{f} X$ quasi isom

morphisms
$$\begin{tikzcd} Y \arrow[dr, "f"] \arrow[d] & \\ Y' \arrow[r, "f'"] & X \end{tikzcd}$$

<u>Prop</u> If X admits a projective resolution then (1) all projective resolutions of X are isomorphic in $K(A)$

(2) Each is an initial object in Quasim/X I.e.,

$$\begin{tikzcd} & X & & Z_2 & \cdots & & P \arrow[llllll, bend right, "\exists!"'] \arrow[dllllll, bend left, "\exists!"] \\ Y \arrow[ur, "f"] & & Z_1 \arrow[ul, "\sim"'] \arrow[ur, "\sim"] & & \arrow[ul] & Z_n \arrow[ur] & \end{tikzcd}$$

__Key point:__ If $h: P \to C$ is a morphism of complexes, P is projective, then h is homotopic to 0:

$$\begin{tikzcd}
C^{-2} \arrow[r] & C^{-1} \arrow[r] & C^{0} \arrow[r] & C^{1} \arrow[r] & \cdots \\
P^{-2} \arrow[r] \arrow[u, "h"] & P^{-1} \arrow[r] \arrow[u, "h"] \arrow[ul, "\exists H"'] & P^{0} \arrow[r] \arrow[u, "h"] \arrow[ul, "\exists H"'] & 0 \arrow[u, "h"] \arrow[ul, "\exists H"'] &
\end{tikzcd}$$

__Pf__ easy.

"Horseshoe Construction" If

$$0 \to X \to Y \to Z \to 0$$

is exact in $\underline{A}$ and 2 of the 3 have projective resolutions, then so does the third.

Hardest case:

$$\begin{tikzcd}
 & & & 0 & \\
 & \text{given} & & \text{given} & \\
0 \arrow[r] & X \arrow[r, "a"] & Y \arrow[r, "b"] & Z \arrow[r] \arrow[uu] & 0 \\
0 \arrow[r] & P_0 \arrow[r, shift left] \arrow[u, "\rho"] & P_0 \oplus Q_0 \arrow[l, shift left, "\pi_1"] \arrow[r, shift left, "\pi_2"] \arrow[u, shift left, "a\rho\pi_1"'] \arrow[u, shift right, "\beta\pi_2"'] & Q_0 \arrow[l, shift left] \arrow[r] \arrow[u] \arrow[ul, "\exists\beta"'] & 0 \\
 & P_1 \arrow[u] & & Q_1 \arrow[from=u] &
\end{tikzcd}$$

need to show 1. surjectivity here

2. exactness at the level of kernels here.

It follows that a short exact sequence of objects produces a "triangle" of complexes (of ~~nego~~ the projective resolutions)

$$0 \to X \xrightarrow{a} Y \xrightarrow{\nu} Z \to 0$$

$$\underset{\text{proj res.}}{P} \xrightarrow{\tilde{a}} \underset{\text{proj res.}}{R} \xrightarrow{\tilde{\nu}} \underset{\text{proj res}}{Q} \xrightarrow{\beta} P[1] \ldots$$

Mapping cones and cylinders

Given $f : C \to D$ of cmplxs,

$\mathrm{Cone}(f) = C(f)$ is given by ....

There is a canonical exact sequence of complexes:

$$0 \to D \longrightarrow C(f) \longrightarrow C[1] \to 0$$

There's also

$$\begin{tikzcd}
 & 0 \arrow[r] & D \arrow[r] & C(f) \arrow[r] & C[1] \arrow[r] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[ur, "f"] & \mathrm{cyl}(f) \arrow[r] \arrow[u] & C(f) \arrow[r] \arrow[u, equal] & 0 &
\end{tikzcd}$$

$$\mathrm{cyl}(f) = C \oplus D \oplus C[1]$$

May 30, 2005 ; Ragnar Buchweitz

Introduction to Maximal Cohen-Macaulay Modules.

1. Definition and examples
2. Ubiquity of MCMs
3. Triangulated structure on $\underline{MCM}$
4 Case of homogeneous hypersurfaces:
   * relation to the Hodge conjecture
   * Semi regularity map & Atiyah-Chern characters.

(1) (R, m, k) R a local commutative Noetherian ring

m: its unique maximal ideal

$k = R/m$: its residue class field

or $R = \bigoplus_{i \geq 0} R_i$ positively graded commutative ring

- $R_0 = k$ a field.
- R finitely generated as an algebra over R_0 (generated in degree 0)

So $R \cong k[x_1 \ldots x_n]/(f_1 \ldots f_m)$

f_j homogeneous,

$x_i \in R_0 = k$

$m \Leftrightarrow R^+ = \bigoplus_{i \geq 1} R_i$

always finitely generated

M an R-module $d :=$ the Krull dim of R.

M is MCM $\Leftrightarrow$ $\exists x_1 \ldots x_d \in M$ s.t.

(i) $x_1 : M \to M$ is injective

(ii) x_{i+1} as multiplication on

$$M / (x_1, \ldots, x_i) \xrightarrow{x_{i+1}}$$

is injective

R is Cohen-Macaulay ring $\Leftrightarrow$ R is MCM as a module over itself.

Examples

(a) $R = K[x_1 \ldots x_n]$

(b) if $f_1 \ldots f_e$ in R form a regular sequence (i.e. $f_i \in M$ f_{i+1} is a non-zero-divisor on $R/(f_1 \ldots f_i)R$)

& if R is CM, then

$R/(f_1 \ldots f_e)R$ is CM

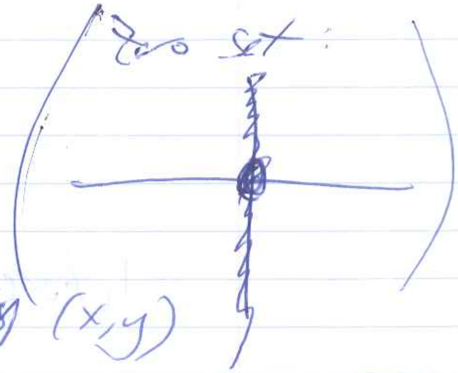
& $\dim \overline{R} = d - e$.

(c) $R = K[x_1, \ldots x_4] \Big/$ 2×2 minors of $\begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$ (zero set: $\{(s^3, ts^2, t^2s,$

Non CM:

"The Emmy Noether Ring"

$$k[x,y]/(x^2, xy) = x(x,y)$$



dim $R = 1$, but no NZD in (x,y)

Deep Thm of Eagon–Hochster

If G is any reductive alg. group acting linearity on a CM ring R, and QCR then R^G is CM.

<u>Examples</u> $R = \mathbb{C}[u,v]$

$G \subseteq Gl(2,\mathbb{C})$ a finite subgroup.

then R^G is CM of dim 2.

eg.,

$$\frac{\mathbb{C}[x_1, \ldots x_4]}{2\times 2 \text{ minors in } \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}} \xrightarrow[\substack{x_1 \to u^3 \\ x_2 \to u^2 v \\ \vdots}]{\varphi} \mathbb{C}[u,v]$$

is a ring homomorphism, its image is all hommogeneous polys of degrees divisible by 3.

The image is

$$\mathbb{C}[u,v]^{\mu_3}$$

where $\mu_3 = \langle w : w^3 = 1 \rangle$ acts by

$$w \longmapsto \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}$$

⋮

R is Gorenstein if one of the following equiv. cond. holds: (R a quotient of $k[x_1, \ldots x_n]$ or $k[[x_1, \ldots x_n]]$.

1. The injective dim of R over itself is finite
2. R is CM and $Ext^i_R(k,R) = \begin{cases} 0 & i \neq \dim \\ k & i = \dim R \end{cases}$
3. The dualizing module ω_R is isomorphic to R.

(if R is not too singular ($\mathrm{codim}_R \, \mathrm{sing}\, R \geq 2$) then $\omega_R = i_*(\Omega^{\dim R}_R |_{R \setminus \mathrm{sing}\, R}) \cong R$

$i : \mathrm{Spec}\, R \setminus \mathrm{sing}\, R \hookrightarrow \mathrm{Spec}\, R$

(4) R is CM and for any regular sequence $x_1 \ldots x_d \in R$ $(d = \dim R)$,

$\overline{R} = R/(x_1 \ldots x_d)R$ is a frob. algebra, i.e., as a module,

$\overline{R} \cong \operatorname{Hom}_k(\overline{R}, k)$

$\Longleftrightarrow$ $\exists$ non-deg. trace pairing

$t : \overline{R} \otimes \overline{R} \to k$ s.th.

$\overline{R}$ factor

$\Longleftrightarrow$ Gorenstein — $\operatorname{Socle} \overline{R}$ is 1-dim.

$\operatorname{Socle} \overline{R} = \{x \in \overline{R} \mid mx = 0\}$

<u>Example</u>:

<u>Ubiquity of MCMs</u>: Let R be a CM ring, N be any f.g. R-module. $\exists$ a free resolution

$$\cdots \to R^{n_2} \to R^{n_1} \xrightarrow{\partial} R^{n_0} \to N \to 0$$

<u>prop</u> R ~~$\mathcal{C}$~~ is CM, N as before,

then $\mathrm{Coker}\left(R^{n_{d+i}} \longrightarrow R^{n_{d+i-1}}\right)$

is MCM for each $i \geq 0$.