Math 1350F - Knot Theory

Fall Semester 2003

Warning - Preliminary Information Only!

Agenda: Use knot theory as an excuse to learning deep and beautiful mathematics.

Instructor: Dror Bar-Natan, drorbn@math.toronto.edu, Sidney Smith 5016G, 416-946-5438. Office hours: Thursdays 12:30-1:30.

Classes: Tuesdays 1-3 at Sidney Smith 5017A and Thursdays 2-3 at Sidney Smith 2128.

Announcements

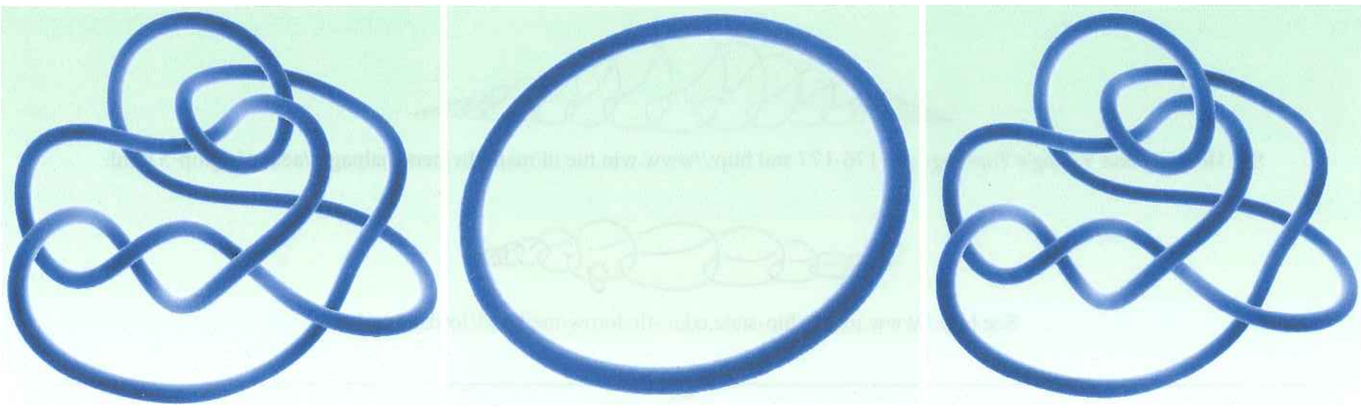
September 8: Welcome back to UofT!

Course Calendar

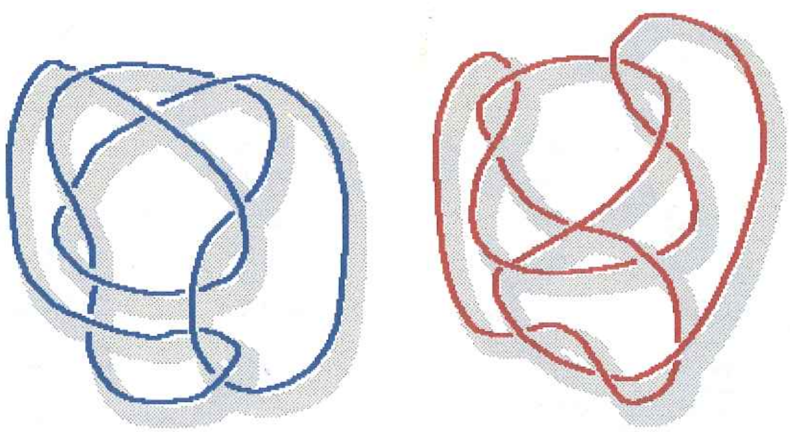
#	Week of ...	
1	September 8	Handout: About This Class Homework Assignment 1
2	September 15	
3	September 22	Our class photo will be taken this week! Our grading policy will be announced this week.
3	September 29	
5	October 6	
6	October 13	Monday is Thanksgiving.
7	October 20	
8	October 27	
8	November 3	
10	November 10	
11	November 17	Dror will be away on Tuesday.
12	November 24	
[illegible]	December 1	

Some Non Obvious Examples

Which two are the same? (rendered using Rob Scharein's KnotPlot)



The Perko Pair: (are these the same?) (taken from http://www.math.cuhk.edu.hk/publect/lecture4/perko.html)



Is this the unknot? (From a book by A.B. Sossinsky. Thanks, Ian Agol)

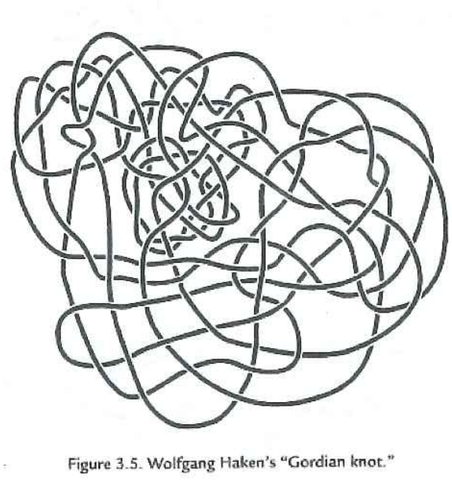
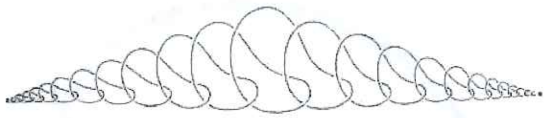


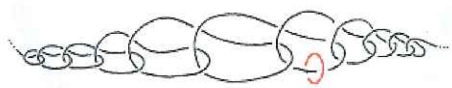
Figure 3.5. Wolfgang Haken's "Gordian knot."

Pathologies

An embedding of an interval in R^3 whose complement is not simply connected:

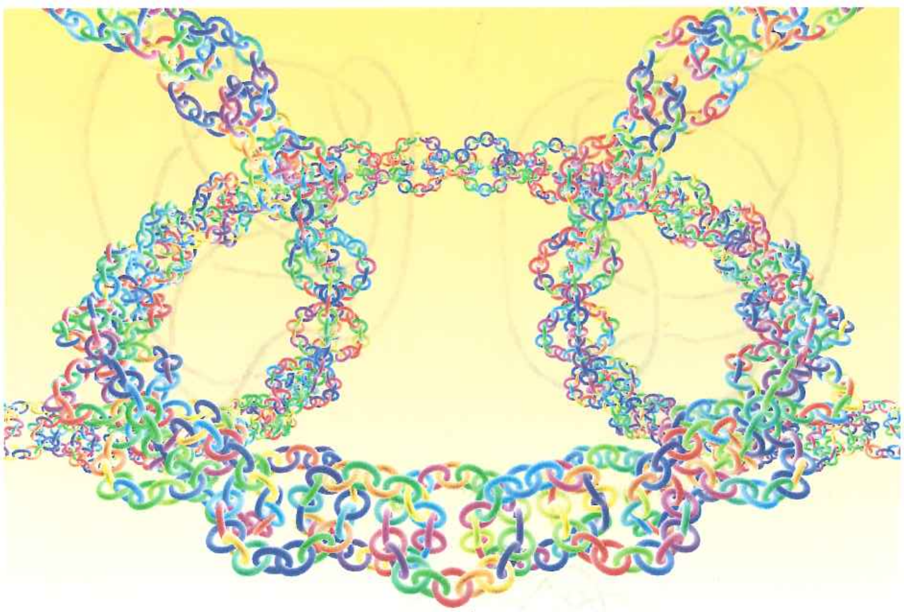


See Hocking and Young's *Topology* pp. 176-177 and http://www.win.tue.nl/math/dw/personalpages/aeb/at/algtop-5.html.



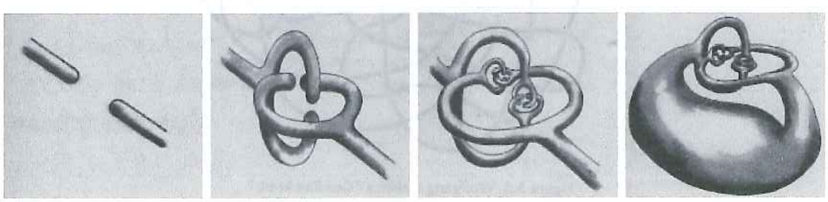
See http://www.math.ohio-state.edu/~fiedorow/math655/Jordan.html.

Antoine's necklace - an embedding of a Cantor set in R^3 whose complement is not simply connected:



See http://www.cs.ubc.ca/nest/imager/contributions/scharein/various/AntoinesNecklace.html.

The Alexander horned sphere - a continuous embedding of a ball in R^3 whose complement is not simply connected:



See http://users.math.uni-potsdam.de/~oeitner/EIGENES/RAEUME/hornsph.htm.

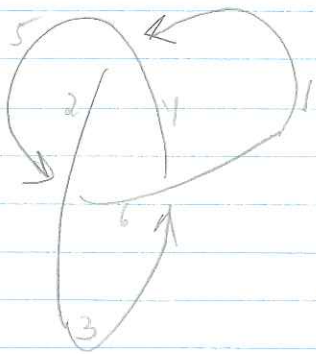
Math 1350F - Knot theory, September 9, 2003

* Some non-obvious examples
* About this class
* Pathologies
* 3-colorings
* Reidemeister's theorem
* The Kauffman bracket & the Jones polynomial

$$\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$$

Math 1350F Knot Theory, September 11, 2003

Restate Reid. theorem



$X_{1524} X_{3146} X_{5362}$

Exercise: Verify that K can be reconstructed from this information

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$\langle \times \rangle = A \langle \asymp \rangle + B \langle)(\rangle$ on the computer.

Fix A, B, d.

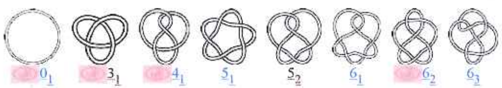
Renormalize.

Dror Bar-Natan: The Knot Atlas:

The Rolfsen Knot Table

Click on a knot to learn more about it!

Knots with 0 through 6 Crossings

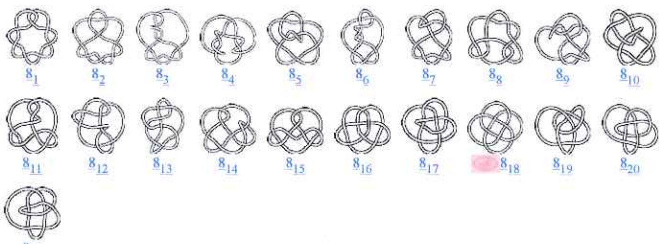


0_1 3_1 4_1 5_1 5_2 6_1 6_2 6_3

Knots with 7 Crossings

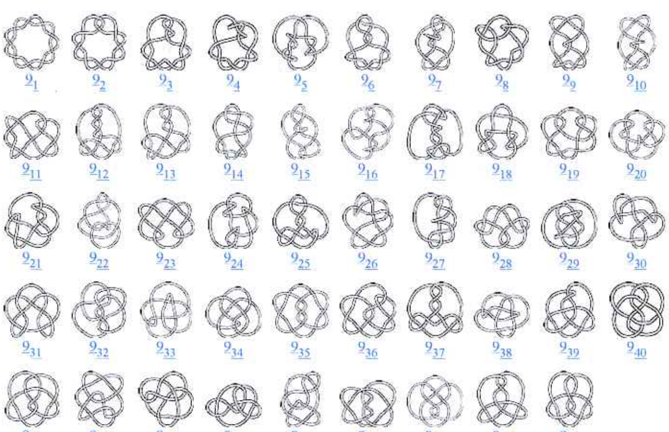
7_1 7_2 7_3 7_4 7_5 7_6 7_7

Knots with 8 Crossings



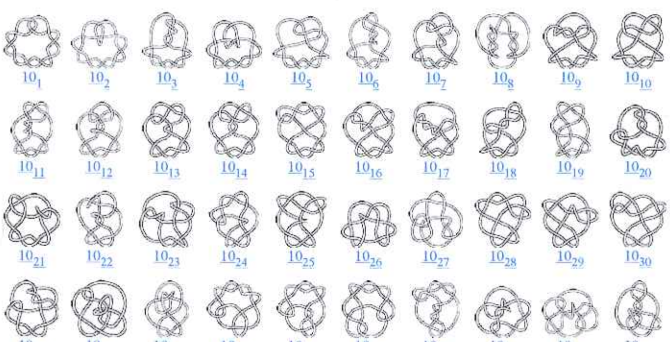
8_1 8_2 8_3 8_4 8_5 8_6 8_7 8_8 8_9 8_{10}
8_{11} 8_{12} 8_{13} 8_{14} 8_{15} 8_{16} 8_{17} 8_{18} 8_{19} 8_{20}
8_{21}

Knots with 9 Crossings

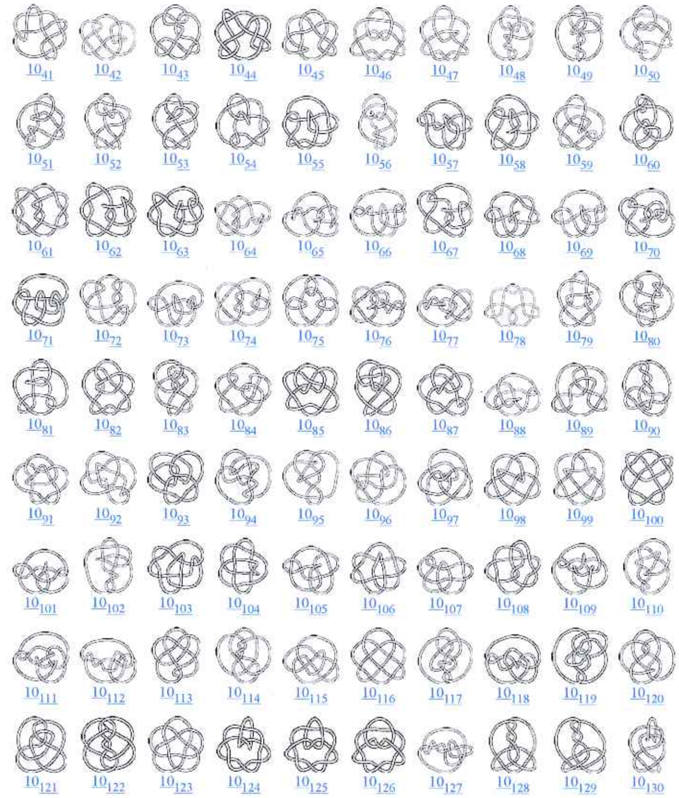


9_1 9_2 9_3 9_4 9_5 9_6 9_7 9_8 9_9 9_{10}
9_{11} 9_{12} 9_{13} 9_{14} 9_{15} 9_{16} 9_{17} 9_{18} 9_{19} 9_{20}
9_{21} 9_{22} 9_{23} 9_{24} 9_{25} 9_{26} 9_{27} 9_{28} 9_{29} 9_{30}
9_{31} 9_{32} 9_{33} 9_{34} 9_{35} 9_{36} 9_{37} 9_{38} 9_{39} 9_{40}
9_{41} 9_{42} 9_{43} 9_{44} 9_{45} 9_{46} 9_{47} 9_{48} 9_{49}

Knots with 10 Crossings

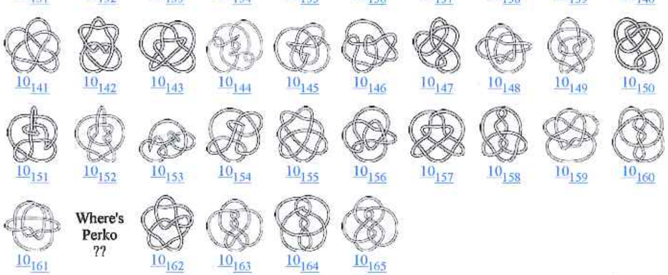


10_1 10_2 10_3 10_4 10_5 10_6 10_7 10_8 10_9 10_{10}
10_{11} 10_{12} 10_{13} 10_{14} 10_{15} 10_{16} 10_{17} 10_{18} 10_{19} 10_{20}
10_{21} 10_{22} 10_{23} 10_{24} 10_{25} 10_{26} 10_{27} 10_{28} 10_{29} 10_{30}
10_{31} 10_{32} 10_{33} 10_{34} 10_{35} 10_{36} 10_{37} 10_{38} 10_{39} 10_{40}



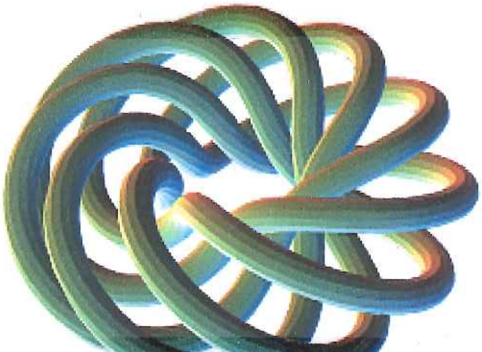
10_{41} 10_{42} 10_{43} 10_{44} 10_{45} 10_{46} 10_{47} 10_{48} 10_{49} 10_{50}
10_{51} 10_{52} 10_{53} 10_{54} 10_{55} 10_{56} 10_{57} 10_{58} 10_{59} 10_{60}
10_{61} 10_{62} 10_{63} 10_{64} 10_{65} 10_{66} 10_{67} 10_{68} 10_{69} 10_{70}
10_{71} 10_{72} 10_{73} 10_{74} 10_{75} 10_{76} 10_{77} 10_{78} 10_{79} 10_{80}
10_{81} 10_{82} 10_{83} 10_{84} 10_{85} 10_{86} 10_{87} 10_{88} 10_{89} 10_{90}
10_{91} 10_{92} 10_{93} 10_{94} 10_{95} 10_{96} 10_{97} 10_{98} 10_{99} 10_{100}
10_{101} 10_{102} 10_{103} 10_{104} 10_{105} 10_{106} 10_{107} 10_{108} 10_{109} 10_{110}
10_{111} 10_{112} 10_{113} 10_{114} 10_{115} 10_{116} 10_{117} 10_{118} 10_{119} 10_{120}
10_{121} 10_{122} 10_{123} 10_{124} 10_{125} 10_{126} 10_{127} 10_{128} 10_{129} 10_{130}

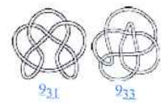
10_{131} 10_{132} 10_{133} 10_{134} 10_{135} 10_{136} 10_{137} 10_{138} 10_{139} 10_{140}

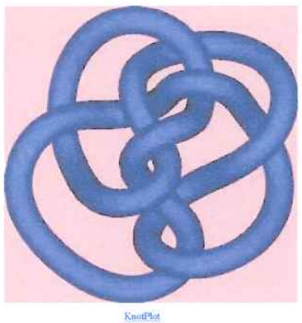


10_{141} 10_{142} 10_{143} 10_{144} 10_{145} 10_{146} 10_{147} 10_{148} 10_{149} 10_{150}
10_{151} 10_{152} 10_{153} 10_{154} 10_{155} 10_{156} 10_{157} 10_{158} 10_{159} 10_{160}
10_{161} Where's Perko ?? 10_{162} 10_{163} 10_{164} 10_{165}

In 1974 K. Perko noticed that the knots labeled 10_{161} and 10_{162} in Rolfsen's tables are in fact the same. In our table we removed his 10_{162} and renumbered the subsequent knots, so that our 10 crossings total is 165, one less than Rolfsen's 166. Read more: 1 2 3 4.

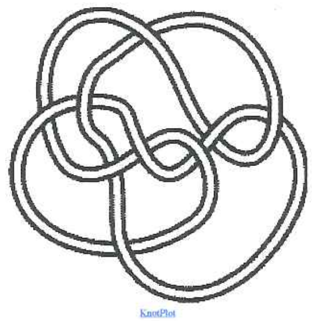






The Knot 9_{32}

Visit 9_{32}'s page at the Knot Server (KnotPlot driven, includes 3D interactive images!)



PD Presentation: $X_{1425} X_{13,18,14,1} X_{3948} X_{9,3,10,2} X_{7,15,8,14} X_{15,11,16,10} X_{5,12,6,13} X_{11,17,12,16} X_{17,7,18,6}$

Alexander Polynomial: $t^{-3} - 6t^{-2} + 14t^{-1} - 17 + 14t - 6t^{2} + t^{3}$

Conway Polynomial: $1 - z^{2} + z^{6}$

Other knots with the same Alexander/Conway Polynomial: {K11n52, K11n124, ...}

Determinant and Signature: {-59, 2}

Jones Polynomial: $-q^{-2} + 4q^{-1} - 6 + 9q - 10q^{2} + 10q^{3} - 9q^{4} + 6q^{5} - 3q^{6} + q^{7}$

Other knots (up to mirrors) with the same Jones Polynomial: {...}

A2 (sl(3)) Invariant: $-q^{-6} + 2q^{-4} + 1 + 3q^{2} - 2q^{4} + 2q^{6} - 2q^{8} - 2q^{14} + 2q^{16} - q^{18} + q^{22}$

V_2 and V_3, the type 2 and 3 Vassiliev invariants: {-1, -2}

Khovanov Homology:
(The squares with yellow highlighting are those on the "critical diagonals", where $j-2r=s+1$ or $j-2r=s+1$, where $s=2$ is the signature of 9_{32}. Nonzero entries off the critical diagonals (if any exist) are highlighted in red.)

$t^r q^j$	r = -3	r = -2	r = -1	r = 0	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6
j = 15										1
j = 13									2	
j = 11								4	1	
j = 9							5	2		
j = 7						5	4			
j = 5					5	5				
j = 3				4	5					
j = 1			3	6						
j = -1		1	3							
j = -3		3								
j = -5	1									

Computer Talk. The data above is also available in a Mathematica readable format. Click to download KnotTheory.m and KnotTheoryData.m, save these files in some directory readable by Mathematica, and check the example Mathematica session below to see how this data can be read: (Mathematica system prompts in blue, human input in red, Mathematica output in black)

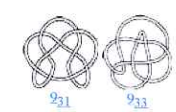
```
In[1]:=    << KnotTheory`
Loading KnotTheory`...
In[2]:=    << KnotTheoryData.m
Loading KnotTheoryData.m...
In[3]:=    PD[Knot[9, 32]]
Out[3]=    PD[X[1, 4, 2, 5], X[13, 18, 14, 1], X[3, 9, 4, 8], X[9, 3, 10, 2],

            >  X[7, 15, 8, 14], X[15, 11, 16, 10], X[5, 12, 6, 13], X[11, 17, 12, 16],

            >  X[17, 7, 18, 6]]
In[4]:=    alex = Alexander[Knot[9, 32], t]
Out[4]=           -3    6    14                 2    3
           -17 + t   - -- + -- + 14 t - 6 t  + t
                        2    t
                       t
In[5]:=    Conway[Knot[9, 32], z]
Out[5]=         2    6
           1 - z  + z
In[6]:=    Select[AllKnots[], (alex === Alexander[#, t])&]
Out[6]=    {Knot[9, 32], Knot[11, NonAlternating, 52], Knot[11, NonAlternating, 124]}
In[7]:=    {KnotDet[Knot[9, 32]], KnotSignature[Knot[9, 32]]}
Out[7]=    {-59, 2}
In[8]:=    Jones[Knot[9, 32], q]
Out[8]=          -2   4            2       3      4      5      6    7
           -6 - q   + - + 9 q - 10 q  + 10 q  - 9 q  + 6 q  - 3 q  + q
                      q
In[9]:=    Select[AllKnots[], (J === Jones[#, q] || (J /. q-> 1/q) === Jones[#, q])&]
Out[9]=    {Knot[9, 32]}
In[10]:=   A2Invariant[Knot[9, 32], q]
Out[10]=        -6   2        2      4      6      8      14      16    18    22
           1 - q   + -- + 3 q  - 2 q  + 2 q  - 2 q  - 2 q   + 2 q   - q   + q
                      4
                     q
In[11]:=   {Vassiliev[2][Knot[9, 32]], Vassiliev[3][Knot[9, 32]]}
Out[11]=   {-1, -2}
In[12]:=   Kh[Knot[9, 32]]
Out[12]=             3     1       3      1     3    3 q     3       5       5  2
           6 q + 4 q  + ----- + ----- + ---- + --- + --- + 5 q  t + 5 q  t + 5 q  t  +
                         5  3    3  2     2    q t    t
                        q  t    q  t    q t

                 7  2      7  3      9  3      9  4      11  4     11  5      13  5
            >  5 q  t  + 4 q  t  + 5 q  t  + 2 q  t  + 4 q   t  + q   t  + 2 q   t  +

                15  6
            >  q   t
```

Math 1350F Knot theory, Sep 16 2003

A comment about linking numbers.

A word about Reid. theorem — The Rolfsen table, torus knots.

9_{32} has $K, rK, \bar{K}, r\bar{K}$ different.

Connect sum of knots. ($\bigcirc$ is $\bigcirc$) — ~~Abelian~~, commutative, associative

Seifert surfaces

Seifert surfaces exist!

Classification of surfaces, genus, Euler characteristic

Knot genus

__Thm__ $g(K_1 + K_2) = g(K_1) + g(K_2)$

__Cor__ $K_1 + K_2 = \bigcirc \Rightarrow K_1 = K_2 = \bigcirc$

__Cor__ $nK \neq mK$ for every knot K.

__Cor__ a knot of genus 1 is prime

__Cor__ Every knot is a sum of prime knots.

Math 1350F Knot Theory, Sep 18 2003

1. Seifert's algorithm.
2. $g(K_1 + K_2) = g(K_1) + g(K_2)$

תקציר ההוכחה (או: למה כיסיתי את כל המקרים?)

נתונים: $K = P + Q$ וגם $K = K_1 + K_2$ ו P ראשוני.

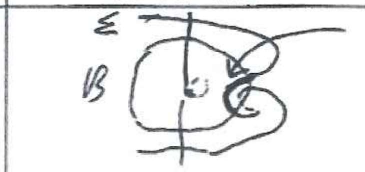
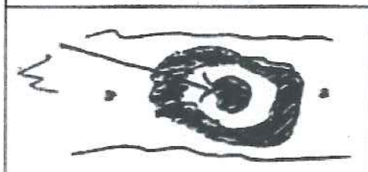
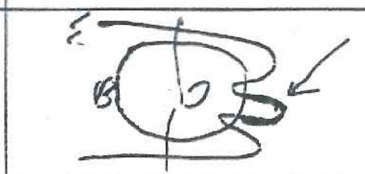
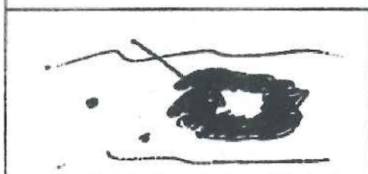
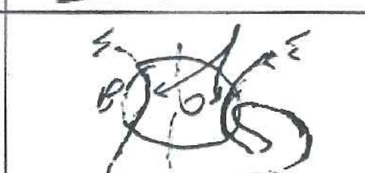
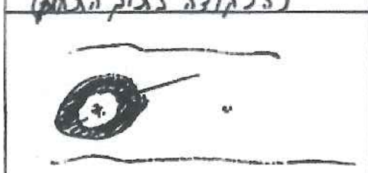
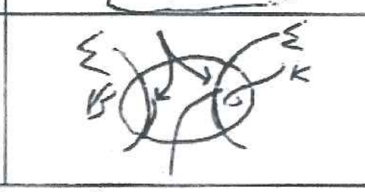
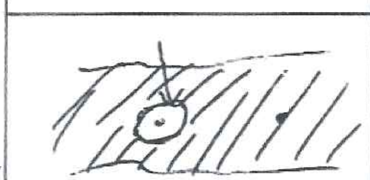
בניה:

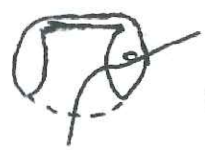
Σ – משטח שמפריד בין K_1 ל K_2.

B – כדור המפריד בין P ו Q (מכיל בתוכו את P)

תזכורת:

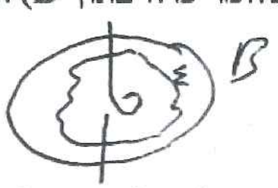
Σ נראה כמו משטח עם מסילות סגורות עליו (שהם החיתוך שלו עם השפה של B) ושתי נקודות מיוחדות (חיתוך עם K). כמו כן הוא צבוע בשחור לבן כאשר שחור הכוונה לחלקים שבתוך B ולבן מחוצה לו. כל מסילה היא גבול בין תחום שחור ללבן.
כל מסילה מחלקת את Σ ל – 2 תחומים. אנחנו נחלק את המקרים לפי איך 2 הנקודות המיוחדות מתפלגות בין שני התחומים. (2 הנקודות באותו צד או בצדדים שונים של המסילה)

חלוקה	תאור המקרה	ציור של B	ציור של Σ	כמו יבולי
2-0	בצד ללא הנקודות אין מסילות וצבעו שחור			מקטינים את B
2-0	בצד ללא הנקודות אין מסילות וצבעו לבן			מגדילים את B
1-1	באחד מהתחומים אין עוד מסילות וצבעו שחור		 (הנקודה בתוך התחום)	מקטינים את B
1-1	באחד מהתחומים אין עוד מסילות וצבעו לבן. כמו כן יש עוד מסילות על Σ			מקטינים את B
1-1	זוהי מסילה יחידה על Σ (כלומר צד אחד לבן וצד שני שחור).			בוחרים את B הקטן מתוך הכדורים החיצוני עם ציפה משולש



להזכירכם בכל אחד מהמקרים אנחנו מקטינים או מגדילים את B במטרה להיפטר מאחת מהמסילות הסגורות שנמצאת על Σ.

כעת נותרו שתי אפשרויות: או שכל Σ צבוע בשחור (כלומר כולו בתוך B) או בלבן. אם הוא לבן סיימנו.



אם הוא שחור אז התמונה נראית ככה:
Σ מחלק את P לשלוש. שניים מהם טריביאלים (מהגדרת הראשוניות) והם כמובן אלו השניים החיצוניים כי K_1 ו K_2 אינם טריביאלים. לכן ניתן להקטין את B ולקבל Σ לבן כולו וללא מסילות. כלומר B מוכל בתוך אחד מצדדי Σ ומכאן נובע המשפט.

Math 1350F Knot Theory, Sep 23 2003

Schonflies Thm

Finish proof of $g(K_1+K_2)=g(K_1)+g(K_2)$

Uniqueness of decomposition into primes

A word about alternating links.

Sep 25 2003

Class Photo!

Finish uniqueness proof.

Class photo!

Math 1350F Knot Theory, Sep 25 2003.

Relative Schonflies

$$\begin{array}{ccc} & & \mathbb{R}^3 \\ & \nearrow^{e} & \downarrow \exists f \\ S^2 & \searrow_{s} & \mathbb{R}^3 \end{array}$$

If e is "right" on some ball, f is id on that ball.

Decomposition systems for γ: $\underline{S} = \bigcup_{i=1}^{n} S_i$,

1. Each S_i intersects γ twice.
2. If you short all S_j's inside S_i, get prime knot inside.
3. If you short all S_j's outside S_i, get ~~any~~ the trivial knot outside.

$\underline{S}, \underline{S}'$ can be made disjoint.

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5. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. 13 (1966), 459–469.
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10. R. S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 65–222.

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The Irreducibility of the 3-Sphere

W. B. R. LICKORISH

1. Introduction

In the theory of 3-dimensional manifolds constant use is necessarily made of the fact that S^3, the 3-dimensional sphere, is irreducible. This fact is usually required in its piecewise linear interpretation, for that seems to be the commonly chosen framework for elementary work with 3-manifolds. The required result is then the following "Schönflies theorem."

THEOREM. *If S^2 is embedded piecewise linearly in S^3, then $S^3 - S^2$ has two components, the closure of each being a piecewise linear ball.*

This theorem was proved by Alexander [1], and a version of his proof is given in [8]. That proof is not, however, readily understood in the context of the standard modern theory of piecewise linear n-manifolds, and the theorem is omitted from the main expositions of that theory ([3], [6], [9], [10]). It is likewise omitted from works on 3-manifolds (e.g., [5], [7]). The purpose of this paper is to give a version of the proof based on handlebody theory. It is hoped that this proof will fill a gap in the literature and that it will bring out the 3-dimensional nature of the proof (an innermost circle argument). That itself is of interest in that the Schönflies problem for S^3 embedded in S^4 is still unsolved in the piecewise linear or smooth sense; a discussion appears in Chapter 3 of [9]. (For locally flat embeddings of S^{n-1} in S^n the result is known to be true in the topological sense for all n [2], and, using the solution to the n-dimensional Poincaré conjecture, in the piecewise linear sense for $n \geq 5$.)

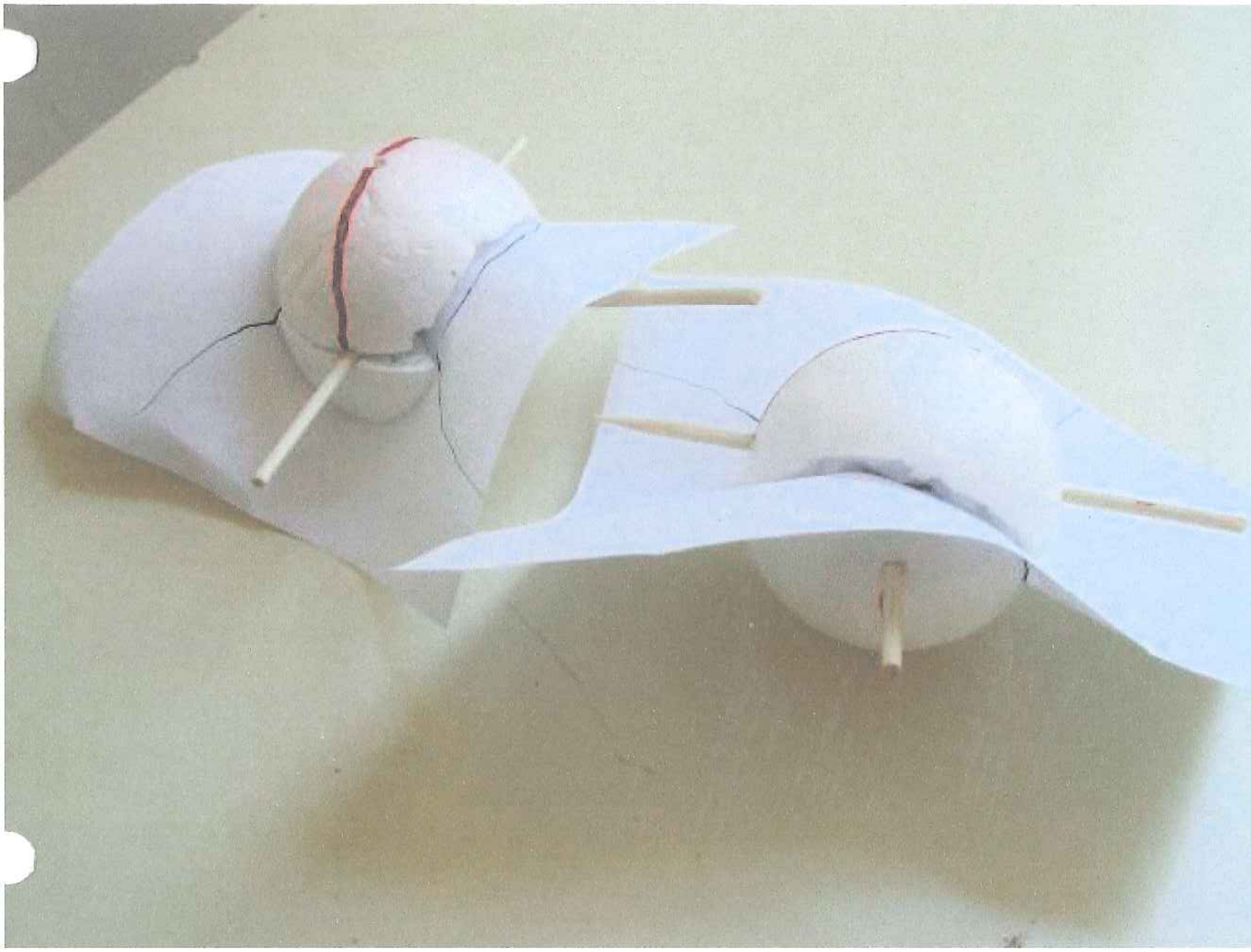
2. Piecewise Linear Preliminaries

A few easily accessible results of piecewise linear topology that will be needed are listed below.

(1) *An S^1, piecewise linearly embedded in S^2, separates S^2 into two piecewise linear discs.*

Received July 14, 1987. Revision received July 22, 1988.
Michigan Math. J. 36 (1989).

A Model by Tom Erez



Math 1350F Knot Theory, Sep 30 2003

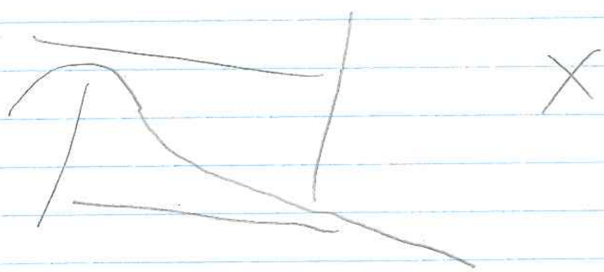
HW Grading policy ; Chap1 Prob 3.

Alternating knots/link

An alternating knot/link is split/prime/knotted iff it looks split/prime/knotted ill

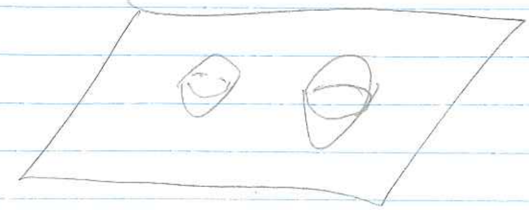
[illegible] * The world view 

* good surfaces.
* every cycle bounds a cap.
* every cap touches each ball at most once.
* No "pointless" caps.



Math 1350F Knot Theory, Oct 2 2003

Show disk



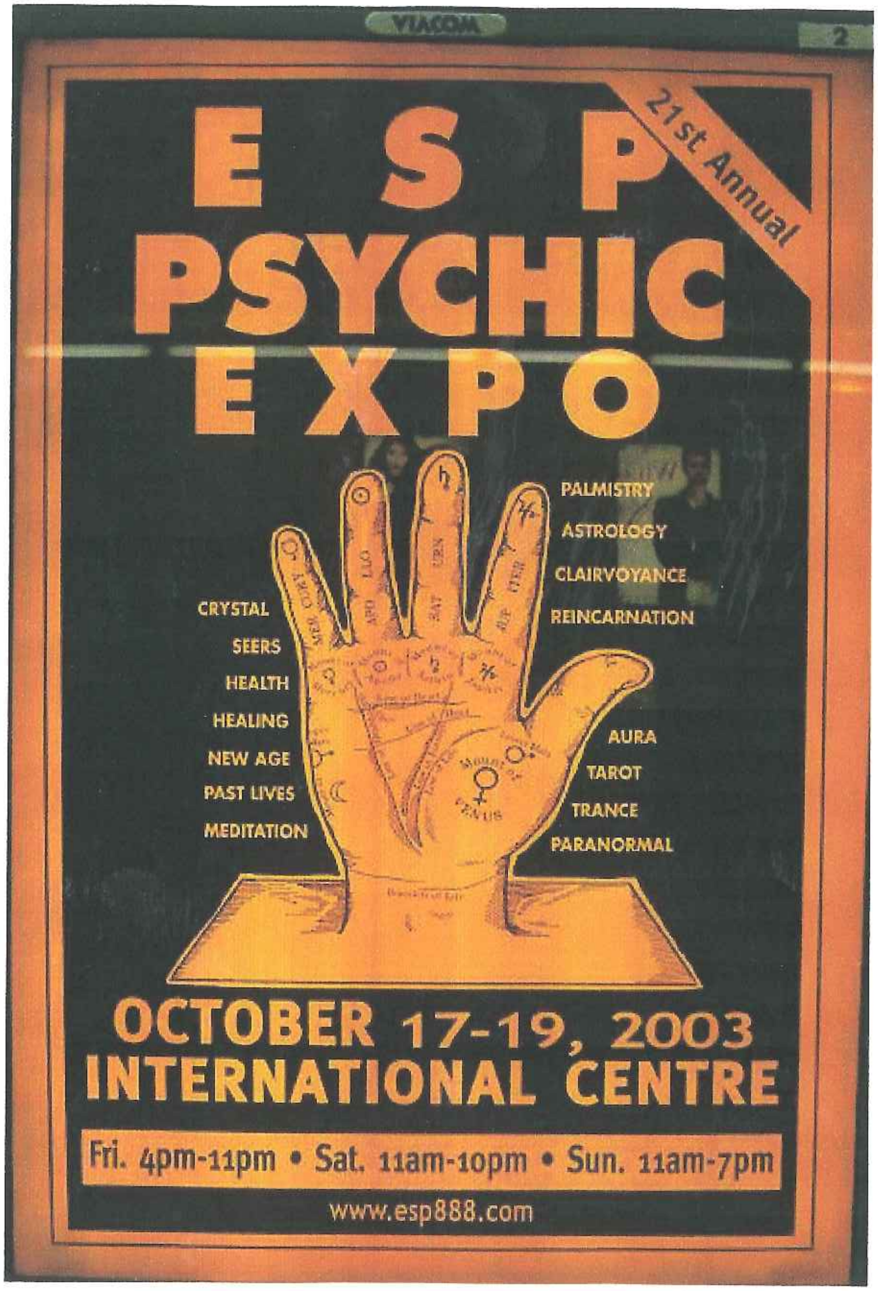
Review proof:

1. S, S^+, S^-, water, air, B_i, S_i
2. "Generic" F
3. Capping all cycles in $F \cap S^\pm$
4. Every cycle γ in $F \cap S^\pm$ intersects every B_i at most once.

Conclusion of the proof.

A Topological Disk

A topological disk, seen on the Toronto Subway, September 30, 2003:



Math 1350F Knot Theory, Oct 7 2003

Last words about alternating knots.

Finite type invariants:

1. Definition

~~Derivatives~~ Differences are cousins of derivatives; Taylor's thm.

Examples: constants.

linking numbers

self-linking / writhe

Jones:

$$t^{-1}V(L_+) - tV(L_-) = (t^{1/2} - t^{-1/2})V(L_0)$$

$$\times \to A\,)(+ A^{-1} \asymp$$

$$V(L) = \left((-A)^{-3w(D)}\langle D\rangle\right)_{A=t^{-1/4}}$$

The top derivative

if V_1, V_2 have equal top derivatives then $V_1 - V_2$ is of lower type

$V^{(m)}$ is "constant"

Chord diagrams (work out examples)

FI, 4T (The Conway example)

The fundamental thm.

Some dimensions of $\mathcal{A}$

Dror Bar-Natan

October 13, 2003

Abstract

We compute the dimensions of $\mathcal{A}_1$ thru $\mathcal{A}_5$ and quote the dimensions of $\mathcal{A}_6$ thru $\mathcal{A}_{12}$.

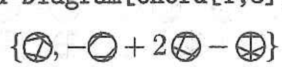
Starting up mathematica [Wo], loading a definitions file and testing the $4T$ relation:

```
Mathematica 4.1 for Linux
Copyright 1988-2000 Wolfram Research, Inc.
 -- Motif graphics initialized --
```

In[1]:= `<< ChordDiagrams.m`

`Loading ChordDiagrams...`

In[2]:= `{d=Diagram[Chord[1,3],Chord[4,6],D4T[5,2,7]], b[d]}`

Out[2]= 

There is only one way to place a single chord on a circle...

In[3]:= `Place[Chord]`

Out[3]= 

and there can be no $4T$ relations in degree 1. Therefore $\dim \mathcal{A}_1 = 1$. Now, there are two ways to place two chords...

In[4]:= `Place[2*Chord]`

Out[4]= 

and one way to place a $4T$ relation symbol and no chords...

In[5]:= `RelationSymbol = Place[D4T]`

Out[5]= 

but the actual relation that corresponds to this symbol is 0...

In[6]:= `Relation = b /@ RelationSymbol`

Out[6]= $\{0\}$

and therefore $\dim \mathcal{A}_1 = 2$. Likewise, there are 5 ways to place three chords...

In[7]:= `Place[3*Chord]`

Out[7]= 

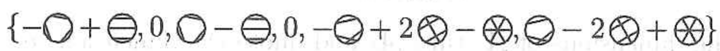
and 6 relations symbols made of one $4T$ symbol and one chord:

In[8]:= `RelationSymbols = Place[D4T+Chord]`

Out[8]= 

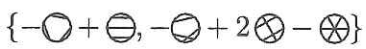
The corresponding relations are...

In[9]:= `Relations = b /@ RelationSymbols`

Out[9]= 

and their linear span is...

In[10]:= `LinearSpan[Relations]`

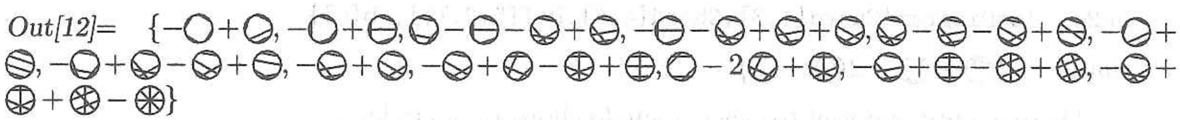
Out[10]= 

As this span is 2 dimensional, we find that $\dim \mathcal{A}_3 = 5-2 = 3$. We now repeat this procedure in degree 4...

In[11]:= `CDs = Place[4*Chord]`

Out[11]= 

In[12]:= `Rels = LinearSpan[b /@ Place[D4T+2*Chord]]`

Out[12]= 

In[13]:= `Length[CDs]-Length[Rels]`

Out[13]= 6

and find that $\dim \mathcal{A}_4 = 6$. Finally,

In[14]:= `Length[Place[5*Chord]]`

Out[14]= 105

In[15]:= `Length[LinearSpan[b /@ Place[D4T+3*Chord]]]`

Out[15]= 95

In[16]:= `105-95`

Out[16]= 10

and thus $\dim \mathcal{A}_5 = 10$.

Working harder and with better programs (see [Ba, Kn]), we learn that

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}_n$	1	1	2	3	6	10	19	33	60	104	184	316	548

A conjectured generating function for the sequence $\dim \mathcal{A}_n$ is at [Br]. At present, the computation of $\dim \mathcal{A}_n$ for general n seems to be beyond our reach.

References

[Ba] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[Br] D. J. Broadhurst, *Conjectured enumeration of Vassiliev invariants,* Open University UK preprint, September 1997, arXiv:q-alg/9709031.

[Kn] J. A. Kneissler, *The number of primitive Vassiliev invariants up to degree twelve,* University of Bonn preprint, June 1997, arXiv:q-alg/9706022.

[Wo] S. Wolfram, *The Mathematica Book,* Cambridge University Press, 1999 and `http://documents.wolfram.com/framesv4/frames.html`.

This handout and the program used in it are available at
`http://www.ma.huji.ac.il/~drorbn/classes/0001/KnotTheory`.

Math 1350F Knot theory, Oct 14 2003.

Review of last class — W.S. of Consts, lk, sl

The conway Weight system

The Jones poly & its W.S.

Handout.

Properties of $\mathcal{A}$: Algebra, co-algebra.

Math 1350F Knot Theory, Oct 16 2003

$\mathcal{A}$ is a commutative associative graded algebra with unit.

$\mathcal{A}$ is a co-commutative co-associative graded algebra with co-unit.

Math 1350F Knot Theory, Oct 21 2003

1. A is an algebra and a co-algebra and the two structures are compatible.

2. The Milnor-Moore Theorem.
3. What is D?
4. A Hopf-Algebra map

$S : A \to A$ mapping FI $\to 0$.

5. Trivalent vertices.

Math 1350F Knot Theory, Oct 23 2003

* Discuss HW Assignment

* Finish trivalent vertices.

Dror Bar-Natan: Classes: 2003-04: Math 1350F - Knot Theory:

Homework Assignment 6: Deframing

Assigned Thursday October 23; due Thursday October 30 in class.

Required reading. Sections 2 and 3 of my paper *On the Vassiliev Knot Invariants.*

Let $\Theta : \mathcal{A} \to \mathcal{A}$ be the multiplication operator by the chord diagram θ, and let $\partial_\theta = \frac{d}{d\theta}$ be the adjoint of multiplication by W_θ on $\mathcal{A}^\star$, where W_θ is the obvious dual of θ in $\mathcal{A}^\star$. Let $P : \mathcal{A} \to \mathcal{A}$ be defined by

$$P = \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{n!} \partial_\theta^n.$$

The following assertions can be verified:

1. $[\partial_\theta, \Theta] = 1$, where $1 : \mathcal{A} \to \mathcal{A}$ is the identity map and where $[A, B] := AB - BA$ for any two operators.
2. P is a degree 0 operator; that is, $\deg Pa = \deg a$ for all $a \in \mathcal{A}$.
3. ∂_θ satisfies Leibnitz' law: $\partial_\theta(ab) = (\partial_\theta a)b + a(\partial_\theta b)$ for any $a, b \in \mathcal{A}$.
4. P is an algebra morphism: $P1 = 1$ and $P(ab) = (Pa)(Pb)$.
5. Θ satisfies the co-Leibnitz law: $\Box \circ \Theta = (\Theta \otimes 1 + 1 \otimes \Theta) \circ \Box$ (why does this deserve the name "the co-Leibnitz law"?).
6. P is a co-algebra morphism: $\eta \circ P = \eta$ (where η is the co-unit of $\mathcal{A}$) and $\Box \circ P = (P \otimes P) \circ \Box$.
7. $P\theta = 0$ and hence $P\langle\theta\rangle = 0$, where $\langle\theta\rangle$ is the ideal generated by θ in the algebra $\mathcal{A}$.
8. If $Q : \mathcal{A} \to \mathcal{A}$ is defined by
$$Q = \sum_{n=0}^{\infty} \frac{(-\Theta)^n}{(n+1)!} \partial_\theta^{(n+1)}$$
then $a = \theta Q a + Pa$ for all $a \in \mathcal{A}$.
9. $\ker P = \langle\theta\rangle$.
10. P descends to a Hopf algebra morphism $\mathcal{A}^r \to \mathcal{A}$, and if $\pi : \mathcal{A} \to \mathcal{A}^r$ is the obvious projection, then $\pi \circ P$ is the identity of $\mathcal{A}^r$. (Recall that $\mathcal{A}^r = \mathcal{A}/\langle\theta\rangle$.)
11. $P^2 = P$.

To be handed in. Verify assertions 4, 5, 7 and 11 above.

Recommended for extra practice. Verify all the other assertions above.

Idea for a good deed. Prepare a beautiful TEX writeup (including the motivation and all the details) of the solution of this assignment for publication on the web. For all I know this information in this form is not available elsewhere.

Math 1350F Knot theory, Oct 28 2003.

Conclusion of $A^t \cong A^c$  $\to$

AS, IHX.

Lie algebras; representations.

Metrics

Math 1350F Knot theory, Oct 30 2003

bases, structure constants, matrix elements.

The w.s. of $(\mathfrak{g}, R)$.

$W_{\mathfrak{gl}(N)}(D)$.

Note taking Volunteer ?

Math 1350F Knot theory, ~~Oct 28~~ Nov 4 2003.

Finish gl(N)

The far colour theorem

Well definedness of $W_L : A \to \mathbb{F}$

Primitives

Math 1350F Knot theory, Nov 6 2003

1. review pf of 4CT
2. Well definededness of $W_L : \mathcal{A} \to \mathbb{F}$
3. Primitives.

Math 1350F Knot Theory, Nov 11 2003.

1. Universal Vassiliev invariants
2. The algebraic approach
   Braids
   generators
   relations
   Syzygies
   Associativities
   Associahedra
   high tech homological algebra.

3.
$$\int_{\Omega^1(\mathbb{R}^3)} \mathcal{D}A\, e^{i\int_{\mathbb{R}^3} A\wedge dA} \int_{\gamma_1} A \cdot \int_{\gamma_2} A \sim lk(\gamma_1, \gamma_2)$$

The Four Colours.
A historical episode, as revealed to Blanche Descartes*

It was in the spring of the year 1892 that Holmes and I endured what he was later pleased to call our "Chromatic Aberration". I cannot give the exact date; my relevant notebook was mislaid in the following year, somewhere in the vicinity of the Reichenbach Falls.

We were in Holmes' chambers at 221B Baker Street, relaxing after a hearty dinner. I lay back in my easy chair puffing at my cigar and contentedly caressing my ancient wound. Holmes, contrarywise, sat stiff and upright. He was staring, so it seemed to me, at some point on the ceiling not far from the royal monogram he had once inscribed, if that be the right word, upon the wall.

He bestirred himself. From the *debris* of our dinner he selected a red radish and a clean white piece of cauliflower. These he put on his empty plate. After some cogitation he added a little pile of green peas. Then he reached into the coal-scuttle and brought forth a shiny black pebble. Half-way to jet I thought, most idly.

He added that lump of coal to the contents of the dinner plate. Then he glared at the contents of that plate, his usually handsome, if acquiline, features contorted in a dreadful frown. After a few minutes he relaxed somewhat, reached for a crayon and began drawing pentagons on the tablecloth. I judged it time to intervene.

"I think you are quite right, Holmes." I said.

He turned towards me. "Eh?"

"I agree with what you were just thinking, That theorem is much deeper than one has hitherto supposed."

He asked in a strange croaking us how I could read his thoughts.

"Hang it, Holmes" I exclaimed, "What with your mooning over those four colours on your plate, and what with those pentagonal diagrams, what

*The author wishes to thank Richard Steinberg for suggesting to her the general idea of this paper.

could you have been thinking about but the Four Colour Theorem? And judging by that portentous frown – ".

I broke off, for my friend was exhibiting some disturbing symptoms. The lower jaw sagged, The eyes seemed to protrude. His visage turned all of a sudden pale and haggard. In a fleeting fancy I envisioned a man confronted with some praeternatural phenomenon. I gazed at him, my first companionable concern rapidly succumbing to professional interest. But before I could move to help him there was an interruption.

I heard footsteps on the stairs. A client no doubt. Holmes got them at all times of the day and night. And here we were in disarray, with the used dinner dishes and the ruined tablecloth!

Holmes too heard the footsteps and he snapped half-way back to normality. But I was not too happy to see him dash to a closet and drag therefrom a gleaming sword. Like so much of his furniture it was a present from high aristocracy. Specifically, if I remember rightly, it came from the Grand Duke of Oberwolffach. It had its own style of beauty but it was a nasty-looking object all the same. I deduced that Holmes had recognized the footsteps and was somewhat distrustful of the footstepper.

There was a tap on the door. "Come in." snarled Holmes. The door opened and our client stood before us.

There was something familiar about that man. Surely I had seen him before? That curious reptilian habit of oscillating his head from side to side? Ah, yes, This was Professor Moriarty, a remarkable man who combined the two full-time occupations of master-criminal and mathematician. In his former aspect he was Holmes' *bete noir*.

Moriarty had raised his eyebrows in an expression of surprise. "Tush, tush," said he, "You know my methods, Holmes. I do not carry out assassinations in person. You may safely put away that skewer. I come as a client, for I have a problem after your own heart."

"Client!" expostulated Holmes. "What about the Earl of Elmira's diamonds? And that business at Breslau? I'll get you for both! And what about –"

"Tush, tush." Interrupted Moriarty, waving an arm in a dismissal manner. "Mr Holmes, let us not waste time on trivialities. My problem is important."

With a thoughtful expression Holmes retreated a few paces and stuck the sword into the cushions of our best settee, a present from the Marchioness of Spitzbergen. "You pique my curiosity." said he. "Let us get this table cleared and we will get to business."

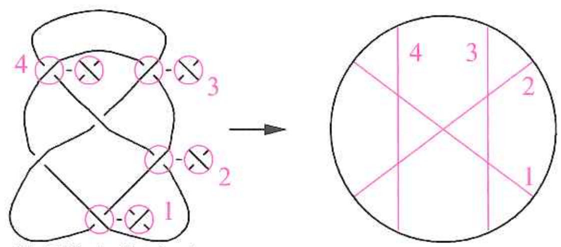
He sounded a bell and soon Mrs. Hudson came in with a tray. "My tablecloth!" she screamed, "Mr. Holmes, if I had known you wanted diagrams I would have lent you Mr. Kempe's paper. Or Mr. Heawood's. I'd

Knotted Trivalent Graphs, Tetrahedra and Associators

HUJI Topology and Geometry Seminar, November 16, 2000

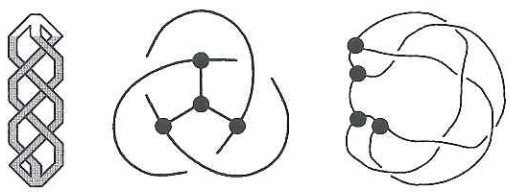
Dror Bar–Natan

Goal: Z:{knots}–>{chord diagrams}/4T so that

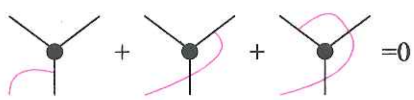


The Miller Institute knot

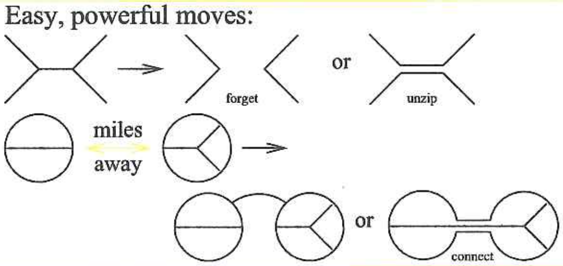
Extend to Knotted Trivalent Graphs (KTG's):

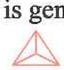


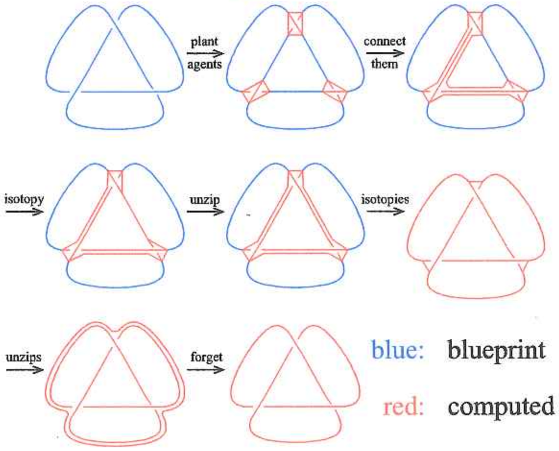
Need a new relation:

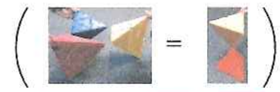


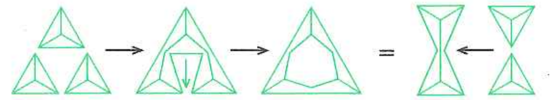
Easy, powerful moves:



Using moves, KTG is generated by ribbon twists and the tetrahedron  :



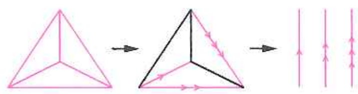
Modulo the relation(s): 

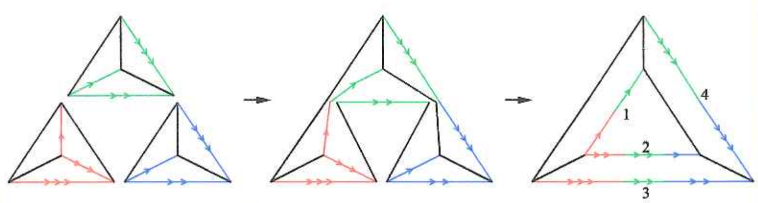


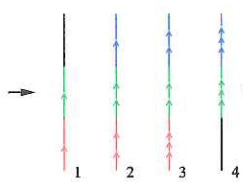
(+more)

Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi Hopf algebras.

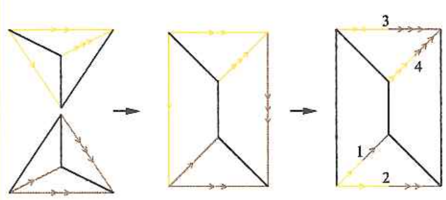
Proof.

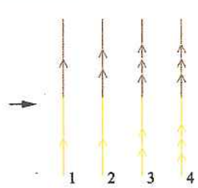
 $:= \Phi \in \mathcal{A}(\uparrow_3)$





$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$





$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

Further directions:

1. Relations with perturbative Chern–Simons theory.
2. Relations with the theory of 6j symbols
3. Relations with the Turaev–Viro invariants.
4. Can this be used to prove the Witten asymptotics conjecture?
5. Does this extend/improve Drinfel'd's theory of associators?

This handout is at http://www.ma.huji.ac.il/~drorbn/Talks/HUJI–001116

No class Tuesday!

Math 135OF Knot Theory, Nov 13 2003

<u>claim</u> $\int_{\Omega^1(\mathbb{R}^3)} \mathcal{D}A\, e^{i\int_{\mathbb{R}^3} A\wedge dA} \int_{\gamma_1} A \int_{\gamma_2} A \sim lk(\gamma_1, \gamma_2)$

Big intro on Gaussian integration:

1. $\int_{-\infty}^{\infty} e^{-x^2/2}$

2. $\int e^{-\lambda x^2/2}$

$\int_{\mathbb{R}^n} e^{-|x|^2/2}$

$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\Lambda_{ij}x^i x^j}$

$\int_{\mathbb{R}^n} \left(\prod x_{i_\alpha}\right) e^{-\frac{1}{2}\Lambda_{ij}x^i x^j}$

$\int_{\mathbb{R}^n} e^{-\frac{1}{2}\Lambda_{ij}x^i x^j + \frac{1}{6}\lambda_{ijk}x^i x^j x^k}$

Math 1350F Knot Theory, Nov 27 2003

* Who's taking this for grade?

Complete the discussion of the linking number:

1. Very flat knots
2. Very condensed volume forms

Deriving (d^{-1})

$Q: V \to V^*$

in our case; $Q = d : \Omega^1 \to \Omega^2$, duality in obvious way

$$\int_{\Omega^1} \longrightarrow \int_{\Omega^1/d\Omega^0} \quad ; V = \frac{\Omega^1}{d\Omega^0} \Rightarrow V^* = \left\{ \begin{matrix} \lambda \in \Omega^2 : \\ d\lambda = 0 \end{matrix} \right\}$$

$C_2(\mathbb{R}^3)$; $\overline{C_2}(\mathbb{R}^3)$, π_1, π_2, π_{12}

Claim: For λ with $d\lambda = 0$

$$d^{-1}\lambda = \pi_{1*}(\pi_{12}^* w \wedge \pi_2^* \lambda)$$

Knots and Feynman Diagrams, Jan 7 2002:

Emergence of FEYNMAN DIAGRAMS

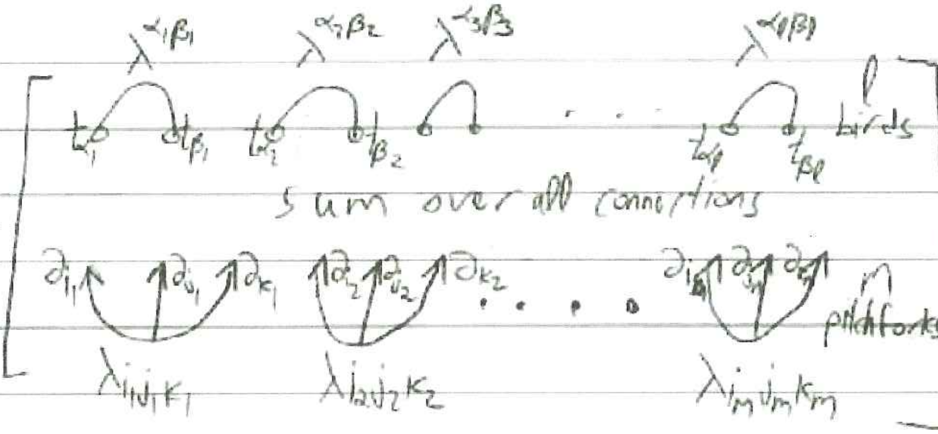
recall: we wish to understand $\int_{\{\text{conn.}\}} \mathcal{D}A\, e^{\frac{ik}{4\pi}\int_{\mathbb{R}^3} tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)}\, hol_\gamma(A)$

(whatever that may mean). As a warmup:

$$\int_{\mathbb{R}^n} dx\, e^{-\frac{1}{2}\lambda_{ij}x^ix^j + \frac{1}{6}\lambda_{ijk}x^ix^jx^k} = \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})}\, e^{\frac{1}{6}\lambda_{ijk}\partial_i\partial_j\partial_k}\, e^{\frac{1}{2}\lambda^{\alpha\beta}t_\alpha t_\beta}\Big|_{t_\alpha=0} \qquad \partial_i = \frac{\partial}{\partial t_i}$$

$$= \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})} \sum_{\substack{m=0\\3m=2l}}^{\infty} \frac{1}{m!\,6^m\,l!\,2^l} \left(\lambda_{ijk}\partial_i\partial_j\partial_k\right)^m \left(\lambda^{\alpha\beta}t_\alpha t_\beta\right)^l$$

$$= \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})} \sum_{\substack{m=0\\3m=2l}}^{\infty} \frac{1}{m!\,6^m\,l!\,2^l}$$



$$= \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})} \sum_{m=0}^{\infty} \frac{1}{m!\,6^m\,l!\,2^m} \sum_{\substack{m\text{-vertex fully marked}\\ \text{Feynman Diagrams } D}} \mathcal{E}(D)$$

$D =$  , ... but fully marked

$$= \frac{(2\pi)^{n/2}}{\det(\lambda_{ij})} \sum_{\substack{\text{unmarked}\\ \text{Feynman}\\ \text{Diagrams}\\ D}} \frac{1}{Aut(D)}\, \mathcal{E}(D)$$

with $\mathcal{E}$:

$\longrightarrow \sum_{\substack{i_1 j_1 k_1\\ i_2 j_2 k_2}} \left(\lambda_{i_1j_1k_1}\lambda_{i_2j_2k_2} \cdot \lambda^{i_1i_2}\lambda^{j_1j_2}\lambda^{k_1k_2}\right)$

Dror Bar-Natan

Math 1350F Knot Theory, Dec 4 2003

* Take home final:

Will be posted by/on Tuesday Dec 9

Will be due Tuesday Jan 6 at noon.

Go over handout.