

QUANTIZATION OF POISSON HOPF ALGEBRAS

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ABSTRACT. We describe a method for quantization of Poisson Hopf algebras in \mathbb{Q} -linear symmetric monoidal categories. It is compatible with tensor products and can also be used to produce braided Hopf algebras. The main idea comes from the fact that nerves of groups are symmetric simplicial sets. Nerves of Hopf algebras then turn out to be braided rather than symmetric and nerves of Poisson Hopf algebras to be infinitesimally braided. The problem is thus solved via the standard machinery of Drinfeld associators.

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1. INTRODUCTION

One of the most natural deformation quantization problems is quantization of Poisson Hopf algebras [3], i.e. of commutative Hopf algebras with a compatible

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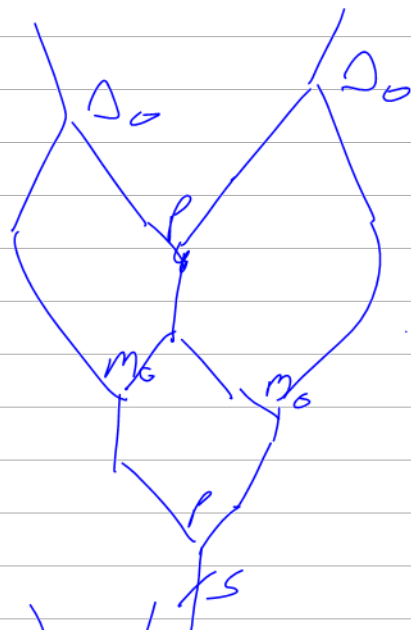
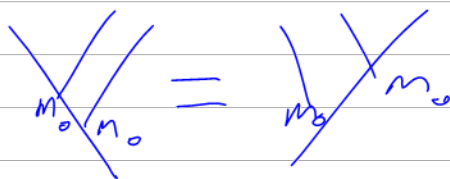
Δ is a morphism of Poisson Alg.

$$\begin{aligned}
 & A \otimes B \\
 & \{a_1, b_1, a_2, b_2\} \\
 & = \{a_1, a_2\} \cdot b_1, b_2 + a_1, a_2 \cdot \{b_1, b_2\}
 \end{aligned}$$

BCH $x, y \in \hbar g[\hbar] = L$

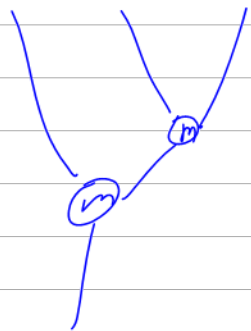
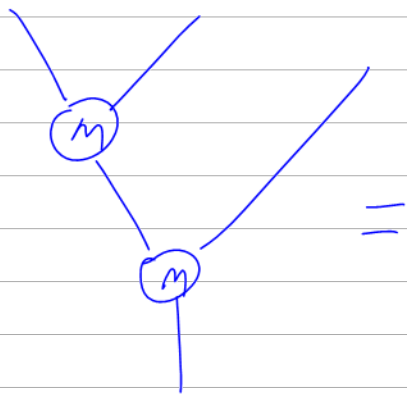
$$Z = \log e^x e^y \in L \in U(g)[\hbar]$$

$$Z = 1^x + 1^y + \frac{1}{2} \begin{array}{c} x \quad y \\ \diagdown \quad / \\ | \\ \diagup \quad \diagdown \\ x \quad y \end{array} + \frac{1}{24} \begin{array}{c} x \quad y \quad x \quad y \\ \diagdown \quad / \quad \diagdown \quad / \\ | \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ x \quad y \quad x \quad y \end{array}$$

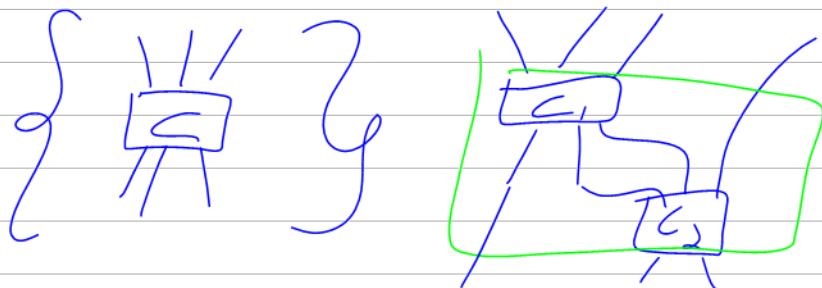


$$1 \otimes 1 \rightarrow H$$

$$m = \sum m_n$$

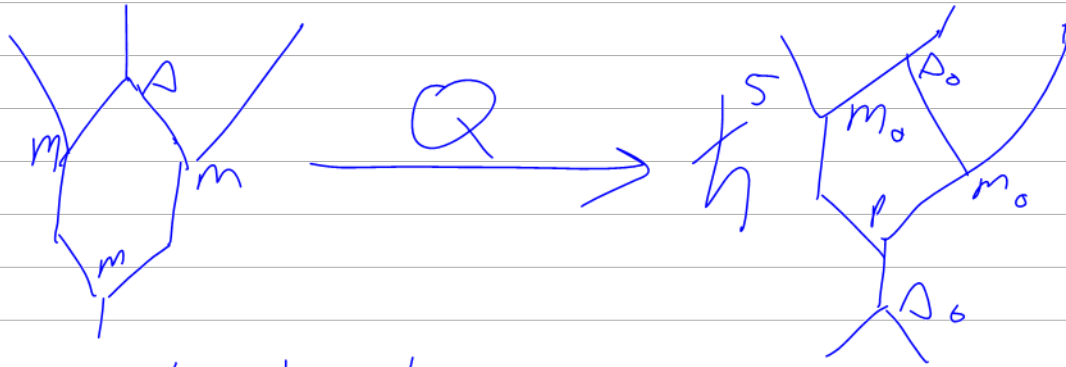


PROP



HOF

PHOF $\frac{h}{h}$



$$\text{PROP} \langle Y_m, \Delta, f \rangle \xrightarrow[\text{cls}]{Q}$$

$$\left. \begin{array}{l} Q(Y_m) \\ Q(\Delta) \end{array} \right\} = \left[\frac{h}{h} \right] \text{graph}$$

Poisson bracket. Given a Poisson Hopf algebra with product m_0 , coproduct Δ_0 , antipode S_0 , Poisson bracket p , unit η and counit ϵ , the problem is to deform m_0 , Δ_0 , and S_0 to

$$m = \sum_{n=0}^{\infty} \hbar^n m_n \quad \Delta = \sum_{n=0}^{\infty} \hbar^n \Delta_n \quad S = \sum_{n=0}^{\infty} \hbar^n S_n$$

so that the result is still a Hopf algebra and so that

$$m - m^{op} = \hbar p + O(\hbar^2).$$

The deformation should be given by universal formulas with rational coefficients, i.e. as a morphism of props

$$\text{Hopf} \rightarrow \text{PoissHopf}(\mathbb{Q})_{\hbar},$$

and thus usable for quantization of Poisson Hopf algebras in arbitrary \mathbb{Q} -linear symmetric monoidal categories and functorial in a rather strong sense. Furthermore, it is natural to demand the quantization to be compatible with tensor products of Hopf algebras; equivalently it means that the morphism of props is compatible with a suitable cocommutative coalgebra enrichment.

Quantization of Poisson Hopf algebras includes, in particular, quantization of Lie bialgebras, which was solved in the seminal work of Etingof and Kazhdan [5]. Nonetheless, the problem of quantization of Poisson Hopf algebras is stronger and appears more natural. } A

We solve this problem using nerves of Hopf algebras. The nerve of a group is not just a simplicial set, but a symmetric simplicial set. A similar statement is true for nerves of commutative Hopf algebras. If we replace the symmetric structure with a braided one, we get nerves of (possibly braided) Hopf algebras. An infinitesimally braided structure corresponds to nerves of Poisson Hopf algebras, and their quantization can thus be obtained by Drinfeld associators (which are machines producing braided structures out of infinitesimally braided ones).

The natural setup for our method is slightly more general: quantization of Poisson Hopf algebras in infinitesimally braided categories to braided Hopf algebras. At the end of the paper we also describe the outcome of our method when applied to quantization of suitable Poisson Hopf algebroids.

2. PRELIMINARIES

2.1. Algebras, coalgebras, Hopf algebras. An *algebra* (or a *monoid*) in a monoidal category \mathcal{C} is an object $A \in \mathcal{C}$ equipped with an associative product $m: A \otimes A \rightarrow A$ and with a unit $\eta: 1_{\mathcal{C}} \rightarrow A$ of m . Similarly a *coalgebra* $C \in \mathcal{C}$ is equipped with a coassociative coproduct $\Delta: C \rightarrow C \otimes C$ and with a counit $\epsilon: C \rightarrow 1_{\mathcal{C}}$.

If \mathcal{C} is a braided monoidal category (BMC) and if $A_1, A_2 \in \mathcal{C}$ are algebras then so is $A_1 \otimes A_2$ with the product [7]

$$\begin{array}{c} A_1 \quad A_2 \\ \diagdown \quad \diagup \\ \quad \quad \quad \\ \diagup \quad \diagdown \\ A_1 \quad A_2 \quad A_1 \quad A_2 \end{array}$$

and with the unit $1_{\mathcal{C}} \cong 1_{\mathcal{C}} \otimes 1_{\mathcal{C}} \xrightarrow{\eta_{A_1} \otimes \eta_{A_2}} A_1 \otimes A_2$. In this way algebras in \mathcal{C} form a monoidal category.

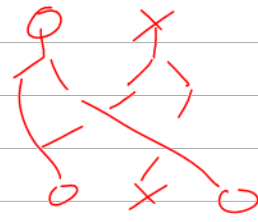
co-Poisson
m, Δ, π: A → A^{⊗2}
Δ - Δ^{op} = ħπ + ...

obj (x o x o o x)

mon (x o x o , o x) =

= braids w/ R234

Algebras: x, o



⇒ x o is an algebra

but non-commutative



(L, b, δ) (L, b) is a Lie algebra
 (L, δ) is a Lie co-alg...

$$\delta[x, y] = (\text{ad}_x \otimes 1) f(y) + (1 \otimes \text{ad}_x) f(y) - (\text{ad}_y \otimes 1) f(x) - (1 \otimes \text{ad}_y) f(x)$$

Quantization: A Hopf alg over $\mathbb{Q}[[\hbar]]$

s.t. $A/\hbar A \cong U(L)$

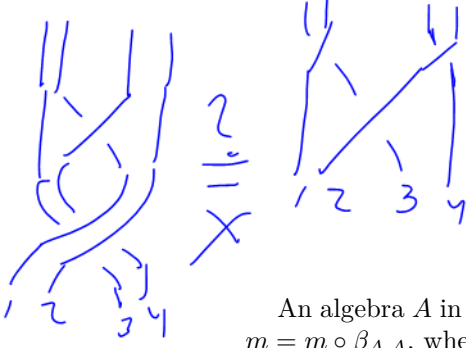
$$\Delta(a) - \Delta^{op}(a) = \hbar f(a) + \dots$$

Construction Given $(L, b, \delta) \subset PHA$
 structure on $U(L) = A$
 m, Δ , as before $\pi(abc \dots) =$

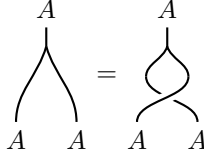
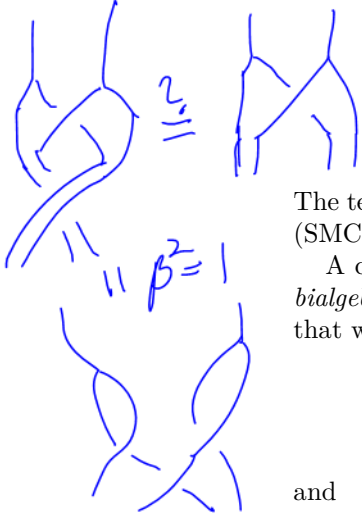
$$\prod (\prod x_i) = \sum_k \Delta(\prod x_i) \delta(x_k) \Delta(\prod x_i)_{k+1}$$

x_1 x_2 x_3 x_4 x_5 x_6
 \int \int \int

SSSTSS
 $L^{\otimes 3} \dots L^{\otimes 3}$

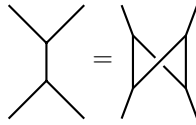


An algebra A in a BMC \mathcal{C} is *commutative* if the product $m: A \otimes A \rightarrow A$ satisfies $m = m \circ \beta_{A,A}$, where $\beta_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ denotes the braiding.



The tensor product of two commutative algebras in a symmetric monoidal category (SMC) is commutative, but in a BMC it is, in general, not true.

A coalgebra H in the monoidal category of algebras in a BMC \mathcal{C} is a (braided) *bialgebra*. Explicitly this means that H is both an algebra and a coalgebra in \mathcal{C} and that we have the identities



and

$$\epsilon \circ \eta = \text{id}_{1e}, \quad \Delta \circ (\eta \otimes \eta) = \eta, \quad \epsilon \circ m = \epsilon \otimes \epsilon$$

(where we tacitly identify $1e \otimes 1e$ with $1e$ via the canonical isomorphism).

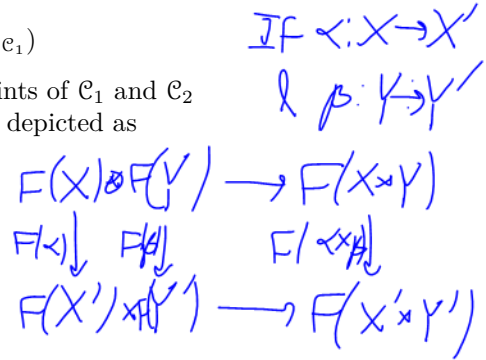
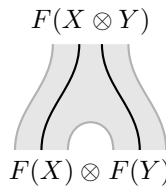
A (braided) *Hopf algebra* in a BMC \mathcal{C} is a bialgebra H with an additional morphism $S: H \rightarrow H$ (the *antipode*) such that

$$m \circ (\text{id} \otimes S) \circ \Delta = m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon.$$

2.2. Lax monoidal functors. A *lax monoidal functor* is a functor $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between two monoidal categories, equipped with natural transformations (called *coherence morphisms*)

$$c_{X,Y}^F: F(X) \otimes F(Y) \rightarrow F(X \otimes Y), \quad c_1^F: 1_{\mathcal{C}_2} \rightarrow F(1_{\mathcal{C}_1})$$

making F compatible with the associativity and the unit constraints of \mathcal{C}_1 and \mathcal{C}_2 (see e.g. [6] for details). Coherence morphisms can be graphically depicted as



A *strong monoidal functor* is a lax monoidal functor whose coherence morphisms are isomorphisms. A *strict monoidal functor* is a strong monoidal functor whose coherence morphisms are identities.

A natural transformation $\alpha_X: F(X) \rightarrow G(X)$ between two lax monoidal functors $F, G: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is *monoidal* if

$$\alpha_{X \otimes Y} \circ c_{X,Y}^F = c_{X,Y}^G \circ (\alpha_X \otimes \alpha_Y) \quad \text{and} \quad \alpha_{1_{\mathcal{C}_2}} \circ c_1^F = c_1^G.$$

Lax monoidal functors form a category with monoidal natural transformations as morphisms.

A lax monoidal functor $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ between two BMCs is *braided* if

$$F(\beta_{X,Y}^{\mathcal{C}_1}) \circ c_{X,Y}^F = c_{Y,X}^F \circ \beta_{F(X),F(Y)}^{\mathcal{C}_2}.$$

Lax monoidal functors send algebras to algebras: If $A \in \mathcal{C}_1$ is an algebra then $F(A) \in \mathcal{C}_2$ is an algebra with the product

$$F(m) \circ c_{A,A}^F = \begin{array}{c} F(A) \\ \text{---} \\ \text{---} \\ \text{---} \\ F(A) \otimes F(A) \end{array} \quad \begin{array}{l} m: A \otimes A \rightarrow A \\ F(A \otimes A) \xrightarrow{c} F(A \otimes A) \rightarrow F(A) \end{array}$$

and with the unit $F(\eta) \circ c_1^F$. Braided lax monoidal functors send commutative algebras to commutative algebras.

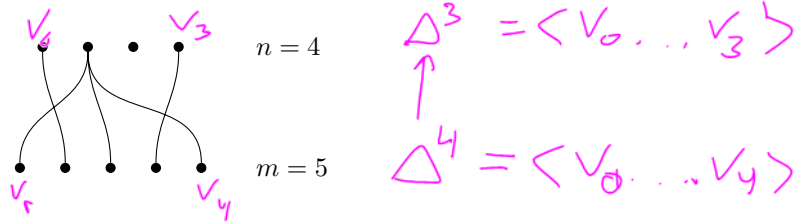
2.3. Props. A *prop* is a strict SMC whose objects are symbols \bullet^n , $n \geq 0$, with the tensor product of objects $\bullet^m \otimes \bullet^n = \bullet^{m+n}$ and with the unit object $1 = \bullet^0$. A *braided prop* is a strict BMC with the same property.

A morphism of (braided or ordinary) props $F: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is a strict braided monoidal functor such that $F(\bullet) = \bullet$.

If \mathcal{P} is an ordinary/braided prop, a \mathcal{P} -algebra in a strict SMC/BMC \mathcal{C} is an object $A \in \mathcal{C}$ together with a strict braided monoidal functor $F: \mathcal{P} \rightarrow \mathcal{C}$ such that $F(\bullet) = A$. (If \mathcal{C} is not strict, F is rather a strong braided monoidal functor, unique up to monoidal isomorphisms which are the identity at the object \bullet . We can specify a particular F by demanding $F(\bullet^n) = A^{\otimes n}$ for a chosen parenthesization of the tensor power $A^{\otimes n}$, and demanding $c_1^F = \text{id}$ and $c_{\bullet^m, \bullet^n}^F: A^{\otimes m} \otimes A^{\otimes n} \rightarrow A^{\otimes(m+n)}$ to be the corresponding composition of the associativity (or unit) constraints of \mathcal{C} .)

Example 2.1. Let Com be the prop of commutative algebras. It is generated by morphisms $\underline{m}: \bullet \bullet \rightarrow \bullet$ and $\underline{\eta}: 1 \rightarrow \bullet$ modulo the relations saying that \underline{m} is commutative and associative and that $\underline{\eta}$ is a unit for \underline{m} . A Com -algebra in a SMC is a commutative algebra.

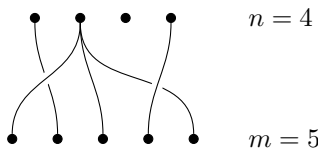
Morphisms $\bullet^m \rightarrow \bullet^n$ in Com can be represented graphically as



with joining lines corresponding to the product $\bullet \bullet \rightarrow \bullet$ and missing lines to the unit $1 \rightarrow \bullet$, i.e. as maps $\{1, \dots, m\} \rightarrow \{1, \dots, n\}$. Composition of morphisms is then equal to the composition of these maps.

Example 2.2. Let BrCom be the braided prop of commutative algebras. It is again generated by two morphisms $\underline{m}: \bullet \bullet \rightarrow \bullet$ (the product) and $\underline{\eta}: 1 \rightarrow \bullet$ (the unit). A BrCom -algebra in a BMC is a commutative algebra.

The morphisms $\bullet^m \rightarrow \bullet^n$ in BrCom are braids with m strands, non-bijectively attached at the top:



$$X \rightarrow Y$$

$$\mathcal{C}(X) \leftarrow \mathcal{C}(Y)$$

$$|F| = \bigsqcup F(\bullet^n) \times \Delta^{n-1}$$

rel.

3. NERVES OF HOPF ALGEBRAS

3.1. **Symmetric simplicial spaces.** An (augmented) symmetric simplicial object in a category \mathcal{S} is a functor

$$\mathbf{Com}^{\text{op}} \rightarrow \mathcal{S}, \quad x \mapsto \mathcal{C}(x)$$

$$F: \mathbf{Com}^{\text{op}} \rightarrow \mathcal{S}$$

$$F(\bullet^n) = \text{"}n\text{-simplices"}$$

Informally speaking, we want to study the case when \mathcal{S} is a "category of spaces". We take the dual point of view and replace "spaces" with commutative algebras. Hence, if \mathcal{C} is a symmetric monoidal category and $\mathbf{ComAlg}(\mathcal{C})$ the category of commutative algebras in \mathcal{C} , our object of interest is the category of functors

$$\mathbf{Com} \rightarrow \mathbf{ComAlg}(\mathcal{C}),$$

i.e. (augmented) symmetric cosimplicial commutative algebras in \mathcal{C} .

Let us observe that every object \bullet^n of \mathbf{Com} is a commutative algebra, being a tensor power of the commutative algebra \bullet , and also that every morphism in \mathbf{Com} is a morphism of algebras. This gives us the following result.

Proposition 3.1. *Suppose \mathcal{C} is a symmetric monoidal category. The category of functors*

$$\mathbf{Com} \rightarrow \mathbf{ComAlg}(\text{Set}^{\text{op}})$$

is isomorphic to the category of symmetric lax monoidal functors

$$\mathbf{Com} \rightarrow \mathbf{ComAlg}(\mathcal{C})$$

$$F: \mathbf{Com} \rightarrow \mathcal{C}$$

$$A_n = F(\bullet^n) \leftarrow A_1^{\otimes n}$$

with monoidal natural transformations as morphisms.

Namely, if $F: \mathbf{Com} \rightarrow \mathcal{C}$ is symmetric lax monoidal then $F(\bullet^n)$ is a commutative algebra in \mathcal{C} as \bullet^n is a commutative algebra in \mathbf{Com} , and thus F becomes a functor $\mathbf{Com} \rightarrow \mathbf{ComAlg}(\mathcal{C})$.

Proof. To get the isomorphism in the opposite direction, if $F: \mathbf{Com} \rightarrow \mathbf{ComAlg}(\mathcal{C})$ is a functor then we can define a symmetric lax monoidal structure on F as follows: $c_{X,Y}^F$ is the composition,

$$F(X) \otimes F(Y) \rightarrow F(X \otimes Y) \otimes F(X \otimes Y) \rightarrow F(X \otimes Y)$$

where the first arrow comes from

$$X = X \otimes 1 \xrightarrow{\text{id}_X \otimes \eta_Y} X \otimes Y \quad \text{and} \quad Y = 1 \otimes Y \xrightarrow{\eta_X \otimes \text{id}_Y} X \otimes Y$$

and the second arrow is the product in $F(X \otimes Y)$, and c_1^F is the unit $1_{\mathcal{C}} \rightarrow F(1)$ of the commutative algebra $F(1)$. \square

3.2. **Nerves of groups and groupoids.** If S is a set, let Pair_S denote the pair groupoid of S , i.e. the groupoid with the set of objects S and with a unique morphism between any two objects.

The (symmetric augmented) *nerve* of a groupoid G is the functor

$$\text{Mor}_{\mathbf{Com}^{\text{op}}}(\mathbb{N}, \mathbb{M})$$

$$X_m^n$$

$$N_G: \mathbf{Com}^{\text{op}} \rightarrow \text{Set}$$

given on objects by

$$N_G(\bullet^n) = \text{Hom}(\text{Pair}_{\{1,2,\dots,n\}}, G).$$

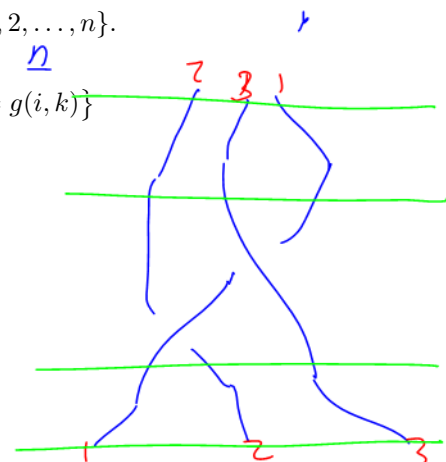
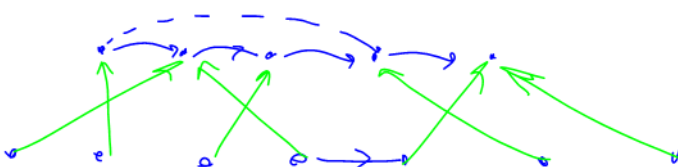
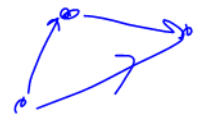
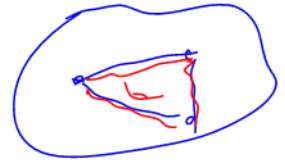
On morphisms it is given by seeing them as maps $\{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$.

If G is a group, this means

$$N_G(\bullet^n) = \{g: \{1, 2, \dots, n\}^2 \rightarrow G \mid g(i, i) = 1 \text{ and } g(i, j)g(j, k) = g(i, k)\}$$

and we have a bijection

$$(1) \quad N_G(\bullet^n) \cong G^{n-1}, \quad g \mapsto (g(i, i+1) \mid 1 \leq i \leq n-1).$$



$$N_G(\bullet^n) = \begin{array}{c} \bullet \xrightarrow{\partial_1} \bullet \xrightarrow{\partial_2} \bullet \xrightarrow{\partial_3} \bullet \xrightarrow{\partial_4} \bullet \end{array}$$

We can recognize nerves of groups among all functors $N: \text{Com}^{\text{op}} \rightarrow \text{Set}$ using (1) as follows. Let us consider the morphisms $\phi_i: \bullet\bullet \rightarrow \bullet^n$

$$\phi_i = \begin{array}{c} 1 \quad \dots \quad i \quad i+1 \quad \dots \quad n \\ \bullet \quad \dots \quad \bullet \quad \bullet \quad \dots \quad \bullet \\ \quad \quad \quad \downarrow \quad \downarrow \end{array} \quad (1 \leq i \leq n-1)$$

and the resulting map

$$(2) \quad (N(\phi_1), \dots, N(\phi_{n-1})) : N(\bullet^n) \rightarrow N(\bullet\bullet)^{n-1}.$$

We then have the following elementary and well known result.

Proposition 3.2. *The assignment $G \mapsto N_G$ is an equivalence of categories between the category of groups and the category of functors $N: \text{Com}^{\text{op}} \rightarrow \text{Set}$ such that (2) is a bijection for every $n \geq 2$ and such that $|N(1)| = |N(\bullet)| = 1$.*

The group corresponding to N is $G = N(\bullet\bullet)$, the product is

$$N\left(\begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}\right) : G \times G \rightarrow G$$

(where we identify $N(\bullet^3)$ with $G \times G$ via (2)) and the unit and the inverse are

$$N\left(\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}\right) \quad \text{and} \quad N\left(\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}\right).$$

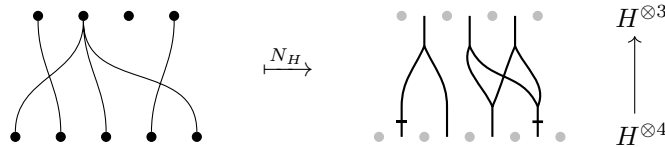
3.3. Nerves of commutative Hopf algebras. By analogy with groups one can define the nerve of a commutative Hopf algebra $H \in \mathcal{C}$ where \mathcal{C} is a symmetric monoidal category (SMC). It is a functor

$$N_H : \text{Com} \rightarrow \text{ComAlg}(\mathcal{C})$$

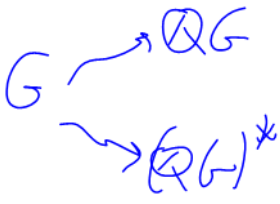
or equivalently, in view of Proposition 3.1, a symmetric lax monoidal functor

$$N_H : \text{Com} \rightarrow \mathcal{C}.$$

On objects it is given by $N_H(\bullet^n) = H^{\otimes(n-1)}$, $N_H(1) = 1_{\mathcal{C}}$ (to simplify the notation we suppose that \mathcal{C} is strict monoidal; if not, one needs to choose a parenthesization of $H^{\otimes(n-1)}$ and use the associativity constraint of \mathcal{C} accordingly). The graphical algorithm for computing $N_H(\phi)$ for $\phi: \bullet^m \rightarrow \bullet^n$ is:



- ▷ put one H between every two consecutive \bullet 's
- ▷ for every consecutive pair $\bullet\bullet$ in \bullet^m , if the order of the pair is reversed in the image, apply S (depicted by \dagger) to the corresponding H
- ▷ if the distance of the two \bullet 's in the image is k , apply iterated Δ to get a morphism $H \rightarrow H^{\otimes k}$ (if $k = 0$, apply ϵ)
- ▷ finally multiply H 's arriving between consecutive \bullet 's in \bullet^n .



$$F(x) \otimes F(y) \rightarrow F(x \otimes y)$$

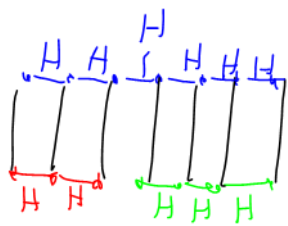
The coherence morphisms of N_H (obtained from Proposition 3.1 and from the commutative algebra structure of $N_H(\bullet^n)$) are as follows: for $m, n > 0$ the morphism $N_H(\bullet^m) \otimes N_H(\bullet^n) \rightarrow N_H(\bullet^{m+n})$ is

$$(3) \quad \text{id}_H^{\otimes(m-1)} \otimes \eta \otimes \text{id}_H^{\otimes(n-1)} : H^{\otimes(m-1)} \otimes H^{\otimes(n-1)} \rightarrow H^{\otimes(m+n-1)}$$

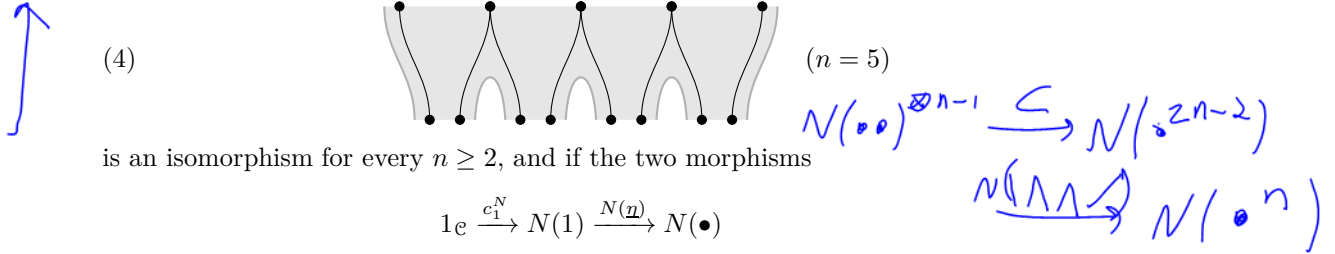
and the remaining coherence morphisms are identities.

If $\mathcal{C} = \text{Set}^{\text{op}}$ and $\otimes = \times$ then H is a group and N_H is its nerve, with the identification $N_H(\bullet^n) = H^{n-1}$ given by (1).

One can recognize nerves of commutative Hopf algebras among all symmetric lax monoidal functors $\text{Com} \rightarrow \mathcal{C}$ by an analogue of the map (2). We shall say that



a braided lax monoidal functor $N: \text{Com} \rightarrow \mathcal{C}$ or $N: \text{BrCom} \rightarrow \mathcal{C}$ satisfies the nerve condition if the morphism $N(\bullet\bullet)^{\otimes(n-1)} \rightarrow N(\bullet^n)$ given by



is an isomorphism for every $n \geq 2$, and if the two morphisms

$$1e \xrightarrow{e_1^N} N(1) \xrightarrow{N(\eta)} N(\bullet)$$

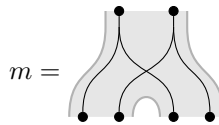
are also isomorphisms. We then have the following minor generalization of Proposition 3.2 (which corresponds to the case $\mathcal{C} = \text{Set}^{\text{op}}$, $\otimes = \times$).

Proposition 3.3. *Let \mathcal{C} be a SMC. The category of commutative Hopf algebras in \mathcal{C} is equivalent to the category of symmetric lax monoidal functors*

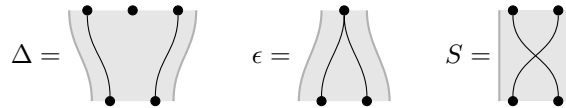
$$N: \text{Com} \rightarrow \mathcal{C}$$

satisfying the nerve condition.

The Hopf algebra corresponding to N is $H = N(\bullet\bullet)$. The commutative algebra structure of H comes from the commutative algebra $\bullet\bullet$, i.e. the product is



and the unit is the composition $1e \rightarrow N(1) \rightarrow N(\bullet\bullet)$. The coproduct, the counit, and the antipode are given by



where we implicitly use the isomorphisms $N(\bullet^3) \cong N(\bullet\bullet)^{\otimes 2}$ and $N(\bullet) \cong 1e$ given by the nerve condition.

The proof is the same as for Proposition 3.2 and we leave its details to the reader.

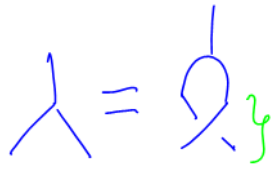
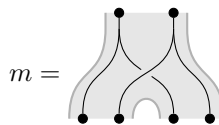
3.4. Nerves of braided Hopf algebras. The previous proposition has a straightforward generalization to the world of noncommutative Hopf algebras (announced in [10, Remark 4]):

Theorem 3.4. *Let \mathcal{C} be a BMC. The category of Hopf algebras with invertible antipodes in \mathcal{C} is equivalent to the category of braided lax monoidal functors*

$$N: \text{BrCom} \rightarrow \mathcal{C}$$

satisfying the nerve condition.

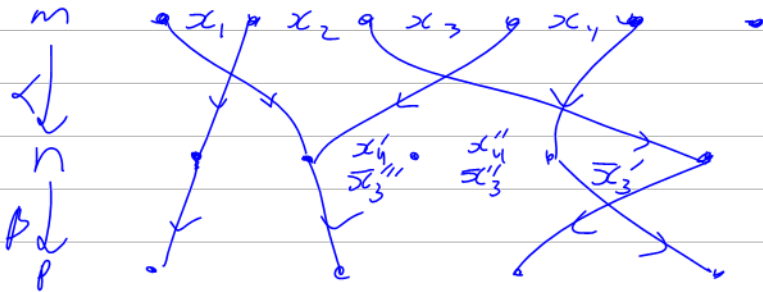
The Hopf algebra corresponding to N is $H = N(\bullet\bullet)$. The algebra structure of H comes from the algebra structure of $\bullet\bullet$ (which is the tensor product of two algebras \bullet), i.e. the product of H is



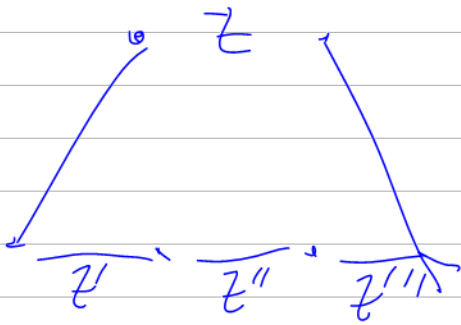
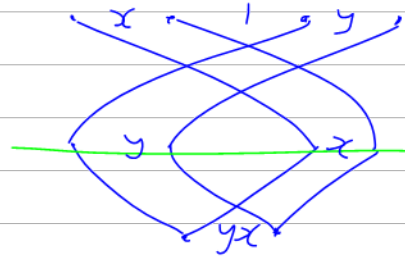
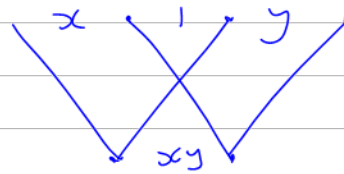
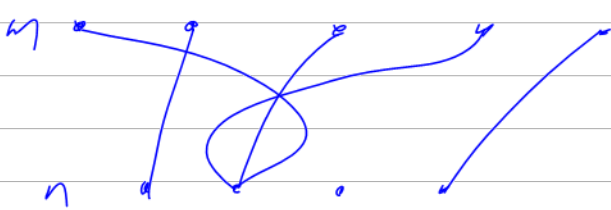
$$\text{Com}(m, n) = \text{map}(\underline{m} \rightarrow \underline{n})$$

Given a commutative Hopf algebra

$$\chi: \underline{m} \rightarrow \underline{n} \quad \chi_*: H^{m-1} \rightarrow H^{n-1}$$



Alg: Prop on NC algebras.

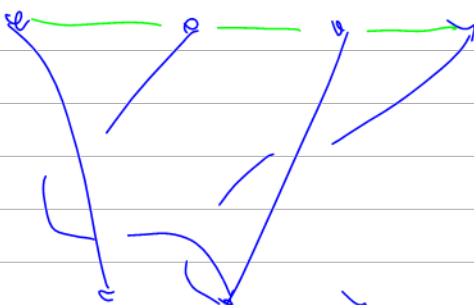
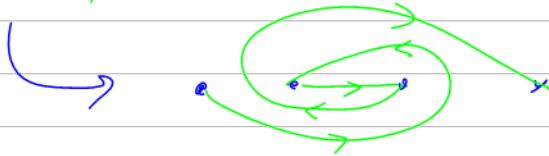


$$\Delta: H \rightarrow H \otimes H$$

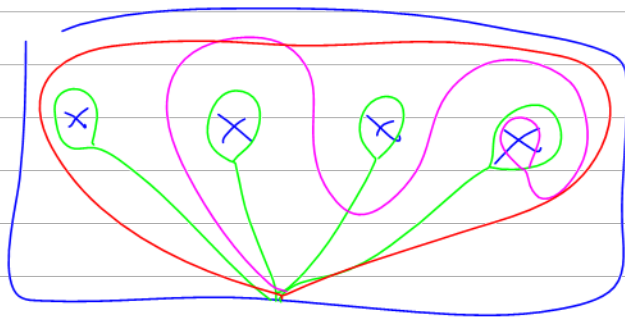
$$\Delta^2 = (\Delta \circ \Delta) = (\Delta \otimes 1) \Delta$$

$$: H \rightarrow H^{\otimes 3}$$

$$\Delta^2(z) = \sum z' \otimes z'' \otimes z'''$$

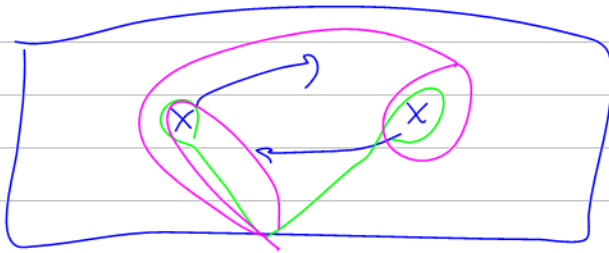


Artin rep $\alpha: B_n \hookrightarrow \text{Aut}(F_n)$



$x_i \rightarrow F_n$
 $\sigma: B_n \rightarrow S_n$
 $\alpha(B)(x_i) = \text{conjugate of } x_{\sigma_i}$

$$\alpha(B)(\prod x_i) = \prod x_i$$



$\alpha(\sigma_i)(x_k) =$
 $\begin{cases} x_i^{-1} x_{i+1} x_i & k=i \\ x_i & k=j \end{cases}$

Given any group G , $\{x_k\}$

$$B_n \hookrightarrow G^n \quad G^n = \text{Hom}(F_n, G)$$

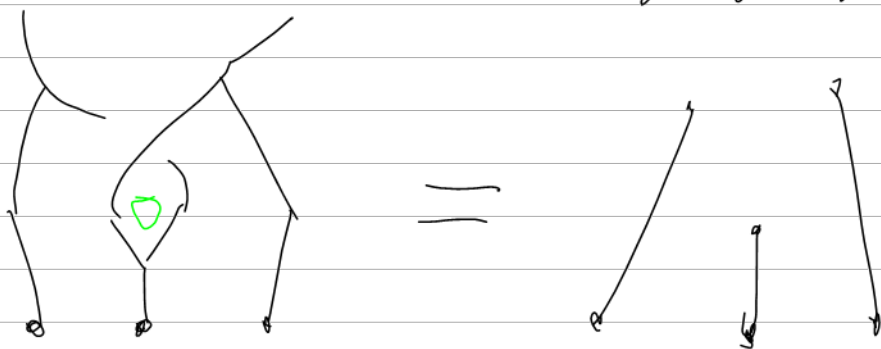
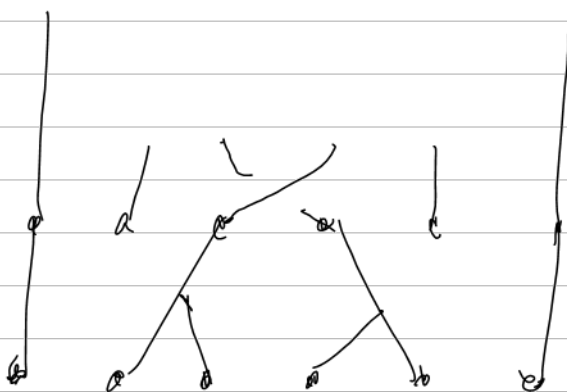
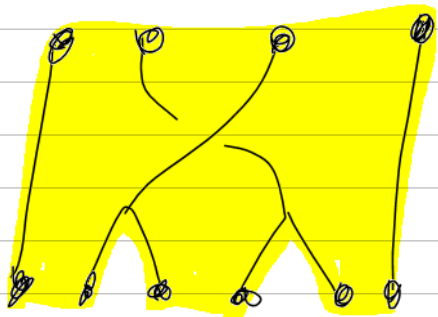
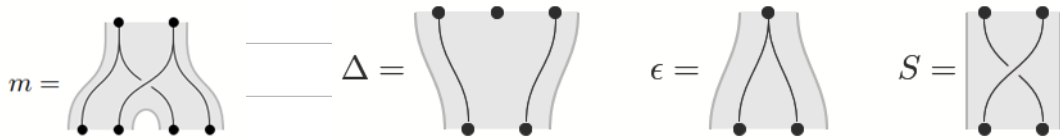
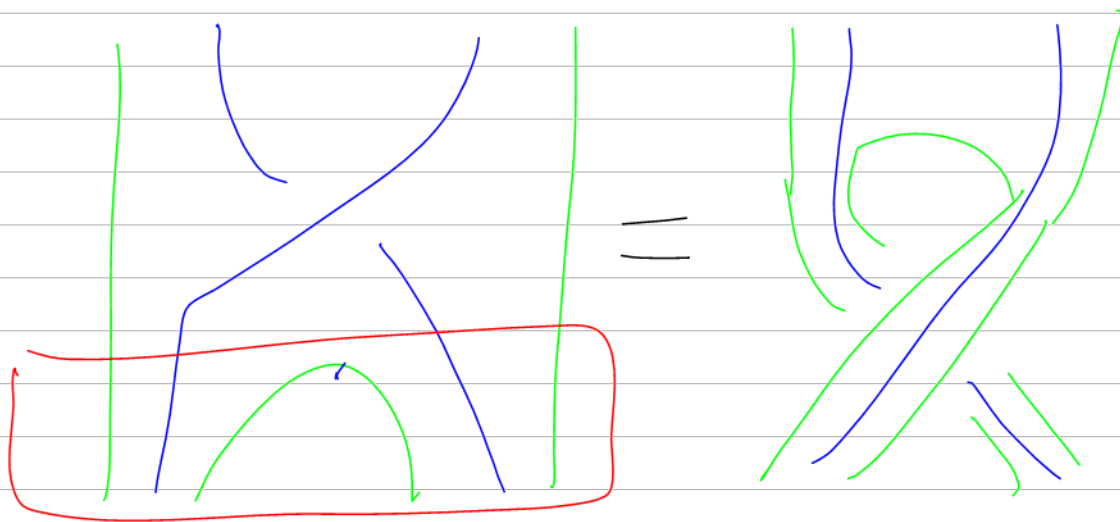
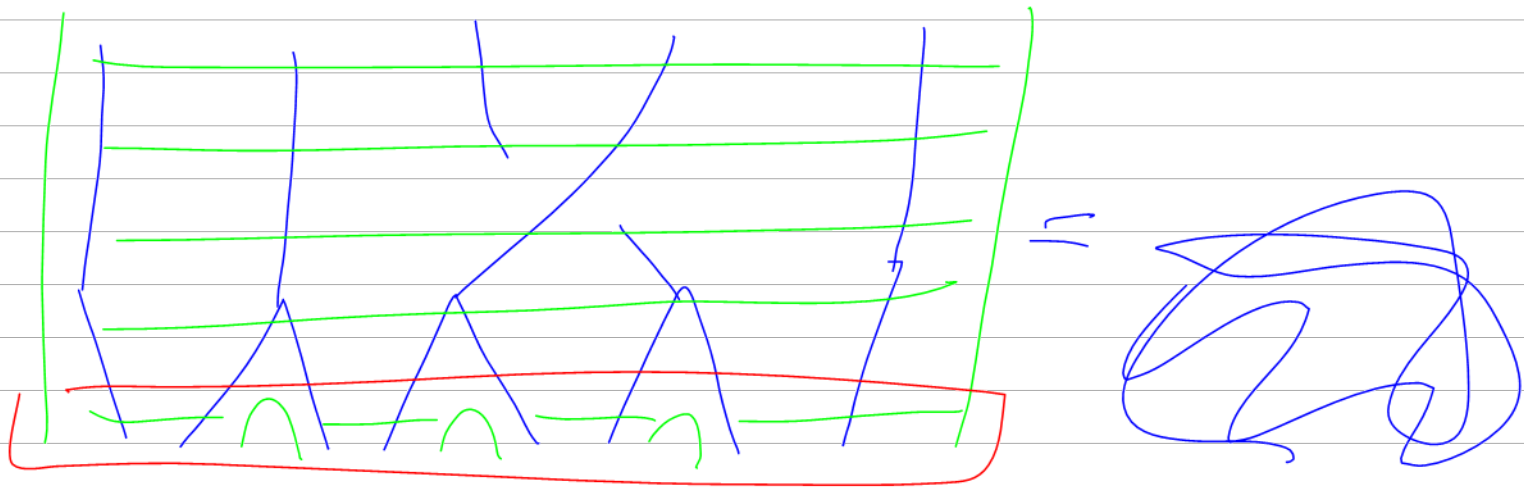
$$\sigma_i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, g_i^{-1} g_{i+1} g_i, g_i, \dots)$$

$$B_n \hookrightarrow Q^n \quad Q \text{ is a quandle}$$

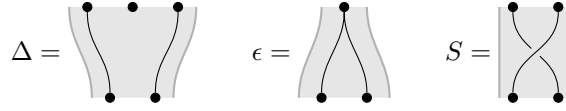
$$B_n \hookrightarrow H^n \quad \text{if } H \text{ is Hopf}$$

$$x^y \mapsto y \cancel{x} y \quad \Delta y = y' \otimes y''$$

$$S(y) x y$$



and the unit is the composition $1_e \rightarrow N(1) \rightarrow N(\bullet\bullet)$. The coproduct, the counit, and the antipode are given by



where we implicitly use the isomorphisms $N(\bullet^3) \cong N(\bullet\bullet)^{\otimes 2}$ and $N(\bullet) \cong 1_e$ given by the nerve condition.

The proof can be found in §6.1. Its main part is the construction of a suitable N out of H , which is a braided version of the above-given construction $H \mapsto N_H$ for commutative Hopf algebras.

4. INFINITESIMAL BRAIDINGS

4.1. Infinitesimally braided categories. If R is a commutative ring, an R -linear category is a category enriched over R -modules, i.e. a category such that $\text{Hom}(X, Y)$ is an R -module for any two objects X and Y and such that the composition of morphisms is R -bilinear.

An R -linear monoidal category is a category which is both R -linear and monoidal and such that the tensor product of morphisms is R -bilinear.

If \mathcal{C} is an R -linear category, let \mathcal{C}_ϵ be the $R[\epsilon]/(\epsilon^2)$ -linear category obtained by extension of scalars, i.e. \mathcal{C}_ϵ has the same objects as \mathcal{C} and

$$\text{Hom}_{\mathcal{C}_\epsilon}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)[\epsilon]/(\epsilon^2).$$

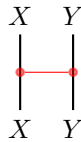
An R -infinitesimally braided monoidal category (R -iBMC) is a symmetric R -linear monoidal category \mathcal{C} with a natural transformation $t_{X,Y}: X \otimes Y \rightarrow X \otimes Y$ called infinitesimal braiding such that \mathcal{C}_ϵ with the modified braiding

$$\beta_{X,Y} := \sigma_{X,Y} \circ (1 + \epsilon t_{X,Y})$$

(where $\sigma_{X,Y}: X \otimes Y \rightarrow Y \otimes X$ is the symmetry in \mathcal{C}) is a braided monoidal category. Moreover, $t_{X,Y}$ is required to satisfy the symmetry condition

$$t_{Y,X} = \sigma_{X,Y} \circ t_{X,Y} \circ \sigma_{Y,X}.$$

The morphism $t_{X,Y}$ will be graphically represented by a horizontal chord



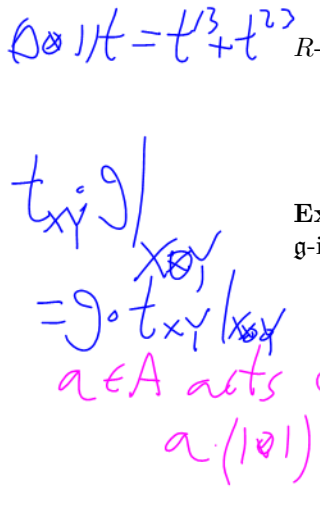
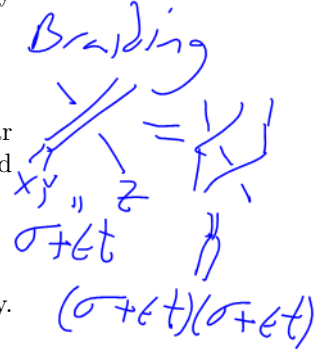
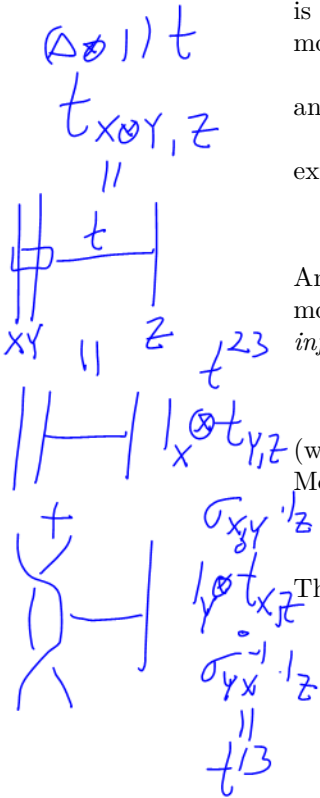
An i -braided lax/strong monoidal functor $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ between two R -iBMCs is an R -linear symmetric lax/strong monoidal functor such that

$$F(t_{X,Y}^{\mathcal{C}_1}) \circ c_{X,Y}^F = c_{X,Y}^F \circ t_{F(X),F(Y)}^{\mathcal{C}_2}.$$

Let us notice that any R -linear SMC becomes an R -iBMC if we set $t_{X,Y} = 0$.

Example 4.1 ([4]). If \mathfrak{g} is a Lie algebra over R and $t \in \mathfrak{g} \otimes \mathfrak{g}$ is symmetric and \mathfrak{g} -invariant then the symmetric monoidal category $U\mathfrak{g}\text{-mod}$ is R -i-braided via

$$t_{X,Y} = \rho_X \otimes \rho_Y(t) \in \text{End}(X \otimes Y).$$



$a \in A$ also acts on $A \otimes A \otimes A$

$$a(1 \otimes 1 \otimes 1)$$

\parallel

$$a(\Delta(1) \otimes 1)$$

\parallel

$$\Delta(a)[\Delta(1) \otimes 1]$$

\parallel

$$\sum a'_i \otimes a''_i (\Delta(1) \otimes 1)$$

\parallel

$$\sum \Delta(a'_i \cdot 1) \otimes a''_i$$

$$\parallel$$

$$(\Delta \otimes 1)(\Delta(a))$$

A

$\Delta \downarrow$

$A \otimes A$

$$\Delta(ba) \stackrel{?}{=} b \Delta(a)$$

$$\parallel$$

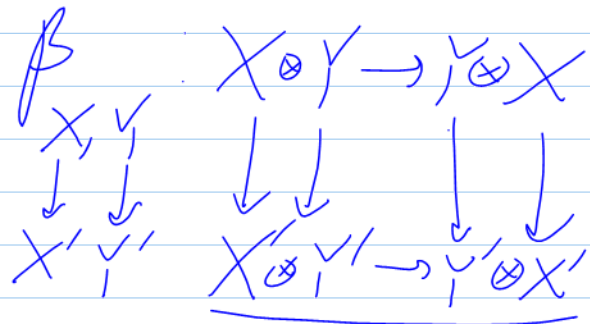
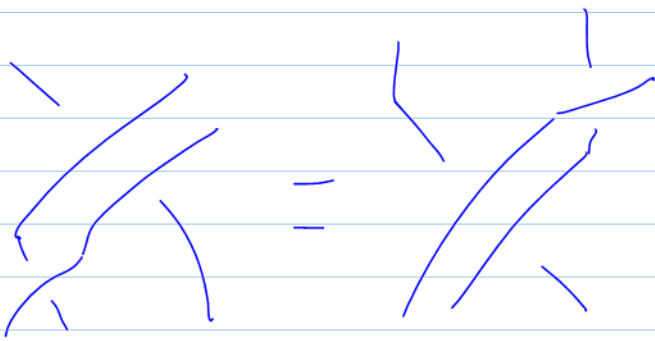
$$(ba)(1 \otimes 1) = b \cdot a(1 \otimes 1)$$

also $\Delta(1) = 1(1 \otimes 1) = 1 \otimes 1$

claim $a(V \otimes W) = \Delta(a)(V \otimes W)$

$$\parallel$$

$$[a(1 \otimes 1)](V \otimes W)$$



$$X \otimes Y \quad Z$$

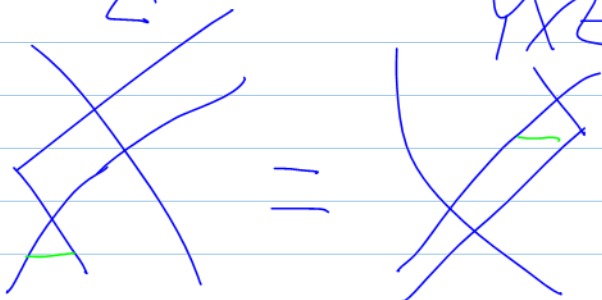
$$\downarrow \beta_{X,Y} \quad \downarrow$$

$$Y \otimes X \quad Z$$

$$XYZ \quad \beta_{X,Y,Z} \rightarrow ZXY$$

$$\downarrow \beta_{X,Y} \quad \downarrow \beta_{X,Y}$$

$$YXZ \quad \beta_{Y,X,Z} \rightarrow ZYX$$



4.2. Drinfeld associators. An iBMC is a SMC with a first order deformation of the symmetry to a braiding. A natural question is whether one can extend this first order deformation to a formal deformation.

If \mathcal{C} is an R -linear category, let \mathcal{C}_\hbar be the $R[[\hbar]]$ -linear category with $\text{Ob}(\mathcal{C}_\hbar) = \text{Ob}(\mathcal{C})$ and with

$$\text{Hom}_{\mathcal{C}_\hbar}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)[[\hbar]].$$

Likewise, if $F: \mathcal{C} \rightarrow \mathcal{C}'$ is an R -linear functor, let $F_\hbar: \mathcal{C}_\hbar \rightarrow \mathcal{C}'_\hbar$ be its (continuous) $R[[\hbar]]$ -linear extension, i.e.

$$F_\hbar\left(\sum f_n \hbar^n\right) = \sum F(f_n) \hbar^n, \quad f_n \in \text{Hom}_{\mathcal{C}}(X, Y).$$

Theorem 4.2 (Drinfeld [4]). *There is an element $\Phi \in \mathbb{Q}\langle\langle x, y \rangle\rangle$ (a Drinfeld associator), $\Phi = 1 + [x, y]/24 + \dots$, with the following property. If $\mathbb{Q} \subset R$ and if \mathcal{C} is a R -iBMC then \mathcal{C}_\hbar with the new braiding $\beta_{X,Y}^{new} = \sigma_{X,Y} \circ \exp(\hbar t_{X,Y}/2)$ and the new associativity constraint*

$$\gamma_{X,Y,Z}^{new} = \gamma_{X,Y,Z} \circ \Phi(\hbar t_{X,Y} \otimes \text{id}_Y, \hbar \gamma_{X,Y,Z}^{-1} \circ (\text{id}_X \otimes t_{Y,Z}) \circ \gamma_{X,Y,Z}),$$

(where $\gamma_{X,Y,Z}: (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ is the associativity constraint of \mathcal{C}) is a BMC which we shall denote by \mathcal{C}_\hbar^Φ .

If $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is an R -i-braided lax/strong monoidal functor then

$$F_\hbar: (\mathcal{C}_1)_\hbar^\Phi \rightarrow (\mathcal{C}_2)_\hbar^\Phi,$$

with the same coherence morphisms, is braided lax/strong monoidal.

Notice that

$$\beta_{X,Y}^{new} = \sigma_{X,Y} \circ (1 + \frac{\hbar}{2} t_{X,Y}) + O(\hbar^2), \quad \gamma_{X,Y,Z}^{new} = \gamma_{X,Y,Z} + O(\hbar^2).$$

Also notice that if $t_{X,Y} = 0$ for all objects $X, Y \in \mathcal{C}$ then $\mathcal{C}_\hbar^\Phi = \mathcal{C}_\hbar$ as a symmetric monoidal category, as then $\beta_{X,Y}^{new} = \sigma_{X,Y}$ and $\gamma_{X,Y,Z}^{new} = \gamma_{X,Y,Z}$.

4.3. Poisson algebras in infinitesimally braided categories. An (infinitesimally braided) *Poisson algebra* in an iBMC \mathcal{C} is a commutative algebra $A \in \mathcal{C}$ together with a biderivation

$$p: A \otimes A \rightarrow A$$

(the ‘‘Poisson bracket’’) satisfying the modified skew-symmetry

$$(5) \quad p + p \circ \sigma_{A,A} = m \circ t_{A,A}$$

(where $m: A \otimes A \rightarrow A$ is the product) and the modified Jacobi identity

$$(6) \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} - \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

(using the notation $p = \blacklozenge$ and $m = \blacktriangle$).

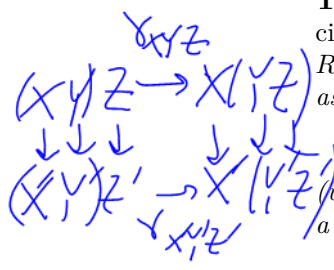
A commutative algebra $A \in \mathcal{C}$ is *strongly commutative* if $m \circ t_{A,A} = 0$. Then A with $p = 0$ is a Poisson algebra.

If $A \in \mathcal{C}$ is a Poisson algebra and if $F: \mathcal{C} \rightarrow \mathcal{C}'$ is an i-braided lax monoidal functor then $F(A) \in \mathcal{C}'$ is a Poisson algebra too.

If $A_1, A_2 \in \mathcal{C}$ are Poisson algebras then so is $A_1 \otimes A_2$, with the Poisson bracket

$$(7) \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

Under this tensor product, Poisson algebras in \mathcal{C} form a monoidal category.



$$\begin{aligned} \star(f,g) &= fg - e^{\frac{\hbar}{2}} gf \\ fg + gf - e^{\frac{\hbar}{2}}(gf + fg) \\ &= \end{aligned}$$

4.3. Poisson algebras in infinitesimally braided categories. An (infinitesimally braided) *Poisson algebra* in an iBMC \mathcal{C} is a commutative algebra $A \in \mathcal{C}$ together with a biderivation

$$p: A \otimes A \rightarrow A$$

(the ‘‘Poisson bracket’’) satisfying the modified skew-symmetry

$$(5) \quad p + p \circ \sigma_{A,A} = m \circ t_{A,A}$$

(where $m: A \otimes A \rightarrow A$ is the product) and the modified Jacobi identity

$$(6) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} - \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}$$

(using the notation $p = \blacktriangleleft$ and $m = \blacktriangleright$).

A commutative algebra $A \in \mathcal{C}$ is *strongly commutative* if $m \circ t_{A,A} = 0$. Then A with $p = 0$ is a Poisson algebra.

If $A \in \mathcal{C}$ is a Poisson algebra and if $F: \mathcal{C} \rightarrow \mathcal{C}'$ is an i-braided lax monoidal functor then $F(A) \in \mathcal{C}'$ is a Poisson algebra too.

If $A_1, A_2 \in \mathcal{C}$ are Poisson algebras then so is $A_1 \otimes A_2$, with the Poisson bracket

$$(7) \quad \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \\ \text{Diagram 11} \end{array}$$

Under this tensor product, Poisson algebras in \mathcal{C} form a monoidal category.

Example 4.3. If $\mathcal{C} = U\mathfrak{g}\text{-mod}$ is as in Example 4.1 and if $1/2 \in R$ then Poisson algebras in \mathcal{C} are the same as \mathfrak{g} -quasi-Poisson algebras defined in [1]. Namely if we split a Poisson bracket p on $A \in \mathcal{C}$ to its anti-symmetric and symmetric parts [Alekseev, Kosmann-Schwarzbach, Meinrenken](#)

$$p = \{, \} + \frac{1}{2} \rho_A \otimes \rho_A(t)$$

then $\{, \}$ is a \mathfrak{g} -invariant biderivation of the commutative algebra A satisfying

$$\{a, \{b, c\}\} + c.p. = -\frac{1}{4} \phi_A(a \otimes b \otimes c)$$

where $\phi_A: A^{\otimes 3} \rightarrow A$ is given by $\phi_A = m^{(3)} \circ \rho_A^{\otimes 3}(\phi)$, where $m^{(3)}: A^{\otimes 3} \rightarrow A$ is the product of 3 elements and $\phi = [t \otimes 1, 1 \otimes t] \in \wedge^3 \mathfrak{g} \subset (U\mathfrak{g})^{\otimes 3}$.

Tensor product of Poisson algebras then corresponds to the fusion product from *op. cit.*

The definition of Poisson algebras and of their tensor product come from the following fact. Suppose that $\mathbb{Q} \subset R$ and let A be an algebra in \mathcal{C}_\hbar^Φ with a product

$$m = \sum_{n=0}^{\infty} \hbar^n m_n \quad (m_n \in \text{Hom}_{\mathcal{C}}(A \otimes A, A))$$

such that m_0 is commutative. If we define $p \in \text{Hom}_{\mathcal{C}}(A \otimes A, A)$ via the braided commutator

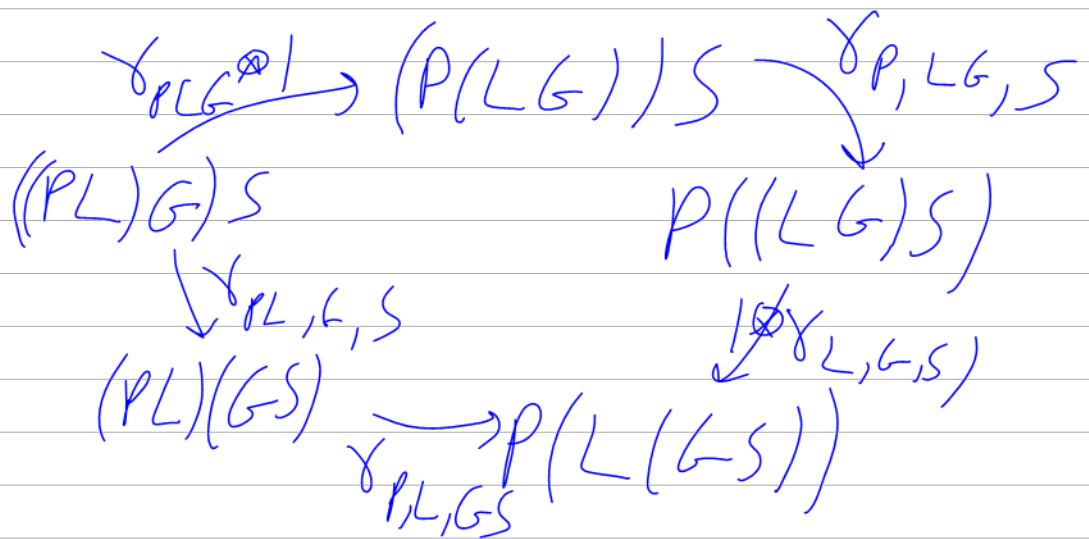
$$(8) \quad m - m \circ \beta_{A,A}^{-1} = \hbar p + O(\hbar^2), \text{ i.e. } p = m_1 - m_1 \circ \sigma_{A,A} + \frac{1}{2} m_0 \circ t_{A,A}$$

then m_0 and p make A to a Poisson algebra in the iBMC \mathcal{C} . We shall then say that m is a *quantization* of the Poisson bracket p .

Moreover, if A_1 and A_2 are two such algebras in \mathcal{C}_\hbar^Φ then the resulting Poisson bracket on $A_1 \otimes A_2$ is given by (7).

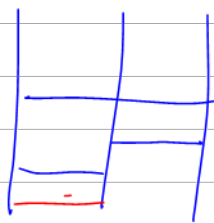
If $A \in \mathcal{C}$ is strongly commutative then it remains, with the same product (i.e. with $m = m_0$) and unit, a commutative algebra in \mathcal{C}_\hbar^Φ .

pretty little girls school



$$\gamma_{PLG} \otimes 1 // \gamma_{P, LG, S} // \gamma_{L, G, S}$$

= - - -




$$C = t_{12} + t_{13} + t_{23} \quad \text{is central.}$$

$$\sum_{i \neq j} t_{ij}$$

$$t_{13} = \frac{0}{0} - t_{12} - t_{23}$$

$$[C, [-, -]]$$

$$a \neq b \neq c \quad b \neq a \neq c$$


If \mathcal{C} is an R -linear SMC with $t_{X,Y} = 0$ (for all objects X, Y) then these notions reduce to the usual definition of Poisson algebras in a linear SMC, and to the usual definition of their tensor products.

Example 4.3. If $\mathcal{C} = U\mathfrak{g}\text{-mod}$ is as in Example 4.1 and if $1/2 \in R$ then Poisson algebras in \mathcal{C} are the same as \mathfrak{g} -quasi-Poisson algebras defined in [1]. Namely if we split a Poisson bracket p on $A \in \mathcal{C}$ to its anti-symmetric and symmetric parts [Aleksseev, Kosmann-Schwarzbach, Meinrenken](#)

$$p = \{, \} + \frac{1}{2}\rho_A \otimes \rho_A(t)$$

then $\{, \}$ is a \mathfrak{g} -invariant biderivation of the commutative algebra A satisfying

$$\{a, \{b, c\}\} + c.p. = -\frac{1}{4}\phi_A(a \otimes b \otimes c)$$

where $\phi_A: A^{\otimes 3} \rightarrow A$ is given by $\phi_A = m^{(3)} \circ \rho_A^{\otimes 3}(\phi)$, where $m^{(3)}: A^{\otimes 3} \rightarrow A$ is the product of 3 elements and $\phi = [t \otimes 1, 1 \otimes t] \in \bigwedge^3 \mathfrak{g} \subset (U\mathfrak{g})^{\otimes 3}$.

Tensor product of Poisson algebras then corresponds to the fusion product from *op. cit.*

The definition of Poisson algebras and of their tensor product come from the following fact. Suppose that $\mathbb{Q} \subset R$ and let A be an algebra in \mathcal{C}_\hbar^Φ with a product

$$m = \sum_{n=0}^{\infty} \hbar^n m_n \quad (m_n \in \text{Hom}_{\mathcal{C}}(A \otimes A, A))$$

such that m_0 is commutative. If we define $p \in \text{Hom}_{\mathcal{C}}(A \otimes A, A)$ via the braided commutator

$$(8) \quad m - m \circ \beta_{A,A}^{-1} = \hbar p + O(\hbar^2), \text{ i.e. } p = m_1 - m_1 \circ \sigma_{A,A} + \frac{1}{2}m_0 \circ t_{A,A}$$

then m_0 and p make A to a Poisson algebra in the iBMC \mathcal{C} . We shall then say that m is a *quantization* of the Poisson bracket p .

Moreover, if A_1 and A_2 are two such algebras in \mathcal{C}_\hbar^Φ then the resulting Poisson bracket on $A_1 \otimes A_2$ is given by (7).

If $A \in \mathcal{C}$ is strongly commutative then it remains, with the same product (i.e. with $m = m_0$) and unit, a commutative algebra in \mathcal{C}_\hbar^Φ .

4.4. Infinitesimally braided props. An *R-infinitesimally braided prop* (or *R-i-braided prop*) is a strict R -iBMC with objects \bullet^n , $n \geq 0$, with the tensor product of objects $\bullet^m \otimes \bullet^n = \bullet^{m+n}$, and with the unit $1 = \bullet^0$.

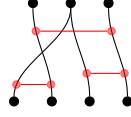
Let \mathcal{A}_n be the Drinfeld-Kohno algebra

$$\mathcal{A}_n := R \langle t_{ij}, 1 \leq i, j \leq n, i \neq j \mid t_{ij} = t_{ji}, [t_{ij} + t_{ik}, t_{jk}] = 0, [t_{ij}, t_{kl}] = 0 \text{ if } i, j, k, l \text{ are all different} \rangle$$

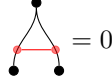
on which S_n acts by permuting the indices. For any R -i-braided prop \mathcal{P} we have a canonical map of R -algebras $S_n \times \mathcal{A}_n \rightarrow \text{End}(\bullet^n)$, with S_n coming from the symmetric monoidal category structure and t_{ij} being the chord connecting the i 'th and j 'th \bullet in \bullet^n .

Example 4.4. Let $\text{iCom}(R)$ be the R -i-braided prop of strongly commutative algebras (initial among strict R -iBMCs with a chosen strongly commutative algebra.) It is generated by morphisms $\underline{m}: \bullet\bullet \rightarrow \bullet$ and $\underline{\eta}: 1 \rightarrow \bullet$ modulo the relations saying that \underline{m} is strongly commutative and associative and that $\underline{\eta}$ is its unit. Morphisms

are then R -linear combinations of “maps with chords”



modulo the strong commutativity relation



and the Drinfeld-Kohno relations (the defining relations of \mathcal{A}_n).

5. POISSON HOPF ALGEBRAS AND THEIR QUANTIZATION

5.1. Poisson Hopf algebras in infinitesimally braided categories. An (infinitesimally braided) *Poisson Hopf algebra* in an iBMC \mathcal{C} is a commutative Hopf algebra $H \in \mathcal{C}$ together with a Poisson bracket $p: H \otimes H \rightarrow H$ such that the coproduct $\Delta: H \rightarrow H \otimes H$ is a morphism of Poisson algebras. In other words, p has to satisfy

(9)

Suppose that $\mathbb{Q} \subset R$ and fix an associator Φ . A *quantization* of a Poisson Hopf algebra

$$(H, m_0, \Delta_0, \eta, \epsilon, S_0, p)$$

in \mathcal{C} is a deformation of its product, coproduct, and antipode

$$m = \sum_{n=0}^{\infty} \hbar^n m_n \quad \Delta = \sum_{n=0}^{\infty} \hbar^n \Delta_n \quad S = \sum_{n=0}^{\infty} \hbar^n S_n$$

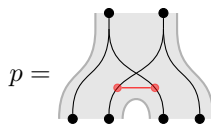
s.t. $(H, m, \Delta, \eta, \epsilon, S)$ is a Hopf algebra in the BMC $\mathcal{C}_{\hbar}^{\Phi}$ and such that (8) holds.

5.2. Nerves of Poisson Hopf algebras. For Poisson Hopf algebras we have the following analogue of Theorem 3.4.

Theorem 5.1. *Let \mathcal{C} be a R -iBMC. The category of Poisson Hopf algebras in \mathcal{C} is equivalent to the category of i -braided lax monoidal functors*

$$N: \text{iCom}(R) \rightarrow \mathcal{C}$$

satisfying the nerve condition. The commutative Hopf algebra corresponding to N is $H = N(\bullet\bullet)$, given by Proposition 3.3 (using the inclusion $\text{Com} \subset \text{iCom}(R)$), and the Poisson bracket on H comes from the Poisson bracket on $\bullet\bullet = \bullet \otimes \bullet$ (where \bullet has $p = 0$), i.e.



The proof can be found in §6.2.

5.3. The “universal quantization functor”.

Proposition 5.2. *There is a braided strong monoidal functor*

$$U_\Phi: \text{BrCom} \rightarrow \text{iCom}(\mathbb{Q})_{\hbar}^\Phi$$

whose reduction mod \hbar is the projection $\text{BrCom} \rightarrow \text{Com}$.

Proof. The algebra $\bullet \in \text{iCom}(\mathbb{Q})$ is strongly commutative, and thus it remains, with the same product and unit, a commutative algebra in $\text{iCom}(\mathbb{Q})_{\hbar}^\Phi$. This gives us a braided strong monoidal functor $\text{BrCom} \rightarrow \text{iCom}(\mathbb{Q})_{\hbar}^\Phi$ sending \bullet to \bullet , and its reduction mod \hbar has clearly the required property. \square

5.4. Quantization of Poisson Hopf algebras. Poisson Hopf algebras can be quantized as follows.

Theorem 5.3. *Let H be a Poisson Hopf algebra in an R -iBMC \mathcal{C} , and let*

$$N: \text{iCom}(R) \rightarrow \mathcal{C}$$

be its nerve. Suppose that $\mathbb{Q} \subset R$, and let Φ be a Drinfeld associator. Then the composed braided lax monoidal functor

$$\text{BrCom} \xrightarrow{U_\Phi} \text{iCom}(\mathbb{Q})_{\hbar}^\Phi \subset \text{iCom}(R)_{\hbar}^\Phi \xrightarrow{N_{\hbar}} \mathcal{C}_{\hbar}^\Phi$$

satisfies the nerve condition. The resulting Hopf algebra structure on $H \in \mathcal{C}_{\hbar}^\Phi$ (cf. Theorem 3.4) is a quantization of the Poisson Hopf algebra $H \in \mathcal{C}$.

Proof. To see that the composed functor \tilde{N} satisfies the nerve condition we reduce it mod \hbar , and get the composition

$$\text{BrCom} \xrightarrow{\text{projection}} \text{Com} \xrightarrow{N} \mathcal{C}$$

which satisfies the nerve condition because N does. Since \tilde{N} is its \hbar -deformation, it also satisfies the nerve condition (which says that some morphisms should be invertible).

We need to check that we get a quantization of the Poisson Hopf algebra. If m is the product on $H \in \mathcal{C}_{\hbar}^\Phi$, we have

$$m - m \circ \beta_{H,H}^{-1} = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} - \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} = \hbar \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \text{---} \end{array} + O(\hbar^2)$$

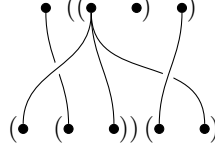
and thus indeed $m - m \circ \beta_{H,H}^{-1} = \hbar p + O(\hbar^2)$ where p is the Poisson bracket of the Poisson Hopf algebra $H \in \mathcal{C}$. \square

If $t_{X,Y} = 0$ for all objects $X, Y \in \mathcal{C}$ then the previous theorem gives us a quantization of H to a Hopf algebra in the SMC \mathcal{C}_{\hbar} .

Remark 5.4. To see how the quantization depends on choices it is better to use parenthesized versions of the props BrCom and iCom .

Following [2], if \mathcal{P} is a (ordinary or braided or R -i-braided) prop, let $\text{Pa}\mathcal{P}$ be its parenthesized version. It is a (symmetric or braided or R -i-braided) non-strict monoidal category equivalent to \mathcal{P} : its objects are fully parenthesised words in \bullet (i.e. expressions built out of \bullet using a non-associative product), and its morphisms are the morphisms of \mathcal{P} forgetting the parenthesization. A typical morphism in

PaBrCom looks as



Theorems 3.4 and 5.1 remain true with PaBrCom and PaiCom in place of BrCom and iCom. The advantage of this setup is that now we have a *unique* strict braided monoidal functor $\text{Pa}U_\Phi: \text{PaBrCom} \rightarrow \text{PaiCom}_\hbar^\Phi$ given by the commutative algebra structure on \bullet and thus the resulting quantization of Poisson Hopf algebras depends only on the associator Φ and not on the choice of U_Φ (which depends on a choice of a parenthesization). A similar argument also gives us an action of the Grothendieck-Teichmueller Lie algebra grr on the prop of Poisson Hopf algebras by derivations.

5.5. Cocommutative coalgebra enrichment. Let us now consider categories enriched over the category of cocommutative R -coalgebras; we shall call them *R -cc enriched categories*.

In an *R -cc enriched BMC* the tensor product of morphisms has to be compatible with the enrichment, and the braiding and the associativity and unit constraints are required to be grouplike, i.e. to be R -cc enriched natural transformations.

An *R -cc enriched iBMC* is an R -cc enriched SMC with an infinitesimal braiding such that $t_{X,Y}$ is primitive for every X, Y (which is equivalent to the deformed braiding $\sigma_{X,Y} \circ (1 + \epsilon t_{X,Y})$ being grouplike).

Example 5.5. The R -i-braided prop $\text{iCom}(R)$ is R -cc enriched, if we demand its generating morphisms $\underline{m}: \bullet\bullet \rightarrow \bullet$ and $\underline{\eta}: 1 \rightarrow \bullet$ to be grouplike.

Example 5.6. Let $\text{iPoissHopf}(R)$ be the R -i-braided prop of Poisson Hopf algebras. It is naturally R -cc enriched, with all the generating morphisms (product, coproduct, unit, counit, antipode) being grouplike, except for the Poisson bracket \underline{p} which satisfies $\Delta \underline{p} = \underline{p} \otimes \underline{m} + \underline{m} \otimes \underline{p}$.

Example 5.7. Let $\text{PoissHopf}(R)$ be the R -linear prop of Poisson Hopf algebras. It is naturally R -cc enriched, in the same way as $\text{iPoissHopf}(R)$.

The R -cc enrichment of the prop $\text{PoissHopf}(R)$ is equivalent to the fact that if H_1 and H_2 are Poisson Hopf algebras in an R -linear SMC \mathcal{C} , then so is $H_1 \otimes H_2$. Let us explain it in a way which works also for braided and i-braided props.

If \mathcal{C}_1 and \mathcal{C}_2 are R -linear categories, let $\mathcal{C}_1 \boxtimes_R \mathcal{C}_2$ be the R -linear category with

$$\text{Ob}(\mathcal{C}_1 \boxtimes_R \mathcal{C}_2) = \text{Ob}(\mathcal{C}_1) \times \text{Ob}(\mathcal{C}_2),$$

$$\text{Hom}_{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2}((X_1, X_2), (Y_1, Y_2)) = \text{Hom}_{\mathcal{C}_1}(X_1, Y_1) \otimes_R \text{Hom}_{\mathcal{C}_2}(X_2, Y_2).$$

If \mathcal{C}_1 and \mathcal{C}_2 are braided R -linear monoidal then so is $\mathcal{C}_1 \boxtimes_R \mathcal{C}_2$: \otimes is given componentwise, i.e. $(X_1, X_2) \otimes (Y_1, Y_2) = (X_1 \otimes Y_1, X_2 \otimes Y_2)$ and similarly for morphisms, $1_{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2} = (1_{\mathcal{C}_1}, 1_{\mathcal{C}_2})$, and we have

$$\gamma^{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2} = \gamma^{\mathcal{C}_1} \otimes_R \gamma^{\mathcal{C}_2} \quad \text{and} \quad \beta^{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2} = \beta^{\mathcal{C}_1} \otimes_R \beta^{\mathcal{C}_2}$$

and likewise for the unit morphisms. Moreover, if \mathcal{C}_1 and \mathcal{C}_2 are R -iBMCs then so is $\mathcal{C}_1 \boxtimes_R \mathcal{C}_2$, with the infinitesimal braiding

$$t^{\mathcal{C}_1 \boxtimes_R \mathcal{C}_2} = t^{\mathcal{C}_1} \otimes_R \text{id} + \text{id} \otimes_R t^{\mathcal{C}_2}.$$

If \mathcal{C} is an R -cc enriched BMC or iBMC then the functor

$$\Delta_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C} \boxtimes_R \mathcal{C},$$

given on objects by $\Delta_{\mathcal{C}}(X) = (X, X)$ and on morphisms by the coalgebra coproduct, is braided (or infinitesimally braided) strictly monoidal.

In particular, if \mathcal{P} is an R -cc enriched prop (or braided prop, or i-braided prop) and if $\mathcal{P} \rightarrow \mathcal{C}_{1,2}$ are two R -linear symmetric (or braided or i-braided) strong monoidal functors, then by composition with $\Delta_{\mathcal{P}}: \mathcal{P} \rightarrow \mathcal{P} \boxtimes_R \mathcal{P}$ we get a symmetric (or braided or i-braided) strong monoidal functor $\mathcal{P} \rightarrow \mathcal{C}_1 \boxtimes_R \mathcal{C}_2$. In other words, if $A_1 \in \mathcal{C}_1$ and $A_2 \in \mathcal{C}_2$ are \mathcal{P} -algebras, then so is $(A_1, A_2) \in \mathcal{C}_1 \boxtimes_R \mathcal{C}_2$.

Finally, if $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ is a SMC, the tensor product functor $\otimes: \mathcal{C} \boxtimes_R \mathcal{C} \rightarrow \mathcal{C}$ is a symmetric strong monoidal functor, and so $A_1 \otimes A_2 \in \mathcal{C}$, being the \otimes -image of the \mathcal{P} -algebra (A_1, A_2) , is a \mathcal{P} -algebra too.

Let us conclude with the compatibility of Drinfeld's construction of the BMC $\mathcal{C}_{\hbar}^{\Phi}$ with the product of categories \boxtimes_R and with R -cc enrichment. Drinfeld [4] showed that Φ can be chosen grouplike (wrt. the coproduct where x and y are primitive); from now on we shall assume that Φ is grouplike. This implies that

$$(\mathcal{C}_1 \boxtimes_R \mathcal{C}_2)_{\hbar}^{\Phi} = (\mathcal{C}_1)_{\hbar}^{\Phi} \boxtimes_{R[[\hbar]]} (\mathcal{C}_2)_{\hbar}^{\Phi} \quad (\text{as } R[[\hbar]]\text{-linear BMCs})$$

and also that if \mathcal{C} is an R -cc enriched iBMC then $\mathcal{C}_{\hbar}^{\Phi}$ is an $R[[\hbar]]$ -cc enriched BMC. (Here we abuse the notation slightly - our $R[[\hbar]]$ -modules are of the form $M[[\hbar]]$ for some R -module M , with the tensor product $M[[\hbar]] \otimes_{R[[\hbar]]} M'[[\hbar]] := (M \otimes_R M')[[\hbar]]$.)

5.6. Quantization in terms of props and compatibility with products. If \mathcal{D} an R -cc enriched BMC, let $\mathcal{D}^{gl} \subset \mathcal{D}$ be the its sub-BMC with all the objects but only the grouplike morphisms. If \mathcal{C} is a BMC we shall say that a braided monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is *grouplike* if its image and coherence morphisms are in \mathcal{D}^{gl} .

Example 5.8. The braided strong monoidal functor $U_{\Phi}: \text{BrCom} \rightarrow \text{iCom}(\mathbb{Q})_{\hbar}^{\Phi}$ is grouplike. This is because the commutative algebra $\bullet \in \text{iCom}(\mathbb{Q})_{\hbar}^{\Phi}$ is a commutative algebra in the subcategory $(\text{iCom}(\mathbb{Q})_{\hbar}^{\Phi})^{gl}$, as \underline{m} and \underline{n} are grouplike morphisms.

Let us now recast Theorem 5.3 into the language of props. Let Hopf be the prop of Hopf algebras and BrHopf the braided prop of Hopf algebras.

Theorem 5.9. *There is a grouplike braided strong monoidal functor*

$$Q_{\Phi}^{br}: \text{BrHopf} \rightarrow \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}, \quad Q_{\Phi}^{br}(\bullet^n) = \bullet^n \quad (\forall n)$$

such that the resulting Hopf algebra structure on $\bullet \in \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$ is a quantization of the Poisson Hopf algebra structure on $\bullet \in \text{iPoissHopf}(\mathbb{Q})$, and a grouplike morphism of props

$$Q_{\Phi}: \text{Hopf} \rightarrow \text{PoissHopf}(\mathbb{Q})_{\hbar}$$

with the same property.

Proof. Since \bullet is a Poisson-Hopf algebra in $\text{iPoissHopf}(\mathbb{Q})$, by Theorem 5.3 we get its quantization in $\text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$, and thus a braided strong monoidal functor

$$Q_{\Phi}^{br}: \text{BrHopf} \rightarrow \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}, \quad Q_{\Phi}^{br}(\bullet^n) = \bullet^n \quad (\forall n).$$

To see that Q_{Φ}^{br} is grouplike, let us notice that the nerve functor corresponding to the Poisson Hopf algebra $\bullet \in \text{iPoissHopf}(\mathbb{Q})$,

$$N: \text{iCom}(\mathbb{Q}) \rightarrow \text{iPoissHopf}(\mathbb{Q}),$$

is R -cc enriched i-braided lax monoidal. As a result

$$N_{\hbar}: \text{iCom}(\mathbb{Q})_{\hbar}^{\Phi} \rightarrow \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$$

is R -cc enriched braided lax monoidal, and thus its composition with U_{Φ} is a grouplike braided lax monoidal functor $\text{BrCom} \rightarrow \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$. As a result, all the defining operations of the resulting braided Hopf algebra $\bullet \in \text{iPoissHopf}(\mathbb{Q})_{\hbar}^{\Phi}$ are grouplike, and thus Q_{Φ}^{br} is grouplike.

Applying the same construction and reasoning to $\text{PoissHopf}(\mathbb{Q})$ we get a grouplike morphism of props $Q_{\Phi}: \text{Hopf} \rightarrow \text{PoissHopf}(\mathbb{Q})_{\hbar}$. \square

The previous theorem puts our quantization of Poisson Hopf algebras to the following form. If $H \in \mathcal{C}$ is a Poisson Hopf algebra in an R -iBMC with $\mathbb{Q} \subset R$, i.e. if we have a \mathbb{Q} -i-braided strong monoidal functor $F: \text{iPoisHopf}(\mathbb{Q}) \rightarrow \mathcal{C}$ with $F(\bullet) = H$, then we compose F_h with Q_Φ^{br} and thus make H to a Hopf algebra in \mathcal{C}_h^Φ . If the infinitesimal braiding of \mathcal{C} vanishes then we can work with PoisHopf and Q_Φ in place of iPoisHopf and Q_Φ^{br} .

If $H \in \mathcal{C}$ is a Poisson Hopf algebra, let $H_h^\Phi \in \mathcal{C}_h^\Phi$ denote the same object H with its new Hopf algebra structure. The fact that Q_Φ^{br} and Q_Φ are grouplike gives us the following.

Corollary 5.10. *Suppose that $H_1 \in \mathcal{C}_1$ and $H_2 \in \mathcal{C}_2$ are Poisson Hopf algebras. Then also $(H_1, H_2) \in \mathcal{C}_1 \boxtimes_R \mathcal{C}_2$ is a Poisson Hopf algebra, and we have the equality of Hopf algebras*

$$(H_{1_h^\Phi}, H_{2_h^\Phi}) = (H_1, H_2)_h^\Phi \in \mathcal{C}_{1_h^\Phi} \boxtimes_{R[[\hbar]]} \mathcal{C}_{2_h^\Phi} = (\mathcal{C}_1 \boxtimes_R \mathcal{C}_2)_h^\Phi.$$

If \mathcal{C} has vanishing infinitesimal braiding, and thus $\mathcal{C}_h^\Phi = \mathcal{C}_h$ is a SMC, and if $H_{1,2} \in \mathcal{C}$ are Poisson Hopf algebras, then also

$$(H_1 \otimes H_2)_h^\Phi = H_{1_h^\Phi} \otimes H_{2_h^\Phi}.$$

6. PROOFS OF THE NERVE THEOREMS

6.1. Nerves of braided Hopf algebras (proof of Theorem 3.4). We can suppose that \mathcal{C} is strict monoidal. We shall prove the theorem in the following form: there is an *isomorphism* of categories between Hopf algebras with invertible antipodes in \mathcal{C} and braided lax monoidal functors $N: \text{BrCom} \rightarrow \mathcal{C}$ which satisfy the *strict nerve condition*: the isomorphisms in the nerve condition are required to be identities.

Let us construct $N = N_H: \text{BrCom} \rightarrow \mathcal{C}$ out of a Hopf algebra H . By the strict nerve condition we have

$$N_H(\bullet^n) = H^{\otimes(n-1)}.$$

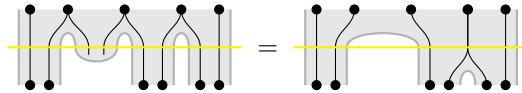
The strict nerve condition also implies that the coherence morphisms

$$N_H(\bullet^m) \otimes N_H(\bullet^n) \rightarrow N_H(\bullet^{m+n})$$

for $m, n > 0$ are

$$\text{id}_H^{\otimes(m-1)} \otimes \eta \otimes \text{id}_H^{\otimes(n-1)}: H^{\otimes(m-1)} \otimes H^{\otimes(n-1)} \rightarrow H^{\otimes(m+n-1)}$$

where $\eta: 1e \rightarrow H$ is the unit of H , as explained for the case $m = 2, n = 3$:



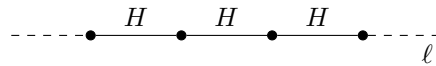
The remaining coherence morphisms are identities.

The main part of the proof is to describe the values of N_H on the morphisms of BrCom . Let us do it and illustrate it with the example

$$N_H\left(\begin{array}{c} \bullet & \bullet & \bullet \\ \diagdown & \diagup & \diagdown \\ \bullet & \bullet & \bullet \end{array}\right): H^{\otimes 3} \rightarrow H^{\otimes 2}.$$

If $\phi: \bullet^m \rightarrow \bullet^n$, we get $N_H(\phi): H^{\otimes(m-1)} \rightarrow H^{\otimes(n-1)}$ as follows:

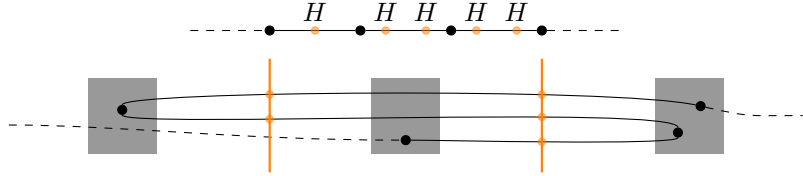
- ▷ We start by drawing m points on a horizontal line ℓ in the plane. There are $m - 1$ intervals between these points, each of them carrying one copy of H (all together representing $H^{\otimes(m-1)}$).



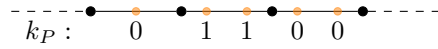
- ▷ The morphism (braid) ϕ can be interpreted as an isotopy of the plane between the identity and a diffeomorphism ψ of \mathbb{R}^2 , such that the isotopy stays the identity outside a large disk. The diffeomorphism ψ brings the m points to n different landing pads, again situated along a horizontal line.



- ▷ Now we draw $n - 1$ vertical lines between the landing pads. We suppose that they intersect $\psi(\ell)$ transversely. If the ψ -image of an interval meets the vertical lines k times, we apply the iterated coproduct $H \rightarrow H^{\otimes k}$ to this copy of H associated to that interval (if $k = 0$, we use the counit). At this point we have one H for each intersection point, i.e. we produced a morphism $H^{\otimes(m-1)} \rightarrow H^{\otimes M}$ where M is the number of the intersections.

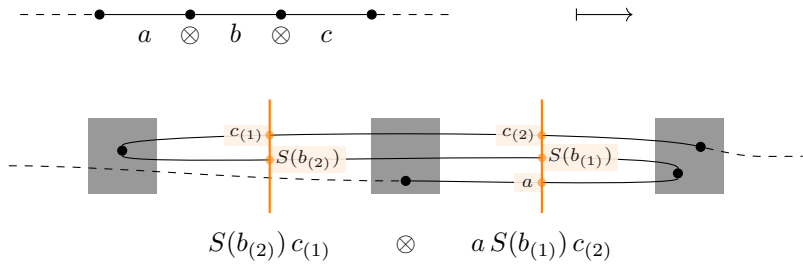


- ▷ For each of the intersection points P we find the total number of half-turns k_P when moving along $\psi(\ell)$ (turns in the positive direction are counted positively, in the negative direction negatively). Then we apply S^{k_P} to the H corresponding to P .

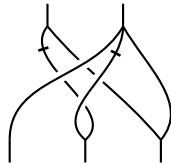


- ▷ We move each of the M H 's by the isotopy (using the braiding in \mathcal{C}), and finally multiply them along the vertical lines from the bottom to the top. Composing all these morphisms we get $N_H(\phi) : H^{\otimes(m-1)} \rightarrow H^{\otimes(n-1)}$. (The multiplication is done by first moving the intersection points to a horizontal position, with the bottom-most points on the left and the top-most on the right, and then multiplying the corresponding H 's.)

In our example we thus have (suppressing the braiding in \mathcal{C} in the formula)

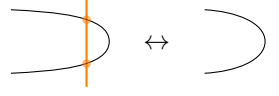


i.e. $N_H(\phi) : H^{\otimes 3} \rightarrow H^{\otimes 2}$ is



To see that $N_H(\phi)$ is well defined, i.e. independent of the details of the isotopy and of the transversals, we need to verify that it is invariant under moves of the

type



This invariance follows from the defining property of the antipode.

The fact that N_H is a functor, i.e. that $N_H(\phi_1 \circ \phi_2) = N_H(\phi_1) \circ N_H(\phi_2)$, follows from H being a bialgebra. Namely, let $m^{(p)}: H^{\otimes p} \rightarrow H$ denote the iterated product

$$m^{(0)} = \eta, \quad m^{(p+1)} = m \circ (m^{(p)} \otimes \text{id}_H).$$

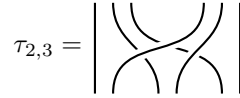
and $\Delta^{(q)}: H \rightarrow H^{\otimes q}$ the iterated coproduct

$$\Delta^{(0)} = \epsilon, \quad \Delta^{(q+1)} = (\Delta^{(q)} \otimes \text{id}_H) \circ \Delta.$$

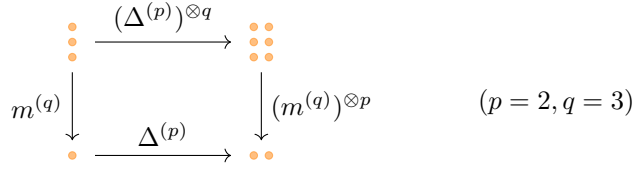
Then for each p, q we have

$$(10) \quad (m^{(q)})^{\otimes p} \circ \tau_{p,q} \circ (\Delta^{(p)})^{\otimes q} = \Delta^{(p)} \circ m^{(q)}$$

where $\tau_{p,q}: H^{\otimes pq} \rightarrow H^{\otimes pq}$ is given by the braid which reshuffles q groups, each made of p strands, into p groups, each made of q strands, by taking the first element of each group together, then second etc., using overcrossings only.



The expression for $N_H(\phi_1 \circ \phi_2)$ differs from the expression for $N_H(\phi_1) \circ N_H(\phi_2)$ only by replacing RHSs of (10) with LHSs, and so they are equal. (If we use the “ Δ is horizontal and m vertical” convention as in the construction of N_H then the identity (10) is the commutativity of the diagram

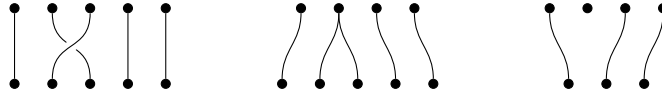


which is much more enlightening in our context.)

Finally, the fact that N_H is braided lax monoidal and that it satisfies the strict nerve condition is evident. Moreover the construction $H \mapsto N_H$ is clearly functorial.

If $N: \text{BrCom} \rightarrow \mathcal{C}$ satisfies the strict nerve condition then checking that $H_N := N(\bullet\bullet)$ (with the operations given in the theorem) is a Hopf algebra is a simple manipulation with diagrams; it is also a special case of Theorem 1 from [10]. Again the construction $N \mapsto H_N$ is functorial.

One easily checks that $H_{N_H} = H$ (as Hopf algebras). To finish the proof, we need to verify that $N_{H_N} = N$, and it is sufficient to do it for the morphisms of the type



as they generate BrCom . Checking it is again a straightforward calculation.

Remark 6.1. The functor N_H can alternatively be constructed as follows. If \mathcal{D} is the category of H -dimodules in \mathcal{C} (the Yetter-Drinfeld category) then H is naturally a commutative algebra in \mathcal{D} , and thus we have a braided strict monoidal functor $F: \text{BrCom} \rightarrow \mathcal{D}$, $F(\bullet^n) = H^{\otimes n}$. H -coinvariants (of the H -coaction on dimodules) is then a braided lax monoidal functor $\mathcal{D} \rightarrow \mathcal{C}$ (in general this functor might not

be defined on whole of \mathcal{D} , but it is defined on the image of F), and we can define N_H as the composition of these two braided lax monoidal functors.

In more detail, H -dimodules are objects X of \mathcal{C} equipped with a (left) H -action and H -coaction, such that for any H -module Y the \mathcal{C} -morphism

$$\tilde{\beta}_{X,Y} : X \otimes Y \rightarrow Y \otimes X, \quad \tilde{\beta}_{X,Y} = \begin{array}{c} Y \quad X \\ \diagdown \quad \diagup \\ X \quad Y \end{array}$$

is a morphism of H -modules. The category \mathcal{D} of H -dimodules is braided via $\tilde{\beta}$. H is an H -comodule via Δ and there is a unique (adjoint) H -module structure on H such that the action $H \otimes Y \rightarrow Y$ is an H -module morphism for every H -module Y . This makes H to an object of \mathcal{D} , and its algebra structure makes it to a commutative algebra in \mathcal{D} . We have

$$\tilde{\beta}_{H,H} = \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array}$$

and for any morphism $\phi: \bullet^p \rightarrow \bullet^q$ in BrCom we get $N_H(\phi) = \kappa_q \circ F(\phi) \circ \iota_p$ where the morphisms ι_p and κ_q are defined below (see Equation (12)).

6.2. Nerves of Poisson Hopf algebras (proof of Theorem 5.1). As in §6.1, we shall suppose that \mathcal{C} is strict monoidal and prove the theorem in the form saying that there is an isomorphism of categories between the category of Poisson Hopf algebras in \mathcal{C} and the category of i-braided lax monoidal functors $N: \text{iCom}(R) \rightarrow \mathcal{C}$ satisfying the strict nerve condition.

The main part of the proof is again a construction of a suitable $N_H: \text{iCom}(R) \rightarrow \mathcal{C}$ out of a Poisson Hopf algebra $H \in \mathcal{C}$. We already know the restriction of N_H to the subcategory $\text{Com} \subset \text{iCom}(R)$, which is the nerve of the commutative Hopf algebra H , and also its coherence morphisms (3).

We shall build the functor N_H in two steps. Let $r \in \text{End}_{\mathcal{C}}(H \otimes H)$ be

$$r = - \begin{array}{c} \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \end{array}$$

and let

$$\tau = r + r^{op} + t_{H,H}$$

where $r^{op} = \sigma_{H,H} \circ r \circ \sigma_{H,H}$.

Lemma 6.2. *There is a unique R -linear symmetric strict monoidal functor*

$$F_H: \text{iCom}(R) \rightarrow \mathcal{C}$$

satisfying $F_H(\bullet) = H$, $F_H(\underline{m}) = m$, $F_H(\underline{\eta}) = \eta$, $F_H(t_{\bullet,\bullet}) = \tau$, where m and η are the product and the unit of H .

On objects the functor F_H is given by $F_H(\bullet^n) = H^{\otimes n}$ and on a typical morphism by

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \quad \diagdown \\ \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \quad \diagup \end{array} \xrightarrow{F_H} \begin{array}{c} \diagdown \quad \diagup \quad \diagdown \\ \text{---} \quad \text{---} \quad \text{---} \\ \diagup \quad \diagdown \quad \diagup \end{array}$$

using the notation $\tau = \begin{array}{c} \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \end{array}$.

Out of F_H we now build N_H as follows. On objects it is defined by

$$(11) \quad N_H(\bullet^n) = H^{\otimes(n-1)}, \quad N_H(1) = 1_{\mathcal{C}}.$$

To define N_H on morphisms, let $\iota_n : N_H(\bullet^n) \rightarrow F_H(\bullet^n)$ be the morphisms

$$\iota_0 = \text{id} : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}, \quad \iota_1 = \eta : 1_{\mathcal{C}} \rightarrow H$$

and for $n > 1$

$$\iota_n = \begin{array}{c} \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ | \quad | \quad | \quad | \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \end{array} \quad (n = 4).$$

Then N_H is determined by the requirement that ι_n 's form a natural transformation $N_H \rightarrow F_H$. If $\kappa_n : F_H(\bullet^n) \rightarrow N_H(\bullet^n)$ satisfy $\kappa_n \circ \iota_n = \text{id}$ then for any morphism $\phi : \bullet^p \rightarrow \bullet^q$ in $\text{iCom}(R)$ we have

$$(12) \quad N_H(\phi) = \kappa_q \circ F_H(\phi) \circ \iota_p.$$

For the morphisms κ_n we can take $\kappa_0 = \text{id}$ and for $n > 0$

$$\kappa_n = \begin{array}{c} | \quad | \quad | \quad | \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ | \quad | \quad | \quad | \end{array} \quad (n = 5)$$

where we use the notation $\epsilon = \mathbf{1}$.

As promised, we have the following result.

Lemma 6.3. *N_H , defined on objects by (11) and on morphisms by (12), is an i -braided lax monoidal functor satisfying the strict nerve condition.*

Let us now prove the two lemmas.

Proof of Lemma 6.2. We need to verify the identities

$$\begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \end{array} = 0, \quad \begin{array}{c} \bullet \text{---} \text{---} \bullet \\ \diagup \quad \diagdown \\ | \quad | \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \diagup \quad \diagdown \\ | \quad | \\ \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \end{array}$$

and that τ_{ij} 's satisfy the Drinfeld-Kohno relations, where $\tau_{ij} \in \text{End}(H^{\otimes n})$ is τ acting on the i 'th and j 'th H in $H^{\otimes n}$.

The first identity follows from (5). The second identity, saying that τ is a biderivation, follows from the fact that both $t_{H,H}$ and r are biderivations.

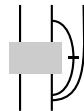
To show the Drinfeld-Kohno relations, let us first prove that τ is a morphism of H -comodules, i.e. that

$$(13) \quad \begin{array}{c} \diagdown \quad \diagup \\ | \quad | \\ \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \diagdown \quad \diagup \\ | \quad | \\ \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \end{array}$$

This follows from

$$\begin{array}{c} \diagdown \quad \diagup \\ | \quad | \\ \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} \bullet \\ | \\ \diagdown \quad \diagup \\ | \quad | \\ \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \bullet \\ | \\ \diagdown \quad \diagup \\ | \quad | \\ \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \end{array} + \begin{array}{c} \bullet \\ | \\ \diagdown \quad \diagup \\ | \quad | \\ \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \end{array}$$

which, in turn, is the identity (9) plugged into the gray rectangle of



Let us introduce the notation

$$\begin{array}{c} | \\ \diagdown \quad \diagup \\ | \end{array} = \begin{array}{c} \bullet \\ | \\ \diagdown \quad \diagup \\ | \end{array} \quad \text{so that} \quad r = - \begin{array}{c} | \\ \diagdown \quad \diagup \\ | \end{array}$$

The Jacobi identity (6) implies after a straightforward calculation

which, in turn, implies

$$(14) \quad \begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \end{array} + \begin{array}{c} \text{diagram 3} \\ \text{diagram 4} \end{array} = \begin{array}{c} \text{diagram 5} \\ \text{diagram 6} \end{array} + \begin{array}{c} \text{diagram 7} \\ \text{diagram 8} \end{array}$$

We can finally prove the non-trivial Drinfeld-Kohno relation $[\tau_{12} + \tau_{13}, \tau_{23}] = 0$. It is the sum of the identities

$$\begin{aligned} [r_{12} + r_{13}, \tau_{23}] &= 0 \\ [r_{12}^{op} + r_{13}^{op}, \tau_{23}] &= 0 \\ [t_{12} + t_{13}, \tau_{23}] &= 0. \end{aligned}$$

The first one is

and so it follows from (13). The second one follows from (14), and the third one from $t_{12} + t_{13} = t_{H, H \otimes H}$ and from the fact that t is a natural transformation. \square

Proof of Lemma 6.3. Let us make H to a left H -comodule via the coaction $\Delta: H \rightarrow H \otimes H$. Then $F_N(\bullet^n) = H^{\otimes n}$ becomes an H -comodule as well, and (13) implies that for any morphism $\phi: \bullet^p \rightarrow \bullet^q$ in $\text{iCom}(R)$ the morphism $F_H(\phi)$ is a morphism of H -comodules.

The morphism ι_n is the equalizer of

$$F_H(\bullet^n) \begin{array}{c} \xrightarrow{\text{coaction}} \\ \xrightarrow{\eta \otimes \text{id}} \end{array} H \otimes F_H(\bullet^n)$$

(i.e. it gives us the coinvariants of the coaction). This shows that N_H is indeed a functor.

The fact that N_H is symmetric lax monoidal with the coherence morphisms c_{\bullet^m, \bullet^n} given by (3) follows from the identity $\iota_{m+n} \circ c_{\bullet^m, \bullet^n} = \iota_m \otimes \iota_n$, which then means that ι_n 's form a monoidal natural transformation $N_H \rightarrow F_H$. The fact that N_H satisfies the strict nerve condition is evident. Finally, the fact that N_H is i -braided follows easily from seeing ι_n as the above-mentioned equalizer. \square

Once N_H is constructed, the rest of the proof is as in §6.1: The construction $H \mapsto N_H$ is functorial in H . If $N: \text{iCom}(R) \rightarrow \mathcal{C}$ satisfies the strict nerve condition then checking that $H_N := N(\bullet\bullet)$ (with the operations given in the theorem) is a Poisson Hopf algebra, i.e. that (9) is satisfied, is a simple manipulation with diagrams. Again the construction $N \mapsto H_N$ is functorial.

Let us verify that $H_{N_H} = H$ as Poisson Hopf algebras, i.e. that from N_H we get back the Poisson bracket p on H . By the definition of N_H and its coherence morphisms we have

It remains to check that $N_{H_N} = N$. The equality is true for morphisms from Com by Proposition 3.3 and one easily checks the equality for a single chord. These morphisms generate $\text{iCom}(R)$, and so the two functors are equal.

7. SEMICOMMUTATIVE HOPF ALGEBROIDS

The main idea of this paper was that groups (or commutative Hopf algebras) have symmetric nerves, and that deforming this symmetric structure to a braided structure provides their quantization. A natural idea is to extend it to other objects having symmetric nerves. The simplest option is to generalize groups to groupoids. The quantum object that we get is “quantum groupoids with a classical base”.

Definition 7.1 (Maltsonitis [8]). *A semicommutative Hopf algebroid over a commutative ring R is a pair of R -algebras B and H , with B commutative, with the following additional structure:*

- ▷ two R -algebra homomorphisms $\eta_L, \eta_R: B \rightarrow Z(H)$ (where $Z(H)$ is the center of H), thus making H to a B - B -bimodule
- ▷ B - B -bimodule morphisms $\Delta: H \rightarrow H \otimes_B H$ and $\epsilon: H \rightarrow B$ making H to a coalgebra in the monoidal category of B - B -bimodules, which are also algebra morphisms
- ▷ an invertible R -algebra anti-homomorphism $S: H \rightarrow H$ such that

$$\begin{aligned} S \circ \eta_L &= \eta_R & S \circ \eta_R &= \eta_L \\ m \circ (\text{id}_H \otimes_B S) \circ \Delta &= \epsilon \circ \eta_L, & m \circ (S \otimes_B \text{id}_H) \circ \Delta &= \epsilon \circ \eta_R \end{aligned}$$

where m is the product on H .

Remark 7.2. For simplicity we gave the definition in the category of R -modules. The definition and also the rest of this section can be generalized to BMCs where $H \otimes_B H$ and its iterations are well defined and behaved. The only change that needs to be done is that $\eta_L: B \rightarrow H$ should be a left-central and $\eta_R: B \rightarrow H$ a right central morphism of algebras.

A *commutative Hopf algebroid* corresponds to the case when H is commutative. A *semicommutative Poisson Hopf algebroid* is a commutative Hopf algebroid together with a Poisson bracket on H with the properties that the maps $\eta_{L,R}$ send B to the Poisson center of H and $\Delta: H \rightarrow H \otimes_B H$ and $\epsilon: H \rightarrow B$ are Poisson algebra morphisms, where the Poisson bracket on B is zero. The *quantization problem* is to deform m , Δ , and S (the rest of the structure is not deformed) so that we obtain a semicommutative Hopf algebroid and so that the deformed m is a quantization of the Poisson bracket on H .

This problem can again be solved using nerves. If $N: \text{BrCom} \rightarrow R\text{-mod}$ or $N: \text{Com} \rightarrow R\text{-mod}$ is a braided lax monoidal functor then $N(\bullet) \in R\text{-mod}$ is a commutative algebra and the map (4) is easily seen to factor through

$$(15) \quad \underbrace{N(\bullet\bullet) \otimes_{N(\bullet)} N(\bullet\bullet) \otimes_{N(\bullet)} \cdots \otimes_{N(\bullet)} N(\bullet\bullet)}_{n-1} \rightarrow N(\bullet^n).$$

We shall say that N satisfies the *Segal condition* [9] (or *groupoid nerve condition*) if (15) is an isomorphism for every n and if $c_1^N: R \rightarrow N(1)$ is an isomorphism.

Theorem 7.3. *The category of semicommutative Hopf algebroids over R is equivalent to the category of braided lax monoidal functors*

$$N: \text{BrCom} \rightarrow R\text{-mod}$$

satisfying the Segal condition. The same is true for commutative Hopf algebroids and symmetric lax monoidal functors

$$N: \text{Com} \rightarrow R\text{-mod}.$$

The Hopf algebroid corresponding to N is

$$H = N(\bullet\bullet) \quad B = N(\bullet).$$

The algebra structure on H and B comes from the algebra structure of $\bullet\bullet$ and \bullet . The remaining operations are given by

where we implicitly use the isomorphism $N(\bullet^3) \cong N(\bullet\bullet) \otimes_{N(\bullet)} N(\bullet\bullet)$ given by the Segal condition.

Finally, semicommutative Poisson Hopf algebroids over R are equivalent to i -braided lax monoidal functors

$$N: \text{iCom}(R) \rightarrow R\text{-mod}$$

satisfying the Segal condition. The Poisson bracket on H comes from the Poisson bracket on $\bullet\bullet$.

The theorem can be proven by a suitable modification of the proofs in §6; we leave the details to the reader.

If $\mathbb{Q} \subset R$, we get from this theorem immediately a solution of the quantization problem: If $N: \text{iCom}(R) \rightarrow R\text{-mod}$ is the nerve of a semicommutative Poisson Hopf algebroid then the composition of braided lax monoidal functors

$$\text{BrCom} \xrightarrow{U_\hbar} \text{iCom}(\mathbb{Q})_{\hbar}^{\Phi} \subset \text{iCom}(R)_{\hbar}^{\Phi} \xrightarrow{N_{\hbar}} R\text{-mod}_{\hbar}$$

satisfies the Segal condition and thus gives us a semicommutative Hopf algebroid.

An interesting question remains whether our method can be applied for quantization of other objects with symmetric nerves (higher groups or groupoids).

REFERENCES

- [1] A. Alekseev, Y. Kosmann-Schwarzbach, E. Meinrenken, Quasi-Poisson manifolds, *Canad. J. Math.* 54 (2002), no. 1, 3–29.
- [2] D. Bar-Natan, On Associators and the Grothendieck-Teichmüller Group I. *Selecta Mathematica* 4 (1998), 183–212.
- [3] V. Drinfeld, Quantum groups, *Proceedings of the International Congress of Mathematicians 1986*, Vol. 1, 798–820, AMS 1987
- [4] V. Drinfeld, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\mathbb{Q}/\mathbb{Q})$, *Algebra i Analiz* 2 (1990), no. 4, 149–181.
- [5] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras I. *Selecta Mathematica* 2 (1996), 1–41.
- [6] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics 5 (second ed.). Springer. ISBN 0-387-98403-8.
- [7] S. Majid. *Algebras and Hopf algebras in braided categories*. In: *Advances in Hopf algebras*, *Lecture Notes in Pure and Appl. Math.* 158, 1994.
- [8] G. Maltsiniotis, Groupoïdes quantiques, *Comptes R Rendus Acad. Sci. Paris* 314, pp. 249–252 (1992)
- [9] G. Segal, Classifying spaces and spectral sequences, *Inst. Hautes Études Sci. Publ. Math.*, vol. 34, pp. 105–112 (1968)
- [10] P. Ševera, Quantization of Lie bialgebras revisited. *Selecta Mathematica* 22 (2016), 1563–1581.

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