Lemma 10.3

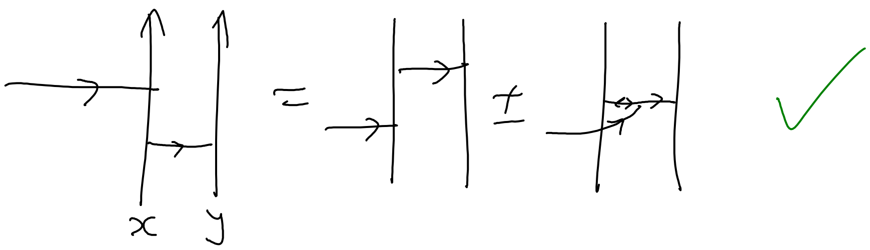
October-03-10
11:10 AM

Lemma 10.3. *(i) The map $r : M_- \otimes M_- \to M_- \otimes M_-$ is acyclic.*
(ii) The maps $r, r^{op} : M_+^* \hat{\otimes} M_- \to M_+^* \hat{\otimes} M_-$ are acyclic.

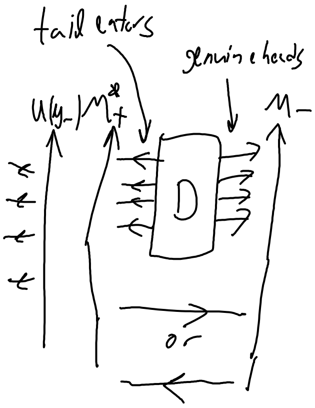
Proof. (i) For any nonnegative integers m, n consider the mapping $\mathfrak{g}_+^{\otimes m} \otimes \mathfrak{g}_+^{\otimes n} \to S\mathfrak{g}_+ \otimes S\mathfrak{g}_+$, given by

$$x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n \to r(x_1 \ldots x_n 1_- \otimes y_1 \ldots y_m 1_-). \tag{10.2}$$

We need to show that this mapping is an acyclic function. We can do this by induction in $N = m + n$. If $N = 0$, the operator is zero and the statement is clear. Assume the statement is proved for $N = K - 1$ and let us prove it for $N = K$. Using the relation $[x \otimes 1 + 1 \otimes x, r] = \delta(x)$, $x \in \mathfrak{g}_+$, we can reduce the question to the case $m = K, n = 0$. In this case, the map is again zero, Q.E.D.



$r, r^{op} : M_+^* \otimes M_- \circlearrowleft$ are acyclic:



(ii) By the same reasoning as in (i), we get the statement for r. For r^{op}, we reduce the question to proving that the map $M_+^* \to M_+^* \hat{\otimes} M_-$ given by $v \to r^{op}(v \otimes 1_-)$ is acyclic.

Let $u = \mathrm{Sym}(y_1 \ldots y_m) 1_+ \in M_+$, $y_1, \ldots, y_m \in \mathfrak{g}_-$. Let us compute the expression

$$X = (u \otimes 1)(r^{op}(v \otimes 1_-)) \in M_-.$$

We get

$$X = -(r^{op}(u \otimes 1))(v \otimes 1_-) = \sum_i \langle L(b^i, y_1, \ldots, y_m) 1_+, v \rangle a_i 1_-, \tag{10.3}$$

where a_i, b^i are dual bases of $\mathfrak{g}_+, \mathfrak{g}_-$, and L is a polynomial of commutators of $b^i, y_1, \ldots, y_m$ over $\mathbb{Q}$ which is symmetric in $b_i, y_1, \ldots, y_m$ and depends only on m.

Using the duality of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, from (10.3) we get

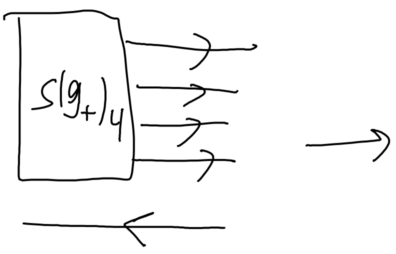
$$X = \sum_i \langle b^i \otimes y_1 \otimes \ldots \otimes y_m, D_L(v) \rangle a_i 1_- \tag{10.4}$$

where $D_L(v) \in S\mathfrak{g}_+$ is a linear combination of iterated cocommutators applied to v. This implies that $r^{op}(v \otimes 1_-)$ is a linear combination of iterated cocommutators applied to v, so the map $v \to r^{op}(v \otimes 1_-)$ is acyclic. $\square$

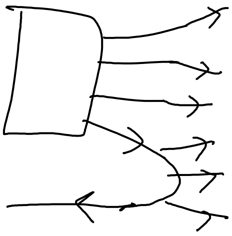
$\xi_+ : S(\mathfrak{g}_-) \to M_+$ $(U(\mathfrak{g}_-))$

So $\xi_+^* : M_+^* \to S(\mathfrak{g}_+)$ by "PBW then sum of gluings".

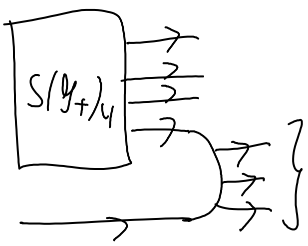
Q What is the $\mathfrak{g}$-modules structure on $S(\mathfrak{g}_+)$ compatible with the above?



a big sum of co-bracket iterations:

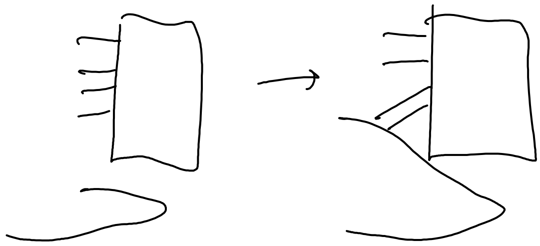


More work needed.

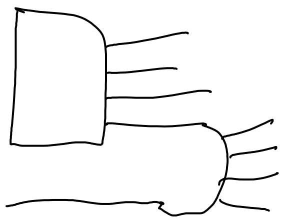


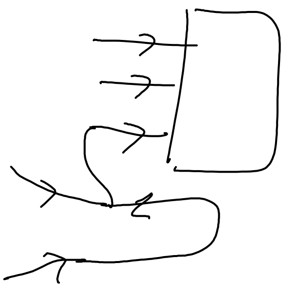
many cobs, one bra.

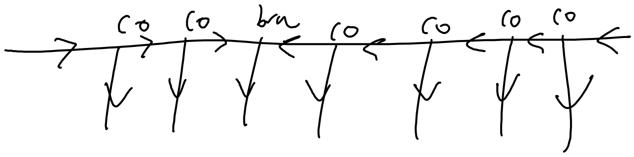
Action:



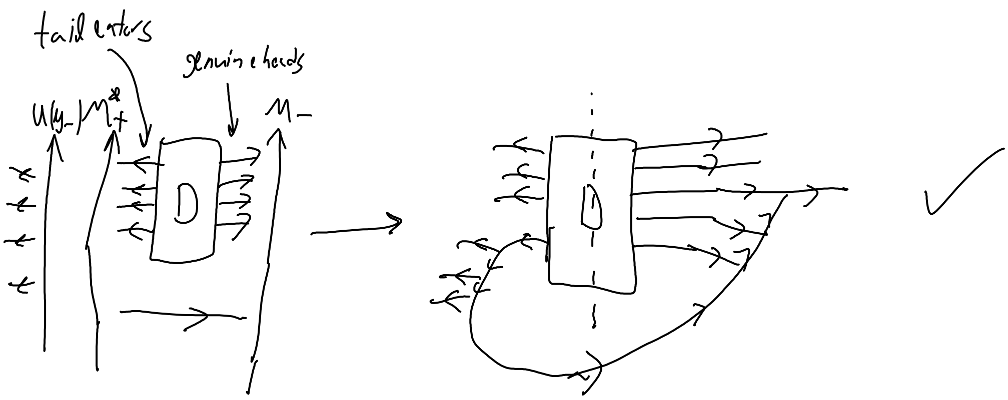
Dual action:

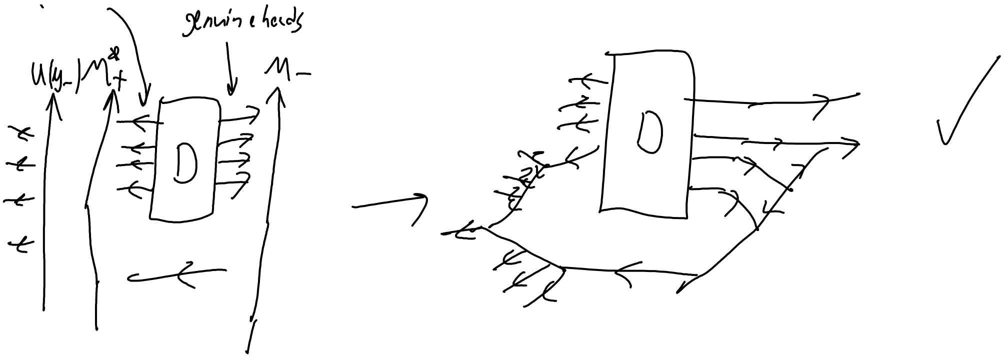






So —




$U(\mathfrak{g}_-)M_+^*$
genuine heads
M_-
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