Etingof-Kazhdan QLB-I, Part II

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states that quasitriangular quantization of Section 6.1 is a functor from the category of quasitriangular Lie bialgebras to the category of quasitriangular topological Hopf algebras over $k[[h]]$. Thus, τ defines a morphism $\hat{\tau} : U_h^{qt}(\mathfrak{g}) \to U_h^{qt}(\mathfrak{a})$. On the other hand, $U_h(\mathfrak{a})$ was constructed as a subalgebra in $U_h^{qt}(\mathfrak{g})$, so we have an embedding $\eta : U_h(\mathfrak{a}) \to U_h^{qt}(\mathfrak{g})$. Consider the morphism $\tau \circ \eta : U_h(\mathfrak{a}) \to U_h^{qt}(\mathfrak{a})$. This morphism is an isomorphism since it equals to 1 modulo h. The theorem is proved. $\square$

Remark. An analogous theorem holds for infinite dimensional Lie bialgebras. Namely, the "usual" quantization of $\mathfrak{a}$ defined in Section 9 is isomorphic to its quasitriangular quantization. The proof is analogous to the finite dimensional case.

6.3. *Representations of $U_h(\mathfrak{g})$.*

Let $\mathfrak{a}$ be a quasitriangular Lie bialgebra (not necessarily finite dimensional). By a representation of $U_h(\mathfrak{a})$ we mean a topologically free $k[[h]]$-module V together with a homomorphism $\pi_V : U_h(\mathfrak{a}) \to \mathrm{End}_{k[[h]]}V$. Representations of $U_h(\mathfrak{g})$ form a braided tensor category, with the trivial associativity morphism and braiding defined by the R-matrix. Denote this category by $\mathcal{R}$.

The functor $F : \mathcal{M}_\mathfrak{a} \to \mathcal{A}$ can be regarded as a functor from $\mathcal{M}_\mathfrak{a}$ to $\mathcal{R}$, since for any $W \in \mathcal{M}_\mathfrak{a}$ the $k[[h]]$-module $F(W)$ is equipped with a natural action of $U_h(\mathfrak{g})$. We denote this new functor also by F. This functor inherits the tensor structure defined by the maps J_{VW}.

Theorem 6.5. *The functor F defines an equivalence of braided tensor categories $\mathcal{M}_\mathfrak{a} \to \mathcal{R}$.*

Proof. The theorem follows from the definition of the functor F, the algebra $U_h(\mathfrak{g})$ and the R-matrix R. $\square$

Part II

7. Drinfeld category for an arbitrary Lie bialgebra.

7.1. *Topological vector spaces.* Recall the definition of the product topology. Let S be a set, T a topological space, and T^S the space of functions from S to T. This space has a natural weak topology, which is the weakest of the topologies in which all the evaluation maps $T^S \to T$, $f \to f(s)$, are continuous. Namely, let B be a basis of the topology on T. For any integer $n \geq 1$, elements $s_1, ..., s_n \in S$, and open sets $U_1, ..., U_n \in B$, define $V(s_1, ..., s_n, U_1, ..., U_n) = \{f \in T^S : f(s_i) \in U_i, i = 1, ..., n\}$. Let $\mathcal{B}$ be the collection of all such sets V. This is a basis of a topology on T^S which is called the weak topology. The obtained topological space is the product of copies of T corresponding to elements of S. If X is any subset in T^S, the weak topology on T^S induces a topology on X. We will call it the weak topology as well.

why this digression?

Let k be a field of characteristic zero with the discrete topology. Let V be a topological vector space over k. The topology on V is called linear if open subspaces of V form a basis of neighborhoods of 0.

Remark. It is clear that in any topological vector space, an open subspace is ?? also closed.

Let V be a topological vector space over k with linear topology. V is called separated if the map $V \to \varprojlim(V/U)$ is a monomorphism, where U runs over open subspaces of V.

Topology on all vector spaces we consider in this paper will be linear and separated. so we will say "topological vector space" for "separated topological vector space with linear topology".

Is it ok to just imagine filtered v.s. with the topology coming from the filtration?

Let M, N be topological vector spaces over k. We denote by $\mathrm{Hom}_k(M, N)$ the space of continuous linear operators from M to N, equipped with the weak topology. It is clear that a basis of neighborhoods of zero in $\mathrm{Hom}_k(M, N)$ is generated by sets of the form $\{A \in \mathrm{Hom}_k(M, N) : Av \in U\}$, where $v \in M$, and $U \subset N$ is an open set.

"$\phi \in \mathrm{Hom}_k(M, N)$ can eat finite much degree"?

In particular, if $N = k$ with the discrete topology, the space $\mathrm{Hom}_k(M, N)$ is the space of all continuous linear functionals on M, which we denote by M^*. It is clear that a basis of neighborhoods of zero in M^* consists of orthogonal complements of finite-dimensional subspaces in M. In particular, if M is discrete then the canonical embedding $M \to (M^*)^*$ is an isomorphism of linear spaces. However, if M is infinite-dimensional, this embedding is not an isomorphism of topological vector spaces since the space $(M^*)^*$ is not discrete.

7.2 *Complete vector spaces*

Let V be a topological vector space over k. V is called complete if the map $V \to \varprojlim(V/U)$ is a epimorphism, where U runs over open subspaces of V.

In particular, if a complete space M has a countable basis of neighborhoods of 0, then there exists a filtration $M = M_0 \supset M_1 \supset ...$, such that $\cap_{n \geq 0} M_n = 0$, and $\{M_n\}$ is a basis of neighborhoods of zero in M. In this case $M = \lim_{n \to \infty} M/M_n$.

Examples. 1. Any discrete vector space is complete.

2. If V is a discrete vector space then the topological space $M = V[[h]]$ of formal power series in h with coefficients in V is a complete vector space.

Let V be a complete vector space, $U \subset V$ an open subspace. Then U is complete and V/U is discrete.

Let V, W be complete vector spaces. Consider the space $V \hat{\otimes} W = \varprojlim V/V_1 \otimes W/W_1$, where the projective limit is taken over open subspaces $V_1 \subset V$, $W_1 \subset W$. It is easy to see that $V \hat{\otimes} W$ is a complete vector space. We call the operation $\hat{\otimes}$ the completed tensor product.

A basis of neighborhoods of 0 in $V \hat{\otimes} W$ is the collection of subspaces $V \hat{\otimes} W_1 + V_1 \hat{\otimes} W$, where V_1, W_1 are open subspaces in V, W.

Example. Let V be a discrete space. Then $V \hat{\otimes} k[[h]] = V[[h]]$.

Complete vector spaces form an additive category in which morphisms are continuous linear operators. This category, equipped with tensor product $\hat{\otimes}$, is a strict symmetric tensor category.

7.3. *Equicontinuous $\mathfrak{g}$-modules.*

Let M be a topological vector space over k, and $\{A_x, x \in X\}$ be a family of elements of $\mathrm{End}M$. We say that the family $\{A_x\}$ is equicontinuous if for every neighborhood of the origin $U \subset M$ there exists another neighborhood of the origin $U' \subset M$ such that $A_x U' = U$ for all $x \in X$. For example, if M is complete and $A \in \mathrm{End}M$ is any continuous linear operator, then $\{\lambda A, \lambda \in k\}$ is equicontinuous.

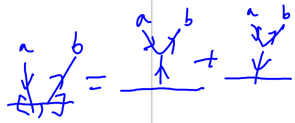
$=$ or $\subset$?

Fix a topological Lie algebra $\mathfrak{g}$.

Definition. Let M be a complete vector space. We say that M is an equicontinuous $\mathfrak{g}$-module if one is given a continuous homomorphism of topological Lie algebras $\pi : \mathfrak{g} \to \mathrm{End}M$, such that the family of operators $\pi(g), g \in \mathfrak{g}$, is equicontinuous.

Example. If M is a complete vector space with a trivial $\mathfrak{g}$-module structure then M is an equicontinuous $\mathfrak{g}$-module.

Let V, W be equicontinuous $\mathfrak{g}$-modules. It is easy to check that $V\hat{\otimes}W$ has a natural structure of an equicontinuous $\mathfrak{g}$-module. Moreover, $(V\hat{\otimes}W)\hat{\otimes}U$ is naturally identified with $V\hat{\otimes}(W\hat{\otimes}U)$ for any equicontinuous $\mathfrak{g}$-modules V, W, U. This means that the category of equicontinuous $\mathfrak{g}$-modules, where morphisms are continuous homomorphisms, is a monoidal category. This category is symmetric since the objects $V\hat{\otimes}W$ and $W\hat{\otimes}V$ are identified by the permutation of components. We denote this category by $\mathcal{M}_0^e$.



7.4. *Lie bialgebras and Manin triples.* Let $\mathfrak{a}$ be a Lie bialgebra over k. We will regard $\mathfrak{a}$ as a topological Lie algebra with the discrete topology. Let $\mathfrak{a}^*$ be the full dual space to $\mathfrak{a}$. The cocommutator defines a Lie bracket on $\mathfrak{a}^*$ which is continuous in the weak topology, so $\mathfrak{a}^*$ has a natural structure of a topological Lie algebra.

Furthermore, the space $\mathfrak{a} \oplus \mathfrak{a}^*$ has a natural topology, and the Lie bracket on $\mathfrak{g}$ defined by (1.1) is continuous in this topology.

$a \in \mathfrak{a}$ $b \in \mathfrak{a}^*$

(1.1) $[a,b] = (ad^* a)b - (1 \otimes b)(\delta(a))$

$\hat{\mathfrak{a}}^* \qquad \hat{\mathfrak{a}}$

Let $\mathfrak{g}$ be a Lie algebra with a nondegenerate invariant inner product $\langle , \rangle$. So far we have no topology on $\mathfrak{g}$. Let $\mathfrak{g}_+, \mathfrak{g}_-$ be isotropic Lie subalgebras in $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space. The inner product $\langle , \rangle$ defines an embedding $\mathfrak{g}_- \to \mathfrak{g}_+^*$. If this embedding is an isomorphism then we equip $\mathfrak{g}$ with a topology, by putting the discrete topology on $\mathfrak{g}_+$ and the weak topology on $\mathfrak{g}_-$. If in addition the commutator in $\mathfrak{g}$ is continuous in this topology then the triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is called a Manin triple.

Is there an asymmetric Filtration on $A^\vee$?

so $\mathfrak{g}$ look like polynomials in $\mathfrak{g}_+$ & power series in $\mathfrak{g}_- \subset \mathfrak{g}_+^*$

To every Lie bialgebra $\mathfrak{a}$ one can associate the corresponding Manin triple $(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*, \mathfrak{a}, \mathfrak{a}^*)$, where the Lie structure on $\mathfrak{g}$ is as above. Conversely, if $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple then $\mathfrak{g}_+$ is naturally a Lie bialgebra: the pairing $\langle , \rangle$ identifies $\mathfrak{g}_+^*$ with $\mathfrak{g}_-$, which defines a commutator on $\mathfrak{g}_+^*$. This commutator turns out to be dual to a 1-cocycle (cf. [Dr1]).

Thus, there is a one-to-one correspondence between Lie bialgebras and Manin triples.

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple. Let $\{a_i, i \in I\}$ be a basis of $\mathfrak{g}_+$, and $b^i \in \mathfrak{g}_-$ be the linear functions on $\mathfrak{a}$ defined by $b^i(a_j) = \delta_{ij}$.

Lemma 7.1. *Let M be an equicontinuous $\mathfrak{g}$-module. Then for any $v \in M$ and any neighborhood of zero $U \subset M$ one has $b^i v \in U$ for all but finitely many $i \in I$.*

Proof. We assume that $\dim \mathfrak{g}_+ = \infty$ (otherwise there is nothing to prove).

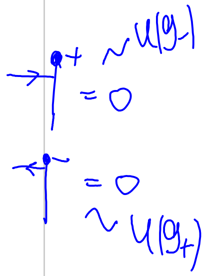
Let $\{i_m \in I : m \geq 1\}$ be any sequence of distinct elements. The $b^{i_m} \to 0$, $m \to \infty$, so $b^{i_m} v \to 0$, $m \to \infty$, for any $v \in M$. This means that $b^i v \in U$ for almost all i. $\square$

7.5. *Examples of equicontinuous $\mathfrak{g}$-modules.*

In this section we will construct examples of equicontinuous $\mathfrak{g}$-modules in the case when $\mathfrak{g}$ belongs to a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$.

so M_+:

M_-:



Consider the Verma modules $M_+ = \mathrm{Ind}_{\mathfrak{g}_+}^{\mathfrak{g}} \mathbf{1}$ $M_- = \mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} \mathbf{1}$, (here $\mathbf{1}$ denotes the trivial 1-dimensional representation). The modules $M_\pm$ are freely generated over $U(\mathfrak{g}_\mp)$ by a vector $1_\pm$ such that $\mathfrak{g}_\pm 1_\pm = 0$, and thus are identified (as vector spaces) with $U(\mathfrak{g}_\mp)$ via $x1_\pm \to x$.

Below we show that the module M_- and the module M_+^* dual to M_+ in an appropriate sense are equicontinuous $\mathfrak{g}$-modules.

Lemma 7.2. *The module M_-, equipped with the discrete topology, is an equicontinuous $\mathfrak{g}$-module.*

is f.d?

Proof. In order to prove the continuity of $\pi_{M_-}(g)$ as a function on $\mathfrak{g}$, we have to check that for any $v \in M_-$ the space $\mathfrak{g}_- v \subset M_-$ is finite dimensional. One may assume that $v = a_{i_1} a_{i_2} ... a_{i_n} 1_-$. We show that $\mathfrak{g}_- v$ is finite dimensional by induction in n. The base of induction is clear since $\mathfrak{g}_- v = 0$ if $n = 0$. Now assume that $v = a_j w$, where $w = a_{i_1} ... a_{i_{n-1}} 1_-$. By the induction assumption, we know that $\mathfrak{g}_- w$ is finite dimensional. For any $b \in \mathfrak{g}_-$ we have $bv = ba_j w = [ba_j]w + a_j bw$. For any $j \in I$ we denote by $W_j \subset \mathfrak{g}_-$ space of all $b \in \mathfrak{g}_-$ such that $(1 \otimes b)(\delta(a_j)) = 0$. It is clear that W_j has finite codimension. For any $b \in W_j$, we have $[ba_j] \in \mathfrak{g}_-$, since $\mathrm{ad}^* b(a_j) = 0$ by the definition of W_j. Therefore, for any $b \in W_j$ $bv = [ba_j]w + a_j bw \in \mathfrak{g}_- w \oplus a_j \mathfrak{g}_- w$. The latter space is finite dimensional, which implies that $W_j v$ is finite dimensional. Since W_j has a finite codimension in $\mathfrak{g}_-$, the space $\mathfrak{g}_- v$ is finite-dimensional. This implies the continuity of the homomorphism $\pi_{M_-} : \mathfrak{g} \to \mathrm{End} M_-$. The equicontinuity condition is trivial. □

Let us now introduce a topology on the space M_+. This topology comes from the identification of M_+ with $U(\mathfrak{g}_-)$. The space $U(\mathfrak{g}_-)$ can be represented as a union of $U_n(\mathfrak{g}_-)$, $n \geq 0$, where $U_n(\mathfrak{g}_-)$ is the set of all elements of $U(\mathfrak{g}_-)$ of degree $\leq n$. Furthermore, for any $n \geq 0$, we have a linear map $\mathfrak{g}_-^{\otimes n} \to U_n(\mathfrak{g}_-)$ given by $x_1 \otimes ... \otimes x_n \to x_1 ... x_n$. This map induces a linear isomorphism $\xi_n : \oplus_{j=0}^n S^j \mathfrak{g}_- \to U_n(\mathfrak{g}_-)$, where $S^j \mathfrak{g}_-$ is the j-th symmetric power of $\mathfrak{g}_-$ (as usual we set $\mathfrak{g}_-^{\otimes 0} = S^0 \mathfrak{g}_- = k$). Since $S^j \mathfrak{g}_-$ has a natural weak topology, coming from its embedding to $(\mathfrak{g}_+^{\otimes j})^*$, the isomorphism ξ_j defines a topology on $U_n(\mathfrak{g}_-)$. Moreover, by the definition, if $m < n$ then $U_m(\mathfrak{g}_-)$ is a closed subspace in $U_n(\mathfrak{g}_-)$. This allows us to equip $U(\mathfrak{g}_-)$, i.e. M_+, with the topology of inductive limit. By the definition, a set $U \subset U(\mathfrak{g}_-)$ is open in this topology if and only if $U \cap U_n(\mathfrak{g}_-)$ is open for all n.

Lemma 7.3. *Let $g \in \mathfrak{g}$. Then $\pi_{M_+}(g)$ is a continuous operator $M_+ \to M_+$.*

Proof. Let $g \in \mathfrak{g}$. We need to show that for any neighborhood of the origin $U \subset M_+$ there exists a neighborhood of the origin $U' \subset M_+$ such that $\pi_{M_+}(g)U' \subset U$.

Let $U \in U(\mathfrak{g}_-)$ be a neighborhood of zero, and $U_n = U \cap U_n(\mathfrak{g}_-)$. To construct U', we need to construct $U'_n = U' \cap U_n(\mathfrak{g}_-)$ such that $U'_n = U'_{n+1} \cap U_n(\mathfrak{g}_-)$. Before giving the construction of U'_n, we make some definitions.

For any neighborhood U of zero, there exists an increasing sequence of finite subsets $T_n \subset I$, $n \geq 1$, such that for any $f \in S^m \mathfrak{g}_-$, $m \leq n$ satisfying the equation $f(a_{i_1}, ..., a_{i_m}) = 0$ for any $i_1, ..., i_m \in T_n$, one has $\xi_n(f) \in U$. Fix such a sequence $\{T_n, n \geq 1\}$.

Let I be as in Section 7.4. For any finite subset $J \subset I$ denote by $S(J)$ the set of all $i \in I$ such that there exists $b \in \mathfrak{g}_-$ and $j \in J$ with the property $[bb^i](a_j) \neq 0$. Since $[bb^i](a_j) = b \otimes b^i(\delta(a_j))$, the set $S(J)$ is finite. Let the sets $S_n(J) \subset I$ be defined recursively by $S_0(J) = J$, $S_n(J) = S(S_{n-1}(J))$.

To construct U', we consider separately the cases $g \in \mathfrak{g}_+$ and $g \in \mathfrak{g}_-$. First consider the case $g \in \mathfrak{g}_-$.

For any elements $x_1, ..., x_n \in \mathfrak{g}_-$ $(n \geq 1)$ consider the element $X = \sum_{\sigma \in S_n} x_{\sigma(1)} ... x_{\sigma(n)}$ in $U_n(\mathfrak{g}_-)$, where S_n is the symmetric group. Consider the element $gX \in U_{n+1}(\mathfrak{g}_-)$. It is easy to see that it is possible to write gX as a linear combination of elements of the form $\sum_{\sigma \in S_m} y_{\sigma(1)} ... y_{\sigma(m)}$, $y_p \in \mathfrak{g}_-$, $0 \leq m \leq n+1$, in such a way that y_p are iterated commutators of g and $x_1, ..., x_n$, and the number of commutators involved in each term y_p does not exceed n.

Now we make a crucial observation.

Claim. Let $J \subset I$ be a finite subset. If for some m, $1 \leq m \leq n$, we have $x_m(a_i) = 0$, for all $i \in S_n(J)$, then every monomial $y_1...y_m$ in the symmetrized expression of gX contains a factor y_p such that $y_p(a_i) = 0$, $i \in J$.

Proof. Clear.

The construction of U' is as follows. For $n \geq 1$, let $U'_n \subset U_n(\mathfrak{g}_-)$ be the span of all elements $\xi_m(f)$, $0 \leq m \leq n$, where $f \in S^m\mathfrak{g}_-$ are such that $f(a_{i_1}, ..., a_{i_m}) = 0$ whenever $i_1, ..., i_m \in S_n(T_{n+1})$. Also, set $U'_0 = 0$ (recall that $\{0\} \subset k$ is a neighborhood of zero since k is discrete). Our observation shows that for any $X \in U'_n$, $gX \in U_{n+1}$, as desired.

Now consider the case $g \in \mathfrak{g}_+$.

Let $R_0(g) \subset I$ be the set of all $i \in I$ such that $b^i(g) \neq 0$. This is a finite set. Define inductively the sets $R_n(g)$ by $R_n(g) = S(R_{n-1}(g))$.

For any finite subsets $K, J \subset I$ denote by $P(K, J)$ the set of all $i \in I$ such that there exists $j \in J$ and $k \in K$ with $[a_k b^i](a_j) \neq 0$. It is clear that if K, J are finite then $P(K, J)$ is finite. Let $P_n(K, J)$ be defined inductively by $P_n(K, J) = P(K, P_{n-1}(K, J))$.

Let $n \geq 1$ be an integer, $X \in U_n(\mathfrak{g}_-)$ be as above, and $K = R_n(g)$. Consider the vector $gX1_+ \in M_+$. Using the relations in M_+, we can reduce this vector to a linear combination of vectors of the form $\sum_{\sigma \in S_m} y_{\sigma_1}...y_{\sigma_m}$, $y_p \in \mathfrak{g}_-$, $0 \leq m \leq n+1$, in such a way that y_p are obtained by iterated commutation of g, $x_1, ..., x_n$. As before, it is easy to see that the resulting symmetrized expression will contain no more than n commutators.

Now let us make a crucial observation.

Claim. Let $J \subset I$ be any finite subset. If for some m, $1 \leq m \leq n$, we have $x_m(a_i) = 0$, for all $i \in S_n(P(K, S_n(J))$, then every monomial $y_1...y_m$ in the symmetrized expression of $gX1_+$ contains a factor y_p such that $y_p(a_i) = 0$, $i \in J$.

Proof. Clear.

The construction of U' is as follows. For $n \geq 1$, let $U'_n \subset U_n(\mathfrak{g}_-)$ be the span of all elements $\xi_m(f)$, $f \in S^m\mathfrak{g}_-$, $0 \leq m \leq n$, such that $f(a_{i_1}, ..., a_{i_m}) = 0$ whenever $i_1, ..., i_m \in S_n(P(K, S_n(T_{n+1})))$. Also, set $U'_0 = 0$. Our observation shows that for any $X \in U'_n$, $gX \in U_{n+1}$, as desired. □

Consider the vector space M_+^* of continuous linear functionals on M_+. By the definition, M_+^* is naturally isomorphic to the projective limit of $U_n(\mathfrak{g}_-)^*$ as $n \to \infty$. As vector spaces, $U_n(\mathfrak{g}_-)^* = (S^j\mathfrak{g}_-)^* = S^j\mathfrak{g}_+$. Therefore, it is natural to put the discrete topology on $U_n(\mathfrak{g}_-)^*$. This equips the module M_+^* with a natural structure of a complete vector space. It is also equipped with a filtration by subspaces $(M_+^*)_n = U_{n-1}(\mathfrak{g}_-)^\perp$, $n \geq 1$, such that $M_+ = \varprojlim M_+^*/(M_+^*)_n$.

Remark. The topology of projective limit on M_+^* does not, in general, coincide with the weak topology of the dual. In fact, it is stronger than the weak topology.

By Lemma 7.3, M_+^* has a natural structure of a $\mathfrak{g}$-module. Namely, the action of $\mathfrak{g}$ on M_+^* is defined to be the dual to the action of $\mathfrak{g}$ on M_+.

Lemma 7.4. *M_+^* is an equicontinuous $\mathfrak{g}$-module.*

Proof. It is easy to see that $a(M_+^*)_n \subset (M_+^*)_n$, $a \in \mathfrak{g}_+$, and $b(M_+^*)_n \subset (M_+^*)_{n-1}$, $b \in \mathfrak{g}_-$. This means that the operators $\pi_{M_+^*}(g)$ are continuous for any $g \in \mathfrak{g}$, and

$\pi_{M_+^*}(\mathfrak{g}) \subset \mathrm{End} M_+^*$ is an equicontinuous family of operators. It remains to show that the assignment $g \to \pi_{M_+^*}(g)$ is continuous for $g \in \mathfrak{g}$. Since $\mathfrak{g}_+$ is discrete, it is enough to check this statement for $g \in \mathfrak{g}_-$.

Let $f \in M_+^*$. Let f_n be the reduction of f modulo $(M_+^*)_n$. We can regard f as an element of $\oplus_{j=0}^n S^j \mathfrak{g}_+$. Let us write f_n in terms of the basis $\{a_i\}$, and let $T_n(f)$ be the set of all $i \in I$ such that a_i is involved in this expression.

Let $S_n(J)$ be as in the proof of Lemma 7.3, and $i \in I \setminus S_n(T_{n+1}(f))$. Then it is easy to see that $b^i f \in (M_+^*)_n$. This shows that for any $n \geq 0$ and any $f \in M_+^*$ $b^i f \in (M_+^*)_n$ for almost all $i \in I$.

Thus, M_+^* is an equicontinuous $\mathfrak{g}$-module. □

Remark. If $\mathfrak{g}_+$ is infinite dimensional then M_+ is not, in general, an equicontinuous $\mathfrak{g}$-module, since the family of operators $\{\pi_{M_+}(g), g \in \mathfrak{g}_+\}$ may fail to be equicontinuous.

7.6. *The Casimir element.*

Consider the tensor product $\mathfrak{a} \otimes \mathfrak{a}^*$. This space can be embedded into $\mathrm{End}\mathfrak{a}$, by $(x \otimes f)(y) = f(y)x$, $x, y \in \mathfrak{a}$, $f \in \mathfrak{a}^*$. This embedding defines a topology on $\mathfrak{a} \otimes \mathfrak{a}^*$, obtained by restriction of the weak topology on $\mathrm{End}\mathfrak{a}$. Let $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ be the completion of $\mathfrak{a} \otimes \mathfrak{a}^*$ in this topology. Since the image of $\mathfrak{a} \otimes \mathfrak{a}^*$ is dense in $\mathrm{End}\mathfrak{a}$, this completion is identified with $\mathrm{End}\mathfrak{a}$.

Lemma 7.5. *Let $V, W \in \mathcal{M}_0^e$. The map $\pi_V \otimes \pi_W : \mathfrak{a} \otimes \mathfrak{a}^* \to End(V\hat{\otimes}W)$ extends to a continuous map $\mathfrak{a}\hat{\otimes}\mathfrak{a}^* \to End(V\hat{\otimes}W)$.*

Proof. Let $x \in V\hat{\otimes}W$ be a vector. It is easy to see that the map $\pi_V \otimes \pi_W(\cdot)x : \mathfrak{a} \otimes \mathfrak{a}^* \to V\hat{\otimes}W$ is continuous. Since the space $V\hat{\otimes}W$ is complete, this map extends to a continuous map $\mathfrak{a}\hat{\otimes}\mathfrak{a}^* \to V\hat{\otimes}W$. This allows us to define a linear map $\pi_V \otimes \pi_W : \mathfrak{a}\hat{\otimes}\mathfrak{a}^* \to \mathrm{End}(V\hat{\otimes}W)$. We would like to show that this map is continuous.

Let $x \in V\hat{\otimes}W$ be a vector, and $n \geq 0$ be an integer. Let $P \subset V\hat{\otimes}W$ be an open subspace, and $U = \{A \in \mathrm{End}(V\hat{\otimes}W) : Ax \in P\}$. Since open sets of this form generate the topology on $\mathrm{End}(V\hat{\otimes}W)$, it is enough to show that there exists a neighborhood of zero $Y \subset \mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ such that $(\pi_V \otimes \pi_W)(Y) \subset U$, i.e. $(\pi_V \otimes \pi_W)(Y)x \subset P$.

We can assume that $P = V_1\hat{\otimes}W + W_1\hat{\otimes}V$, where V_1, W_1 are open subspaces of V, W. By the equicontinuity of $\pi_V(g), \pi_W(g)$, $g \in \mathfrak{g}$, there exist open subspaces $V_2 \subset V$, $W_2 \subset W$ such that $\pi_V(\mathfrak{g})V_2 \subset V_1$, $\pi_W(\mathfrak{g})W_2 \subset W_1$. Let $y \in V \otimes W$ be a vector in the usual tensor product of V and W such that $y - x \in V_2\hat{\otimes}W + V\hat{\otimes}W_2$. Then for any $t \in \mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ $(\pi_V \otimes \pi_W)(t)(y-x) \in P$, so it is enough to find Y satisfying the condition $(\pi_V \otimes \pi_W)(Y)y \subset P$.

We have $y = \sum_{j=1}^m v_j \otimes w_j$, $v_j \in V, w_j \in W$. Let $X \subset \mathfrak{a}$ be a finite-dimensional subspace such that for any $b \in X^\perp \subset \mathfrak{a}^*$ $bw_j \in W_1$ for $j = 1, ..., m$. Such a subspace exists by Lemma 7.1. The set $Y = \mathfrak{a}\hat{\otimes}X^\perp$ (the completion of $\mathfrak{a} \otimes X^\perp$ in $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$) is open in $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$, and $(\pi_V \otimes \pi_W)(Y)y \subset P$, as desired. This shows the continuity of $\pi_V \otimes \pi_W$ on $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$. □

Let $r \in \mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ be the vector corresponding to the identity operator under the identification $\mathfrak{a}\hat{\otimes}\mathfrak{a}^*$ with $\mathrm{End}\mathfrak{a}$. Let $r^{op} \in \mathfrak{a}^*\hat{\otimes}\mathfrak{a}$ be the element obtained from r by permutation of the components. We define the Casimir element $\Omega \in \mathfrak{a}\hat{\otimes}\mathfrak{a}^* \oplus \mathfrak{a}^*\hat{\otimes}\mathfrak{a}$. to be the sum $r + r^{op}$.

It is easy to see that, $r = \sum a_i \otimes b^i$, $r^{op} = \sum b^i \otimes a_i$, $\Omega = \sum(a_i \otimes b^i + b^i \otimes a_i)$.

Let V, W be equicontinuous $\mathfrak{g}$-modules, and denote by $\pi_V : \mathfrak{g} \to \mathrm{End}V$, $\pi_W : \mathfrak{g} \to \mathrm{End}W$ the corresponding linear maps. Let $\Omega_{VW} = \pi_V \otimes \pi_W(\Omega)$. This endomorphism of $V\hat{\otimes}W$ is well defined and continuous by Lemma 7.5. Moreover, it is easy to see that Ω_{VW} commutes with $\mathfrak{g}$, so it is an endomorphism of $V\hat{\otimes}W$ as an equicontinuous $\mathfrak{g}$-module.

Remark. Although the Casimir operator $\Omega = \sum(a_i \otimes b^i + b^i \otimes a_i)$ is defined in the product of any two equicontinuous $\mathfrak{g}$-modules $V\hat{\otimes}W$, the Casimir element $C = \sum(a_i b^i + b^i a_i)$ in general (for $\dim \mathfrak{a} = \infty$) has no meaning as an operator in an equicontinuous $\mathfrak{g}$-module V.

7.7. *Drinfeld category.* Let $\mathcal{M}^e$ denote the category whose objects are equicontinuous $\mathfrak{g}$-modules, and $\mathrm{Hom}_{\mathcal{M}^e}(U, W) = \mathrm{Hom}_{\mathfrak{g}}(U, W)[[h]]$. This is an additive category. For brevity we will later write Hom for $\mathrm{Hom}_{\mathcal{M}^e}$.

Define a structure of a braided monoidal category on $\mathcal{M}^e$ analogously to Section 1.4, using an associator Φ and the functor $\hat{\otimes}$. As before, we identify $\mathcal{M}^e$ with a strict category and forget about positions of brackets.

Let γ be the functorial isomorphism defined by $\gamma_{XY} = \beta_{YX}^{-1} \in \mathrm{Hom}(X \otimes Y, Y \otimes X)$, $X, Y \in \mathcal{M}^e$. It is easy to check that γ is a braiding on $\mathcal{M}^e$. We will need the braiding γ in our construction below.

8. The fiber functor.

8.1. *The category of complete $k[[h]]$-modules.* ?

Let V be a complete vector space over k. Then the space $V[[h]] = V\hat{\otimes}k[[h]]$ of formal power series in h with coefficients in V is also a complete vector space. Moreover, $V[[h]]$ has a natural structure of a topological $k[[h]]$-module. We call a topological $k[[h]]$-module complete if it is isomorphic to $V[[h]]$ for some complete V.

Let $\mathcal{A}^c$ be the category of complete $k[[h]]$-modules, where morphisms are continuous $k[[h]]$-linear maps. It is an additive category. Define the tensor structure on $\mathcal{A}^c$ as follows. For $V, W \in \mathcal{A}^c$ define $V\tilde{\otimes}W$ to be the quotient of the completed tensor product $V\hat{\otimes}W$ by the image of the operator $h \otimes 1 - 1 \otimes h$. It is clear that for $V, W \in \mathcal{A}^c$, $V\tilde{\otimes}W$ is also in $\mathcal{A}^c$. The category $\mathcal{A}^c$ equipped with the functor $\tilde{\otimes}$ is a symmetric monoidal category.

Let CVect be the category of complete vector spaces. We have the functor of extension of scalars, $V \mapsto V[[h]]$, acting from CVect to $\mathcal{A}^c$. This functor respects the tensor product, i.e. $(V\hat{\otimes}W)[[h]]$ is naturally isomorphic to $V[[h]]\tilde{\otimes}W[[h]]$.

8.2. *Properties of the Verma modules.*

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple, and $\mathcal{M}^e$ be the Drinfeld category associated to $\mathfrak{g}$. Let M_+, M_- be the Verma modules over $\mathfrak{g}$ defined in Section 7.5.

Recall that the modules $M_\pm$ are identified with $U(\mathfrak{g}_\mp)$. Thus, we can define the maps $i_\pm : M_\pm \to M_\pm \otimes M_\pm$ given by comultiplication in the universal enveloping algebras $U(\mathfrak{g}_\mp)$. These maps are $U(\mathfrak{g})$-intertwiners, since they are $U(\mathfrak{g}_\pm)$-intertwiners and map the vector $1_\pm$ to the $\mathfrak{g}_\mp$-invariant vector $1_\pm \otimes 1_\pm$.

Let M_+^* be as in Section 7.5, and $f, g \in M_+^*$. Consider the linear functional $M_+ \to k$ defined by $v \to (f \otimes g)(i_+(v))$. It is easy to check that this functional is continuous, so it belongs to M_+^*. Define the map $i_+^* : M_+^* \otimes M_+^* \to M_+^*$ by $i_+^*(f \otimes g)(v) = (f \otimes g)(i_+(v))$, $v \in M_+$. It is clear that i_+^* is continuous, so it extends to a morphism in $\mathcal{M}^e$: $i_+^* : M_+^*\hat{\otimes}M_+^* \to M_+^*$.

Let $V \in \mathcal{M}$. Consider the space $\mathrm{Hom}_{\mathfrak{g}}(M_-, M_+^*\hat{\otimes}V)$, where $\mathrm{Hom}_{\mathfrak{g}}$ denotes the

$\mathcal{M}^e$?

set of continuous homomorphisms. Equip this space with the weak topology (see Section 7.1).

Lemma 8.1. *The complete vector space $Hom_{\mathfrak{g}}(M_-, M_+^*\hat{\otimes}V)$ is isomorphic to V. The isomorphism is given by $f \to (1_+ \otimes 1)(f(1_-))$, $f \in Hom_{\mathfrak{g}}(M_-, M_+^*\hat{\otimes}V)$.*

Proof. By the Frobenius reciprocity, $\mathrm{Hom}_{\mathfrak{g}}(M_-, M_+^*\hat{\otimes}V)$ is isomorphic, as a topological vector space, to the space of invariants $(M_+^*\hat{\otimes}V)^{\mathfrak{g}_-}$, via $f \to f(1_-)$. Consider the space $\mathrm{Hom}_k(M_+, V)$ of continuous homomorphisms from M_+ to V, equipped with the weak topology, and the map $\phi : (M_+^*\hat{\otimes}V) \to \mathrm{Hom}_k(M_+, V)$, given by $\phi(f \otimes v)(x) = f(x)v$, $u \in M_+^*$, $x \in M_+$, $v \in V$. It is clear that ϕ is injective and continuous.

Claim. The map ϕ restricts to an isomorphism $(M_+^*\hat{\otimes}V)^{\mathfrak{g}_-} \to \mathrm{Hom}_{\mathfrak{g}_-}(M_+, V)$.

Proof. It is clear that $\phi((M_+^*\hat{\otimes}V)^{\mathfrak{g}_-}) \subset \mathrm{Hom}_{\mathfrak{g}_-}(M_+, V)$. So it is enough to show that any continuous $\mathfrak{g}_-$-intertwiner $g : M_+ \to V$ is of the form $\phi(g')$, $g' \in (M_+^*\hat{\otimes}V)^{\mathfrak{g}_-}$, where g' continuously depends on $\mathfrak{g}$.

Let $X \subset V$ be an open subspace. Then for any $\mathfrak{g}_-$-intertwiner $g : M_+ \to V$ and $n \geq 1$ the image of $g(U_n(\mathfrak{g}_-)1_+)$ in V/V_m is finite-dimensional. This shows that $g = \phi(g')$ for some $g' \in (V \otimes M_+^*)^{\mathfrak{g}_-}$. It is clear that g' is continuous in g. The claim is proved.

By the Frobenius reciprocity, the space $\mathrm{Hom}_{\mathfrak{g}_-}(M_+, V)$ is isomorphic to V as a topological vector space, via $f \to f(1_+)$. The lemma is proved. □

8.3. *The forgetful functor.*

Let $F : \mathcal{M}^e \to \mathcal{A}^c$ be a functor given by $F(V) = \mathrm{Hom}(M_-, M_+^*\hat{\otimes}V)$. Lemma 8.1 implies that this functor is naturally isomorphic to the "forgetful" functor which associates to every equicontinuous $\mathfrak{g}$-module M the complete $k[[h]]$-module $M[[h]]$. The isomorphism between these two functors is given by $f \to (1_+ \otimes 1)(f(1_-))$, for any $f \in F(M)$. Denote this isomorphism by τ.

8.4. *Tensor structure on the functor F.*

From now on, when no confusion is possible, we will denote the tensor product in the categories $\mathcal{M}^e$ and $\mathcal{A}^c$ by $\otimes$, instead of $\hat{\otimes}$ and $\tilde{\otimes}$.

Define a tensor structure on the functor F constructed in Section 8.3.

For any $v \in F(V)$, $w \in F(W)$ define $J_{VW}(v \otimes w)$ to be the composition of morphisms:

$$\begin{aligned} &M_- \xrightarrow{i_-} M_- \otimes M_- \xrightarrow{v\otimes w} M_+^* \otimes V \otimes M_+^* \otimes W \xrightarrow{1\otimes\gamma_{23}\otimes 1} \\ &M_+^* \otimes M_+^* \otimes V \otimes W \xrightarrow{i_+^*\otimes 1\otimes 1} M_+^* \otimes V \otimes W, \end{aligned} \tag{8.1}$$

where γ_{23} denotes the braiding γ acting in the second and third components of the tensor product. That is,

$$J_{VW}(v \otimes w) = (i_+^* \otimes 1 \otimes 1) \circ (1 \otimes \gamma_{23} \otimes 1) \circ (v \otimes w) \circ i_-. \tag{8.2}$$

Proposition 8.2. *The maps J_{VW} are isomorphisms and define a tensor structure on the functor F.*

Proof. It is obvious that J_{VW} is an isomorphism since it is an isomorphism modulo h.

To prove the associativity of J_{VW}, we need the following result.

Lemma 8.3 $(i_- \otimes 1) \circ i_- = (1 \otimes i_-) \circ i_-$ in $\mathrm{Hom}(M_-, M_-^{\otimes 3})$; $(i_+^* \otimes 1) \circ i_+^* = (1 \otimes i_+^*) \circ i_+^*$ in $\mathrm{Hom}(M_+^*, (M_+^*)^{\otimes 3})$.

Proof. The proof of the first identity coincides with the proof of Lemma 2.3 in Part I. To prove the second identity, define $M_+ \hat{\otimes} M_+ \hat{\otimes} M_+$ to be space of continuous linear functionals on $M_+^* \hat{\otimes} M_+^* \hat{\otimes} M_+^*$. Since the operators $\Omega_{ij} \in \mathrm{End}_{\mathfrak{g}}(M_+^* \hat{\otimes} M_+^* \hat{\otimes} M_+^*)$ are continuous, one can define the dual operators $\Omega_{ij}^* \in \mathrm{End}_{\mathfrak{g}}(M_+ \hat{\otimes} M_+ \hat{\otimes} M_+)$, and hence the operator Φ^* dual to Φ. It is easy to show analogously to the proof of Lemaa 2.3 that $\Phi^*(1_+ \otimes 1_+ \otimes 1_+) = 1_+ \otimes 1_+ \otimes 1_+$, which implies the second identity of Lemma 8.3.

Now we can finish the proof of the proposition. We need to show that for any $v \in F(V), w \in F(W), u \in F(U)$ $J_{V\otimes W,U} \circ (J_{VW} \otimes 1)(v \otimes w \otimes u) = J_{V,W\otimes U} \circ (1 \otimes J_{WU})(v \otimes w \otimes u)$, i.e.

$$(i_+^* \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{34,5} \circ (i_+^* \otimes 1 \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{23} \circ (v \otimes w \otimes u) \circ (i_- \otimes 1) \circ i_- =$$
$$(8.3)$$
$$(i_+^* \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{23} \circ (1 \otimes 1 \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{45} \circ (v \otimes w \otimes u) \circ (1 \otimes i_-) \circ i_-$$

in $F(V \otimes W \otimes U)$, where $\gamma_{23,4}$ means the braiding applied to the product of the second and the third factors and to the fourth factor. Because of Lemma 8.3 and commutation relation of $\gamma_{23,4}$ and $i_+^* \otimes 1 \otimes 1 \otimes 1 \otimes 1$, identity (8.3) is equivalent to the identity

$$\begin{aligned} &(i_+^* \otimes 1 \otimes 1 \otimes 1) \circ (i_+^* \otimes 1 \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{34,5} \circ \gamma_{23} = \\ (8.4) \qquad &(i_+^* \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{23} \circ (1 \otimes 1 \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{45} \end{aligned}$$

in $\mathrm{Hom}(M_+^* \otimes V \otimes M_+^* \otimes W \otimes M_+^* \otimes U, M_+^* \otimes V \otimes W \otimes U)$.

To prove this equality, we observe that the functoriality of the braiding implies the identity

$$(8.5) \qquad \gamma_{23} \circ (1 \otimes 1 \otimes i_+^* \otimes 1 \otimes 1) = (1 \otimes i_+^* \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{2,34}.$$

Using (8.5) and the identity $(i_+^* \otimes 1) \circ i_+^* = (1 \otimes i_+^*) \circ i_+^*$, which follows from Lemma 8.3, we reduce (8.4) to the identity $\gamma_{34,5}\gamma_{23} = \gamma_{2,34}\gamma_{45}$, which follows directly from the braiding axioms. □

We will call the functor F equipped with the tensor structure defined above *the fiber functor.*

9. Quantization of Lie bialgebras.

9.1. *The algebra of endomorphisms of the fiber functor.*

Let $H = \mathrm{End}(F)$ be the algebra of endomorphisms of the functor F, with a topology defined by the ideal $hH \subset H$. It is clear that H is a topological algebra over $k[[h]]$ (see Part I, Section 3.1).

Let H_0 be the algebra of endomorphisms of the forgetful functor $\mathcal{M}_0^e \to CVect$. It follows from Lemma 8.1 that the algebra H is naturally isomorphic to $H_0[[h]]$.

Let $F^2 : \mathcal{M}^e \times \mathcal{M}^e \to \mathcal{A}^c$ be the bifunctor defined by $F^2(V, W) = F(V) \otimes F(W)$. Let $H^2 = \mathrm{End}(F^2)$. It is clear that $H^2 \supset H \otimes H$ but $H^2 \neq H \otimes H$ unless $\mathfrak{g}$ is finite dimensional.

The algebra H has a natural "comultiplication" $\Delta : H \to H^2$ defined by $\Delta(a)_{V,W}(v \otimes w) = J_{VW}^{-1} a_{V \otimes W} J_{VW}(v \otimes w)$, $a \in H$, $v \in F(V), w \in F(W)$ where a_V denotes the action of a in $F(V)$. We can also define the counit on H by $\varepsilon(a) = a_{\mathbf{1}} \in k[[h]]$, where $\mathbf{1}$ is the neutral object.

A topological algebra A over $k[[h]]$ is said to be a topological bialgebra if it is equipped with a coproduct $\Delta : A \to A \otimes A$ (where $\otimes$ is the tensor product in $\mathcal{A}$) and a counit $\varepsilon : A \to k[[h]]$ which are $k[[h]]$-linear, continuous, and satisfy the standard axioms of a bialgebra.

We will need the following statement.

Proposition 9.1. *Let $A \subset H$ be a topological subalgebra such that $\Delta(A) \subset A \otimes A$. Then (A, Δ, ε) is a topological bialgebra over $k[[h]]$.*

The proof is straightforward.

Remark. For infinite-dimensional $\mathfrak{g}$, the algebra H equipped with the topology defined by the ideal hH is not a topological bialgebra since $\Delta(H)$ is not a subset of $H \otimes H$.

In the following sections we construct a quantum universal enveloping algebra $U_h(\mathfrak{g}_+)$, which is a quantization of the Lie bialgebra $\mathfrak{g}_+$, in the sense of Drinfeld (see [Dr1] and Part I, Section 3.1). Namely, the algebra $U_h(\mathfrak{g}_+)$ is obtained as a subalgebra of H such that $\Delta(A) \subset A \otimes A$.

9.2. *The algebra $U_h(\mathfrak{g}_+)$.*

Let $x \in F(M_-)$. Define the endomorphism $m_+(x)$ of the functor F as follows. For any $V \in \mathcal{M}^e$, $v \in F(V)$, define the element $m_+(x)v \in F(V)$ to be the composition of the following morphisms in $\mathcal{M}^e$: $m_+(x)v = (i_+^* \otimes 1) \circ (1 \otimes v) \circ x$. This defines a linear map $m_+ : F(M_-) \to H$. Denote the image of this map by $U_h(\mathfrak{g}_+)$.

It is easy to see that for any $a \in U(\mathfrak{g}_+)$ $\tau(m_+(a1_-)v) \equiv a\tau(v) \bmod h$, which implies that m_+ is an embedding.

Proposition 9.2. *$U_h(\mathfrak{g}_+)$ is a subalgebra in H.*

Proof.

Using Lemma 8.3, for any $x, y \in F(M_-)$, $V \in \mathcal{M}^e$, $v \in F(V)$ we obtain

$$\begin{aligned} m_+(x)m_+(y)v = (i_+^* \otimes 1) \circ (1 \otimes i_+^* \otimes 1) \circ (1 \otimes 1 \otimes v) \circ (1 \otimes y) \circ x = \\ (i_+^* \otimes 1) \circ (i_+^* \otimes 1 \otimes 1) \circ (1 \otimes 1 \otimes v) \circ (1 \otimes y) \circ x = \\ (i_+^* \otimes 1) \circ (1 \otimes v) \circ (i_+^* \otimes 1) \circ (1 \otimes y) \circ x = \\ (i_+^* \otimes 1) \circ (1 \otimes v) \circ z, \end{aligned} \tag{9.1}$$

where $z = (i_+^* \otimes 1) \circ (y \otimes 1) \circ x \in F(M_-)$.

So by the definition we get $m_+(x) \circ m_+(y) = m_+(z)$. □

Note that the algebra $U_h(\mathfrak{g}_+)$ is a deformation of the algebra $U(\mathfrak{g}_+)$. Indeed, we can define a linear isomorphism $\mu : U(\mathfrak{g}_+)[[h]] \to U_h(\mathfrak{g}_+)$ by $\mu(a) = m_-(a1_-)$, $a \in U(\mathfrak{g}_+)[[h]]$. This isomorphism has the property $\mu(ab) = \mu(a) \circ \mu(b) \bmod h^2$, which follows from the fact that $\Phi \equiv 1 \bmod h$, but in general $\mu(ab) \neq \mu(a) \circ \mu(b)$.

The subalgebra $U_h(\mathfrak{g}_+)$ has a unit which is equal to $\mu(1)$, $1 \in U(\mathfrak{g}_+)$. To check this, it is enough to observe that $\mu(1)$ is invertible and check the identity $\mu(1)^2 = \mu(1)$.

9.3. *The coproduct on $U_h(\mathfrak{g}_+)$.*

Proposition 9.3. *The algebra $U_h(\mathfrak{g}_+)$ is closed under the coproduct Δ, i.e. $\Delta(U_h(\mathfrak{g}_+)) \subset U_h(\mathfrak{g}_+) \otimes U_h(\mathfrak{g}_+)$, and for any $x \in F(M_-)$ one has*

$$\Delta(m_+(x)) = (m_+ \otimes m_+)(J^{-1}_{M_- M_-}((1 \otimes i_-) \circ x)). \tag{9.2}$$

Proof. Let $x \in F(M_-)$, $V, W \in \mathcal{M}^e$, $v \in V$, $w \in W$. By the definition of Δ and m_+, the element $\Delta(m_+(x)) \in H^2$ is uniquely determined by the identity

$$\begin{aligned}(i_+^* \otimes 1 \otimes 1) \circ (1 \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{34} \circ (1 \otimes v \otimes w) \circ R(1 \otimes i_-) \circ x = \\ (i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ \Delta(m_+(x))(v \otimes w) \circ i_-\end{aligned} \tag{9.3}$$

in $F(V \otimes W)$.

The element $X = J^{-1}_{M_- M_-}((1 \otimes i_-)x) \in F(M_-) \otimes F(M_-)$ is, by the definition, uniquely determined by the identity

$$(1 \otimes i_-) \circ x = (i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ X \circ i_- \tag{9.4}$$

in $F(M_- \otimes M_-)$. Therefore, to prove formula (9.2), it is enough to prove the equality obtained by substitution of $(i_+^* \otimes 1 \otimes i_+^* \otimes 1) \circ (1 \otimes v \otimes 1 \otimes w) \circ X$ instead of $\Delta(m_+(x))(v \otimes w)$ in (9.3):

$$\begin{aligned}(i_+^* \otimes 1 \otimes 1) \circ (1 \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{34} \circ (1 \otimes v \otimes w) \circ (1 \otimes i_-) \circ x = \\ (i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ (i_+^* \otimes 1 \otimes i_+^* \otimes 1) \circ (1 \otimes v \otimes 1 \otimes w) \circ X \circ i_-\end{aligned} \tag{9.5}$$

in $F(V \otimes W)$.

Using the functoriality of the braiding and Lemma 8.3, we obtain

$$\begin{aligned}(i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ (i_+^* \otimes 1 \otimes i_+^* \otimes 1) \circ (1 \otimes v \otimes 1 \otimes w) = \\ (i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ (i_+^* \otimes 1 \otimes i_+^* \otimes 1) \circ \gamma^{-1}_{23,4} \circ (1 \otimes 1 \otimes v \otimes w) \circ \gamma_{23} = \\ (i_+^* \otimes 1 \otimes 1) \circ (i_+^* \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{3,45} \circ \gamma^{-1}_{23,4} \circ (1 \otimes 1 \otimes v \otimes w) \circ \gamma_{23} = \\ (i_+^* \otimes 1 \otimes 1) \circ (i_+^* \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{45}\gamma_{23}^{-1} \circ (1 \otimes 1 \otimes v \otimes w) \circ \gamma_{23} =\end{aligned}$$

$$(i_+^* \otimes 1 \otimes 1) \circ (i_+^* \otimes 1 \otimes 1 \otimes 1) \circ (1 \otimes i_+^* \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{45}\gamma_{23}^{-1} \circ (1 \otimes 1 \otimes v \otimes w) \circ \gamma_{23} \tag{9.6}$$

in $\mathrm{Hom}(M_+^* \otimes M_- \otimes M_+^* \otimes M_-, M_+^* \otimes V \otimes W)$. It is easy to see that $i_+^* \circ \gamma = i_+^*$, so using Lemma 8.3 again, we get from (9.6):

$$\begin{aligned}(i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ (i_+^* \otimes 1 \otimes i_+^* \otimes 1) \circ (1 \otimes v \otimes 1 \otimes w) = \\ (i_+^* \otimes 1) \circ (1 \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{34} \circ (1 \otimes v \otimes w) \circ (i_+^* \otimes 1 \otimes 1) \circ \gamma_{23}\end{aligned} \tag{9.7}$$

Substituting (9.7) into the right hand side of (9.5) and using (9.4), we get

$$\begin{aligned}(i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ (i_+^* \otimes 1 \otimes i_+^* \otimes 1) \circ (1 \otimes v \otimes 1 \otimes w) \circ X \circ i_- = \\ (i_+^* \otimes 1 \otimes 1) \circ (1 \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{34} \circ (1 \otimes v \otimes w) \circ (i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ X \circ i_- = \\ (i_+^* \otimes 1 \otimes 1) \circ (1 \otimes i_+^* \otimes 1 \otimes 1) \circ \gamma_{34} \circ (1 \otimes v \otimes w) \circ (1 \otimes i_-) \circ x\end{aligned} \tag{9.8}$$

in $F(V \otimes W)$, which proves (9.2). The proposition is proved. $\square$

Corollary 9.4. *The algebra $U_h(\mathfrak{g}_+)$, equipped with the coproduct Δ, is a quantized universal enveloping algebra.*

Proof.. It follows from Lemma 9.1 and Propositions 9.2, 9.3 that $U_h(\mathfrak{g}_+)$ is a topological bialgebra over $k[[h]]$ isomorphic to $U(\mathfrak{g}_-)[[h]]$ as a topological $k[[h]]$-module, and such that $U_h(\mathfrak{g}_+)/hU_h(\mathfrak{g}_+)$ is isomorphic to $U(\mathfrak{g}_+)$ as a bialgebra. This implies that $U_h(\mathfrak{g})$ has an antipode, because the antipode exists mod$h$. Thus, $U_h(\mathfrak{g}_+)$ is a quantized universal enveloping algebra. □

9.4. *The algebra $U_h(\mathfrak{g}_+)$ is a quantization of $\mathfrak{g}_+$.*

Proposition 9.5. *The algebra $U_h(\mathfrak{g}_+)$ is a quantization of the Lie bialgebra $\mathfrak{g}_+$.*

Proof. Let $x \in U_h(\mathfrak{g}_+)$ be such that there exists $x_0 \in \mathfrak{g}_+ \subset U(\mathfrak{g}_+)$ satisfying the condition $x \equiv x_0 \bmod h$.

It is easy to show that for any $V, W \in M$

$$\tau_{V\otimes W}^{-1} \circ J_{VW} \circ (\tau_V \otimes \tau_W) = 1 + hr/2 + O(h^2) \tag{9.9}$$

in $\mathrm{End}(V \otimes W)$. From (9.9) and the definition of coproduct, analogously to the proof of Proposition 3.6 in Part I, it is easy to obtain the congruence

$$h^{-1}(\Delta(x) - \Delta^{op}(x)) \equiv \delta(x_0) \bmod h. \tag{9.10}$$

which means that $U_h(\mathfrak{g}_+)$ is the quantization of $\mathfrak{g}_+$. □

Thus, we have proved the following theorem, which answers question 1.1 in [Dr3].

Theorem 9.6. *Let $\mathfrak{a}$ be a Lie bialgebra over k. Then there exists a quantized universal enveloping algebra $U_h(\mathfrak{a})$ over k which is a quantization of $\mathfrak{a}$.*

9.5. *The isomorphism between two constructions of the quantization.*

Let us compare the results of the previous sections to the results of Part I. In Part I, we showed the existence of quantization for any finite dimensional Lie bialgebra. Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a finite-dimensional Manin triple. Let $U_h(\mathfrak{g}_+)$ denote the quantization of $\mathfrak{g}_+$ constructed in this section, and by $\tilde{U}_h(\mathfrak{g}_+)$ the quantization constructed in Part I.

Proposition 9.7. *The quantized universal enveloping algebras $U_h(\mathfrak{g}_+)$, $\tilde{U}_h(\mathfrak{g}_+)$ are isomorphic.*

Proof. If $\mathfrak{g}$ is finite-dimensional, then M_+ is an equicontinuous $\mathfrak{g}$-module. Let $\tilde{F} : \mathcal{M}^e \to \mathcal{A}^c$ be the functor defined by $\tilde{F}(V) = \mathrm{Hom}(M_+ \otimes M_-, V)$, $V \in \mathcal{M}^e$. The tensor structure on $\tilde{F}$ can be defined as in Part I.

Let $\sigma \in \mathrm{Hom}(\mathbf{1}, M_+^* \otimes M_+)$ be the canonical element. Consider the morphism $\chi : \tilde{F} \to F$, defined as follows. For any $V \in M$, $v \in \tilde{F}(V)$, define $\chi_V(v) \in F(V)$ as the composition $\chi_V(v) = (1 \otimes v) \circ (\sigma \otimes 1)$. It is obvious that χ is an isomorphism of additive functors.

Claim. χ is an isomorphism of tensor functors.

Proof. The statement is equivalent to the identity

$$\begin{aligned}&(1 \otimes v \otimes w) \circ \beta_{34} \circ (1 \otimes i_+ \otimes i_-) \circ (\sigma \otimes 1) = \\ &(i_+^* \otimes 1 \otimes 1) \circ \gamma_{23} \circ (1 \otimes v \otimes 1 \otimes w) \circ (\sigma \otimes 1 \otimes \sigma \otimes 1) \circ i_-,\end{aligned} \tag{9.11}$$

which should be satisfied in $\mathrm{Hom}(M_-, M_+^*\otimes V\otimes W)$ for any $V,W\in\mathcal{M}^e$, $v\in\tilde F(V)$, $w\in\tilde F(W)$. Using the identity $(1\otimes v\otimes 1\otimes w)\circ\gamma_{23}=\gamma_{23,4}\circ(1\otimes 1\otimes v\otimes w)$, we reduce (9.11) to the identity

$$\begin{aligned}&\beta_{34}\circ(1\otimes i_+\otimes i_-)\circ(\sigma\otimes 1)=\\ &(i_+^*\otimes 1\otimes 1\otimes 1\otimes 1)\circ\gamma_{23,4}\circ(\sigma\otimes 1\otimes\sigma\otimes 1)\circ i_-\end{aligned}\tag{9.12}$$

in $\mathrm{Hom}(M_-, M_+^*\otimes M_+\otimes M_-\otimes M_+\otimes M_-)$. Moving β_{34} from left to right and interchanging β_{34}^{-1} with $i_+^*\otimes 1\otimes 1\otimes 1\otimes 1$, so that (9.12) is equivalent to the identity:

$$\begin{aligned}&(1\otimes i_+\otimes i_-)\circ(\sigma\otimes 1)=\\ &(i_+^*\otimes 1\otimes 1\otimes 1\otimes 1)\circ\beta_{45}^{-1}\gamma_{23,4}\circ(\sigma\otimes 1\otimes\sigma\otimes 1)\circ i_-\end{aligned}\tag{9.13}$$

in $\mathrm{Hom}(M_-, M_+^*\otimes M_+\otimes M_+\otimes M_-\otimes M_-)$. It is clear that $\gamma_{1,23}\circ(1\otimes\sigma)=\sigma\otimes 1$ in $\mathrm{Hom}(M_-, M_-\otimes M_+^*\otimes M_+)$. Therefore, using the relations $\gamma_{23,4}\gamma_{3,45}^{-1}=\gamma_{23}\gamma_{45}^{-1}$, and $\beta\gamma=1$, we reduce (9.13) to

$$(1\otimes i_+)\circ\sigma=(i_+^*\otimes 1\otimes 1)\circ\gamma_{23}\circ(\sigma\otimes\sigma)\tag{9.14}$$

in $\mathrm{Hom}(\mathbf{1}, M_+^*\otimes M_+\otimes M_+)$. Since $i_+^*\circ\gamma=i_+^*$, we can rewrite (9.14) as

$$(1\otimes i_+)\circ\sigma=(i_+^*\otimes 1\otimes 1)\circ\gamma_{12,3}\circ(\sigma\otimes\sigma).\tag{9.15}$$

Using the equality $\gamma_{12,3}\circ(\sigma\otimes 1)=1\otimes\sigma$, we reduce (9.15) to

$$(1\otimes i_+)\circ\sigma=(i_+^*\otimes 1\otimes 1)\circ(1\otimes\sigma\otimes 1)\circ\sigma.\tag{9.16}$$

To prove this equality, we compute the image of $1\in\mathbf{1}$ under right hand side of (9.16). In this calculation, we can ignore the action of the associator because for any representations V_1,V_2,V_3 of $\mathfrak{g}$ the associator acts trivially on the $\mathfrak{g}$-invariants in $V_1\otimes V_2\otimes V_3$. The calculation yields that 1 goes to $(1\otimes i_+)(\sigma(1))$, which proves (9.16). The claim is proved.

Let $\mathcal{M}\subset\mathcal{M}^e$ be the full subcategory of discrete $\mathfrak{g}$-modules, and $\tilde U_h(\mathfrak{g})=\mathrm{End}(\tilde F|_{\mathcal{M}})$ be the quantization of $\mathfrak{g}$ constructed in Part I. It is easy to show that the homomorphism of topological Hopf algebras $\mathrm{End}\tilde F\to\tilde U_h(\mathfrak{g})$ defined by restriction from $\mathcal{M}^e$ to $\mathcal{M}$ is an isomorphism, since both algebras are canonically isomorphic to $U(\mathfrak{g})[[h]]$. This means that the morphism χ defined above induces an isomorphism of topological Hopf algebras $\tilde U_h(\mathfrak{g})\to U_h(\mathfrak{g})$. It is easy to check that this isomorphism maps $\tilde U_h(\mathfrak{g}_+)$ onto $U_h(\mathfrak{g}_+)$, which proves the proposition. □

10. Universality and functoriality of the quantization of Lie bialgebras.

10.1. *Acyclic functions.*

Let V be vector space over k. For any integers $m,n\geq 0$, let $H_{mn}=\mathrm{Hom}(V^{\otimes m},V^{\otimes n})$ be the space of tensors of rank m,n on V.

Let $B=\oplus_{m,n\geq 0}H_{mn}$. We have two binary operations on B: the tensor product and the composition. (If the composition makes no sense, we set it to zero).

Let $m_1, ..., m_r, n_1, ..., n_r$ be nonnegative integers, and $W = \oplus_{i=1}^r H_{m_i n_i}$. Let $p_i : W \to H_{m_i n_i}$ be the natural projections.

Let X be a subset of W and Y_X be the space of all functions from X to B. Denote by A_X the smallest subspace in Y_X closed under composition and tensor product, and satisfying the following conditions:

(i) $p_i|_X \in A_X$, $i = 1, ..., r$;

(ii) If $\sigma \in Y_X$ is a permutation operator from $S_p \subset H_{pp}$, regarded as a constant function on X, then $\sigma \in A_X$.

We call an element of A_X an acyclic function on X.

!?

10.2. *Universal quantization.*

Let $\mathfrak{g}$ be a Lie bialgebra over k, and $U_h(\mathfrak{g})$ be its quantization constructed in Chapter 9. Recall that as a $k[[h]]$-module, $U_h(\mathfrak{g})$ was identified with $U(\mathfrak{g})[[h]]$, which, in turn, we identify with $S\mathfrak{g}[[h]]$ in the standard way. Therefore, the multiplication map $\mu : U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g}) \to U_h(\mathfrak{g})$ splits in a direct sum of linear maps $\mu_{mn}^p : S^m\mathfrak{g} \otimes S^n\mathfrak{g} \to S^p\mathfrak{g}[[h]]$. Similarly, the coproduct $\Delta : U_h(\mathfrak{g}) \to U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ splits in a direct sum of linear maps $\Delta_p^{mn} : S^p\mathfrak{g} \to S^m\mathfrak{g} \otimes S^n\mathfrak{g}[[h]]$. All these linear maps are functions of the commutator $[,]$ and cocommutator δ of the Lie bialgebra $\mathfrak{g}$.

Now consider the setting of Section 10.1, with $m_1 = 2, m_2 = 1, n_1 = 1, n_2 = 2$, $W = \mathrm{Hom}(V \otimes V, V) \oplus \mathrm{Hom}(V, V \otimes V)$, $X \subset W$ the set of all pairs $([,], \delta) \in W$ satisfying the axioms of a Lie bialgebra. To every Lie bialgebra $\mathfrak{g} \in X$ ($\mathfrak{g} = (V, [,], \delta)$), we have associated a quantized universal enveloping algebra $U_h(\mathfrak{g})$, which is identified with $SV[[h]]$ as a $k[[h]]$-module. Thus, we can regard μ_{mn}^p, Δ_p^{mn} as functions on X with values in $H_{m+n,p}[[h]]$, $H_{p,m+n}[[h]]$, respectively.

Including continuity?

Theorem 10.1. *The coefficients of the h-expansion of μ_{mn}^p, Δ_p^{mn}, are acyclic functions on X.*

Remark. Drinfeld calls a quantization having this property a universal quantization. Thus, the quantization of Lie bialgebras constructed in Chapter 9 is universal.

Theorem 10.1 implies functoriality of quantization. Namely, let LBA denote the category of Lie bialgebras over k, and QUEA denote the category of quantum universal enveloping algebras over $k[[h]]$.

Theorem 10.2. *There exists a functor $Q : LBA \to QUEA$ such that for any $\mathfrak{g} \in LBA$ we have $Q(\mathfrak{g}) = U_h(\mathfrak{g})$.*

Proof. On objects, the functor Q is already defined. Now let us define it on morphisms. Let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of Lie bilagebras. It defines a linear map $Q(f) : S\mathfrak{g}_1[[h]] \to S\mathfrak{g}_2[[h]]$. By Theorem 10.1, this map defines a homomorphism of Hopf algebras $U_h(\mathfrak{g}_1) \to U_h(\mathfrak{g}_2)$. The theorem is proved. □.

10.3. *Proof of Theorem 10.1.* Let $\mathfrak{g}_+$ be a Lie bialgebra, $U_h(\mathfrak{g}_+)$ be its quantization. We will use the notation of Section 9.

By the definition, $U_h(\mathfrak{g}_+) = F(M_-) = \mathrm{Hom}(M_-, M_+^*\hat{\otimes}M_-)$. We have the identifications $\xi_\pm : S\mathfrak{g}_\mp \to M_\pm$ given by

$$\xi_\pm(\mathrm{Sym}(x_1 \otimes ... \otimes x_l)) = \mathrm{Sym}(x_1...x_l)1_\pm, x_i \in \mathfrak{g}_\mp, \tag{10.1}$$

and $\theta : U_h(\mathfrak{g}_+) \to M_-[[h]]$ by

$$\theta(x) = (1_+ \otimes 1)(x1_-).$$

They give us identifications $\eta = \theta^{-1}\xi_- : S\mathfrak{g}_+[[h]] \to U_h(\mathfrak{g}_+)$, and $\xi_+^* : M_+^* \to \hat{S}\mathfrak{g}_+$ (here hat denotes the completion by degree). From now on we fixed these identifications and thus regard the spaces $M_-, M_+^*, U_h(\mathfrak{g}_+)$ as sums of spaces of the form $S^m\mathfrak{g}_+$ or $S^m\mathfrak{g}_+[[h]]$. This allows us to make sense of the statement that certain maps between tensor products of these spaces, depending on $[,], \delta$, are acyclic functions (on X).

To prove the theorem, we need to show that the maps μ, Δ, S are acyclic.

To implement the proof, we need a few Lemmas.

Let $r \in \mathfrak{g}_+ \otimes \mathfrak{g}_-$ be the classical r-matrix, so that $\Omega = r + r^{op}$.

$M_- \cong U(\mathfrak{g}_+)$

Read! Is all content already here? (probably not...) (see sep. page)

Lemma 10.3. *(i) The map $r : M_- \otimes M_- \to M_- \otimes M_-$ is acyclic.*

(ii) The maps $r, r^{op} : M_+^* \hat{\otimes} M_- \to M_+^* \hat{\otimes} M_-$ are acyclic.

Proof. (i) For any nonnegative integers m, n consider the mapping $\mathfrak{g}_+^{\otimes m} \otimes \mathfrak{g}_+^{\otimes n} \to S\mathfrak{g}_+ \otimes S\mathfrak{g}_+$, given by

(10.2) $$x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n \to r(x_1 \ldots x_n 1_- \otimes y_1 \ldots y_m 1_-).$$

We need to show that this mapping is an acyclic function. We can do this by induction in $N = m + n$. If $N = 0$, the operator is zero and the statement is clear. Assume the statement is proved for $N = K - 1$ and let us prove it for $N = K$. Using the relation $[x \otimes 1 + 1 \otimes x, r] = \delta(x)$, $x \in \mathfrak{g}_+$, we can reduce the question to the case $m = K, n = 0$. In this case, the map is again zero, Q.E.D.

(ii) By the same reasoning as in (i), we get the statement for r. For r^{op}, we reduce the question to proving that the map $M_+^* \to M_+^* \hat{\otimes} M_-$ given by $v \to r^{op}(v \otimes 1_-)$ is acyclic.

Let $u = \mathrm{Sym}(y_1 \ldots y_m)1_+ \in M_+$, $y_1, \ldots, y_m \in \mathfrak{g}_-$. Let us compute the expression

$$X = (u \otimes 1)(r^{op}(v \otimes 1_-)) \in M_-.$$

We get

(10.3) $$X = -(r^{op}(u \otimes 1))(v \otimes 1_-) = \sum_i \langle L(b^i, y_1, \ldots, y_m)1_+, v\rangle a_i 1_-,$$

where a_i, b^i are dual bases of $\mathfrak{g}_+, \mathfrak{g}_-$, and L is a polynomial of commutators of $b^i, y_1, \ldots, y_m$ over $\mathbb{Q}$ which is symmetric in $b_i, y_1, \ldots, y_m$ and depends only on m.

Using the duality of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, from (10.3) we get

(10.4) $$X = \sum_i \langle b^i \otimes y_1 \otimes \ldots \otimes y_m, D_L(v)\rangle a_i 1_-$$

where $D_L(v) \in S\mathfrak{g}_+$ is a linear combination of iterated cocommutators applied to v. This implies that $r^{op}(v \otimes 1_-)$ is a linear combination of iterated cocommutators applied to v, so the map $v \to r^{op}(v \otimes 1_-)$ is acyclic. $\square$

For any $x \in M_-$, let $\psi_x : M_- \to M_+^* \hat{\otimes} M_-$ be the morphism such that $(1_+ \otimes 1)(\psi_x 1_-) = x$.

Lemma 10.4. *The map $M_-\otimes M_- \to M_+^*\hat{\otimes}M_-$ defined by $x\otimes y \to \psi_x y$ is acyclic.*

Proof. For any $x\in M_-$, $y\in U(\mathfrak{g}_+)$ we have $\psi_x y 1_- = \Delta_0(y)\psi_x 1_-$, where Δ_0 is the coproduct in $U(\mathfrak{g}_+)$. Since the map Δ_0 is obviously acyclic, it suffices to show that the assignment $x\to\psi_x 1_-$ is an acyclic map $M_-\to M_+^*\hat{\otimes}M_-$.

Let $z\in U(\mathfrak{g}_-)$. Since the vector $\psi_x 1_-$ is $\mathfrak{g}_-$-invariant, we have

$$(z1_+\otimes 1)(\psi_x 1_-) = (1_+\otimes S_0(z))(\psi_x 1_-) = S_0(z)x,$$

where S_0 is the antipode of $U(\mathfrak{g}_-)$. Let $z=\mathrm{Sym}(z_1\otimes ...\otimes z_m), x=\mathrm{Sym}(x_1\otimes ...\otimes x_n)1_-$, $z_i\in\mathfrak{g}_-$, $x_i\in\mathfrak{g}_+$. Computing the product $S_0(z)x$, we see that it is a linear combination of products of expressions of the form x_i and $Z=[\mathrm{ad}^*x_{i_1}...\mathrm{ad}^*x_{i_r}z_j, x_i]_+$, applied to 1_-, where $[za]_+$ denotes the $\mathfrak{g}_+$-component of $[za]$. Using the identity $[za]_+=\langle 1\otimes z,\delta(a)\rangle$, we can rewrite Z in the form $Z=\pm\langle 1\otimes z_j,(\mathrm{ad}x_{i_r}...\mathrm{ad}x_{i_1}\otimes 1)\delta(x_i)\rangle$. This shows that the product $S_0(z)x$, regarded as an element of $S\mathfrak{g}_+$, can be represented as a linear combination of summands of the form

$$\langle z_1\otimes ...\otimes z_m\otimes 1^{\otimes s}, x'\rangle,$$

where $x'\in S\mathfrak{g}_+$ is a polylinear symmetric function of $x_1,...,x_n$, such that the assignment $\mathrm{Sym}(x_1\otimes ...\otimes x_n)\to x'$ is acyclic. This proves the acyclicity of $\psi_x 1_-$. □

Now we can show the acyclicity of the product in $U_h(\mathfrak{g}_+)$. According to Chapter 9, for any $x,y\in M_-$,

$$x*y = (1_+\otimes 1_+\otimes 1)(\Phi^{-1}(1\otimes\psi_y)\psi_x 1_-) \tag{10.5}$$

Since Φ^{-1} is a noncommutative formal series of $h\Omega_{12}$, $h\Omega_{23}$, the acyclicity of the map $x\otimes y\to x*y$ follows from Lemmas 10.3 and 10.4.

To prove the acyclicity of the coproduct Δ, consider the linear operator $\mathcal{J}\in\mathrm{End}_k(M_-\otimes M_-)[[h]]$ defined by

$$\mathcal{J}(x\otimes y) = (1_+\otimes 1_+\otimes 1\otimes 1)(\Phi_{1,2,34}^{-1}\Phi_{2,3,4}\gamma_{23}\Phi_{2,3,4}^{-1}\Phi_{1,2,34}(\psi_x 1_-\otimes\psi_y 1_-)). \tag{10.6}$$

According to Proposition 9.3, the coproduct on $U_h(\mathfrak{g}_+)$, (when $U_h(\mathfrak{g}_+)$ is identified with M_-), is written in the form

$$\Delta(x) = \mathcal{J}^{-1}i_-(x). \tag{10.7}$$

The map J is acyclic by Lemmas 10.3 and 10.4. Therefore, Δ is acyclic.

10.4. *Universal quantization of quasitriangular Lie bialgebras.*

Let $\mathfrak{g}$ be a quasitriangular Lie bialgebra over k, and $U_h^{qt}(\mathfrak{g})$ be its quasitriangular quantization constructed in Chapter 6. As a $k[[h]]$-module, $U_h^{qt}(\mathfrak{g})$ was identified with $U(\mathfrak{g})[[h]]$, which we identify with $S\mathfrak{g}[[h]]$ in the standard way. Therefore, the multiplication map $\mu: U_h^{qt}(\mathfrak{g})\hat{\otimes}U_h^{qt}(\mathfrak{g})\to U_h^{qt}(\mathfrak{g})$ splits in a direct sum of linear maps $\mu_{mn}^p: S^m\mathfrak{g}\otimes S^n\mathfrak{g}\to S^p\mathfrak{g}[[h]]$. Similarly, the coproduct $\Delta: U_h^{qt}(\mathfrak{g})\to U_h^{qt}(\mathfrak{g})\hat{\otimes}U_h^{qt}(\mathfrak{g})$ splits in a direct sum of linear maps $\Delta_p^{mn}: S^p\mathfrak{g}\to S^m\mathfrak{g}\otimes S^n\mathfrak{g}[[h]]$, and the quantum R-matrix R splits in ta direct sum of $R_{mn}\in S^m\mathfrak{g}\otimes S^n\mathfrak{g}[[h]]$. All these linear maps are functions of the commutator $[,]$ and the classical r-matrix r of $\mathfrak{g}$.

inputs outputs

Now consider the setting of Section 10.1, with $m_1 = 2, m_2 = 0, n_1 = 1, n_2 = 2$, $W = \mathrm{Hom}(V \otimes V, V) \oplus V \otimes V$, $X \subset W$ the set of all pairs $([,], r) \in W$ satisfying the axioms of a quasitriangular Lie bialgebra. To every quasitriangular Lie bialgebra $\mathfrak{g} \in X$ ($\mathfrak{g} = (V, [,], r)$), we have associated a quantized universal enveloping algebra $U_h^{qt}(\mathfrak{g})$, which is identified with $SV[[h]]$ as a $k[[h]]$-module. Thus, we can regard $\mu_{mn}^p, \Delta_p^{mn}, R_{mn}$ as functions on X with values in $H_{m+n,p}[[h]]$, $H_{p,m+n}[[h]]$, $H_{0,m+n}[[h]]$, respectively.

Theorem 10.5. *The coefficients of the h-expansion of $\mu_{mn}^p, \Delta_p^{mn}, R_{mn}$ are acyclic functions on X.*

R is acyclic (in r of the form , allowing brackets)

[This may be an empty statement, for my purposes.] or not. . . .

Proof. The proof is analogous to the proof of Theorem 10.1 .□

Theorem 10.5 implies functoriality of quasitriangular quantization. Namely, let QTLBA denote the category of quasitriangular Lie bialgebras over k, and QTQUEA denote the category of quasitriangular quantum universal enveloping algebras.

Theorem 10.6. *There exists a functor $Q^{qt} : QTLBA \to QTQUEA$ such that for any $\mathfrak{g} \in QTLBA$ we have $Q^{qt}(\mathfrak{g}) = U_h^{qt}(\mathfrak{g})$.*

Proof. The proof is analogous to the proof of Theorem 10.2 .□

10.5. *Universal quantization of classical r-matrices.*

Let A be an associative algebra over k with unit, and $r \in A \otimes A$ be a solution of the classical Yang-Baxter equation. In Chapter 5, we assigned to (A, r) a solution of the quantum Yang-Baxter equation $R(r) \in A \otimes A[[h]]$.

Consider the setting of Section 10.1, with $m_1 = 2, m_2 = 0, m_3 = 0, n_1 = 1, n_2 = 1, n_3 = 2$, $W = \mathrm{Hom}(V \otimes V, V) \oplus V \oplus V \otimes V$, $X \subset W$ the set of all triples $(*, 1, r) \in W$ such that $*$ is an associative product, 1 is a unit, and r satisfies the classical Yang-Baxter equation. To every $A \in X$ ($A = (V, *, 1, r)$), we have associated a quantum R-matrix $R(r) \in A \otimes A[[h]]$. Thus, we can regard R as a function on X with values in $H_{02}[[h]]$.

Theorem 10.7. *The coefficients of the h-expansion of R are acyclic functions on X.*

Proof. The theorem follows from Theorem 10.5. □

Theorem 10.5 implies functoriality of quantization of classical r-matrices.

Namely, let us call a *classical Yang-Baxter algebra* a pair (A, r), where A is an associative algebra with unit over k, and $r \in A \otimes A$ satisfies the classical Yang-Baxter equation, and a *quantum Yang-Baxter algebra* a pair (A, R), where A is an associative algebra with unit over k, and $R \in A \otimes A[[h]]$ satisfies the quantum Yang-Baxter equation. Let CYBA, QYBA denote the categories of classical, respectively quantum, Yang-Baxter algebras. Morphisms in these categories are algebra homomorphisms preserving the unit and r (respectively, R).

Theorem 10.8. *There exists a functor $Q^{YB} : CYBA \to QYBA$ such that for any $(A, r) \in CYBA$ we have $Q^{YB}(A, r) = (A, R(r))$.*

Proof. The proof is analogous to the proof of Theorem 10.2. □

10.6. *Quantization over a complete local $\mathbb{Q}$-algebra*

Let K be a local Artinian or pro-Artinian commutative algebra over $\mathbb{Q}$, and I be the maximal ideal in K. Let $k = K/I$ (it is a field of characteristic 0). Let LBA(K), QTLBA(K), QUEA(K), QTQUEA(K) be the categories of Lie bialgebras, quasitriangular Lie bialgebras, quantum universal enveloping algebras, quasitriangular quantum universal enveloping algebras over K which are topologically free as K-modules and cocommutative modulo I. Let CYBA(K), QYBA(K) be the categories of classical Yang-Baxter algebras, quantum Yang-Baxter algebras, which are topologically free as K-modules and trivial modulo I (i.e. $r = 0, R = 1$ modulo I).

In Section 10.2, we showed that μ, δ are series of acyclic functions of $[,], h\delta$ with *rational* coefficients. Similarly, in Section 10.4, we showed that μ, Δ, R are series of acyclic functions with rational coefficients of $[,], hr$, and in Section 10.5 that R is a series of acyclic functions with rational coefficients of $*, 1, hr$. Therefore, we can use these formulas to define quantization over K (the series will converge in the topology of K). This quantization over K possesses the same functorial properties as the original quantization over $k[[h]]$. Thus, we obtain quantization functors $Q_A : LBA(K) \to QUEA(K)$, $Q_K^{qt} : QTLBA(K) \to QTQUEA(K)$, $Q_K^{YB} : CYBA(K) \to QYBA(K)$.

Appendix: computation of the product in $U_h(\mathfrak{a})$ modulo h^3.

To illustrate the proof of Theorem 10.1, here we compute the product in the quantization $U_h(\mathfrak{a})$ of a Lie bialgebra a modulo h^3. In the text below we always assume summation over repeated indices.

Let $\{a_i, i \in I\}$ be a basis of $\mathfrak{a}$, and $\{b^i\}$ be the topological basis of $\mathfrak{a}^*$ dual to $\{a_i\}$. Let us write down the commutation relations for the Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$:

$$[a_i a_j] = c_{ij}^k a_k, [b^i b^j] = f_k^{ij} b^k, [a_i b^j] = f_i^{jk} a_k - c_{ik}^j b^k. \tag{A1}$$

Let $1_+^* \in M_+^*$ be the functional on M_+^* defined by $1_+^*(x1_+) = \varepsilon(x)$, $x \in U(\mathfrak{a})$.

Let $\{(M_+^*)_n\}$ be the filtration of M_+^* which was defined in Chapter 7.

For $x \in U(\mathfrak{a})$, let $\psi_x : M_- \to M_+^* \hat{\otimes} M_-$ be the $\mathfrak{g}$-intertwiner such that

$$\psi_x 1_- \equiv 1_+^* \otimes x1_- \mod (M_+^*)_1.$$

For $x, y \in U(\mathfrak{a})$, we defined the quantized product $z = y \circ x$ to be the element of $U(\mathfrak{a})[[h]]$ such that the operator ψ_z is the composition

$$M_- \xrightarrow{\psi_x} M_+^* \hat{\otimes} M_- \xrightarrow{1 \otimes \psi_y} M_+^* \hat{\otimes} (M_+^* \hat{\otimes} M_-) \xrightarrow{\Phi^{-1}}$$
$$(M_+^* \hat{\otimes} M_+^*) \hat{\otimes} M_- \xrightarrow{i_+^* \otimes 1} M_+^* \hat{\otimes} M_-. \tag{A2}$$

We want to compute the product $a_q \circ a_p$ modulo h^3. We fix elements $\rho_i \in M_+^*$, $i \in I$, such that $\rho_i(1_+) = 0$, $\rho_i(b^j 1_+) = \delta_i^j$. These elements are uniquely defined modulo $(M_+^*)_2$.

Let $w^i \in M_-$ be the vectors such that

$$\psi_{a_p} 1_- \equiv 1_+^* \otimes a_p 1_- + \rho_i \otimes w^i \mod (M_+^*)_2 \otimes M_-. \tag{A3}$$

We must have $b^j \psi_{a_p} 1_- = 0$ for all j, so $1_+^* \otimes b^j a_p 1_- + b^j \rho_i \otimes w^i = 0$. But $b^j \rho_i(1_+) = \rho_i(-b^j 1_+) = -\delta_i^j$, so we get $w^i = b^i a_p 1_- = -f_p^{ik} a_k 1_-$.

Thus we get

$$\psi_{a_p}1_- \equiv 1_+^* \otimes a_p 1_- - f_p^{ik}\rho_i \otimes a_k 1_- \mod (M_+^*)_2 \hat{\otimes} M_-. \tag{A4}$$

Using (A4), we get

$$\begin{aligned} &\psi_{a_q} a_r 1_- \equiv (a_r \otimes 1 + 1 \otimes a_r)\psi_{a_q} 1_- \equiv \\ &1_+^* \otimes a_r a_q 1_- - f_q^{ik} a_r \rho_i \otimes a_k - f_q^{ik}\rho_i \otimes a_r a_k 1_- \mod (M_+^*)_2 \hat{\otimes} M_-. \end{aligned} \tag{A5}$$

We have

$$a_r \rho_i (b^j 1_+) = -\rho_i (a_r b^j 1_+) = \rho_i (c_{rk}^j b^k 1_+) = c_{ri}^j, \tag{A6}$$

Thus, substituting (A6) into (A5), we get

$$\psi_{a_q} a_r 1_- \equiv 1_+^* \otimes a_r a_q 1_- - f_q^{ik} c_{ri}^j \rho_j \otimes a_k 1_- - f_q^{ik}\rho_i \otimes a_r a_k 1_- \mod (M_+^*)_2 \hat{\otimes} M_-. \tag{A7}$$

In particular, we have

$$\begin{aligned} &(1 \otimes \psi_{a_q})\psi_{a_p} 1_- \equiv 1_+^* \otimes 1_+^* \otimes a_q a_p 1_- \\ &-c_{pi}^j f_q^{ik} 1_+^* \otimes \rho_j \otimes a_k 1_- - f_q^{ik} 1_+^* \otimes \rho_i \otimes a_p a_k 1_- - f_p^{ik}\rho_i \otimes 1_+^* \otimes a_k a_q 1_- + \\ &f_p^{ik} c_{kl}^j f_q^{ls} \rho_i \otimes \rho_j \otimes a_s 1_- + f_p^{ik} f_q^{ls} \rho_i \otimes \rho_l \otimes a_k a_s 1_- \\ &\mod (M_+^*)_2 \hat{\otimes} M_+^* \hat{\otimes} M_- + M_+^* \hat{\otimes} (M_+^*)_2 \hat{\otimes} M_-. \end{aligned} \tag{A8}$$

The definition of an associator implies

$$\Phi = 1 + \frac{h^2}{24}[t_{12}, t_{23}] + O(h^3). \tag{A9}$$

(see [Dr2],[Dr4]). This means that the part of the h^2-coefficient of $\Phi_{V_1V_2V_3}^{-1}$ which belongs to $\mathfrak{a}^* \otimes \mathfrak{a}^* \otimes \mathfrak{a}$ is $\frac{1}{24} c_{ij}^k b^i \otimes b^j \otimes a_k$.

Now let us apply Φ^{-1} to both sides of (A8). We want to compute the answer in the form $1_+^* \otimes 1_+^* \otimes u + ...$, $u \in M_-[[h]]$. To do this, we only need to use the last two terms on the r.h.s. of (A8) and the $\mathfrak{a}^* \otimes \mathfrak{a}^* \otimes \mathfrak{a}$-part of the quadratic term of Φ. The calculation gives

$$\begin{aligned} &\Phi^{-1}(1 \otimes \psi_{a_q})\psi_{a_p} 1_- \equiv 1_+^* \otimes 1_+^* \otimes u \mod (M_+^*)_1 \hat{\otimes} M_+^* \hat{\otimes} M_- + M_+^* \hat{\otimes} (M_+^*)_1 \hat{\otimes} M_-, \\ &u = a_q a_p 1_- + \frac{h^2}{24}(f_p^{ik} f_q^{ls} c_{kl}^j c_{ij}^m a_m a_s + f_p^{in} f_q^{ls} c_{il}^r a_r a_n a_s) 1_- \end{aligned} \tag{A10}$$

This shows that

$$a_q \circ a_p = a_q a_p + \frac{h^2}{24}(f_p^{ik} f_q^{ls} c_{kl}^j c_{ij}^m a_m a_s + f_p^{in} f_q^{ls} c_{il}^r a_r a_n a_s) + O(h^3). \tag{A11}$$

This formula is analogous to the formula deduced by Drinfeld [Dr3] (equation 1.1).

It is easy to see that this formula contains only acyclic monomials. Therefore, this formula is universal.

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