QUANTIZATION OF LIE BIALGEBRAS, I

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Abstract

In the paper [Dr3] V.Drinfeld formulated a number of problems in quantum group theory. In particular, he raised the question about the existence of a universal quantization for Lie bialgebras, which arose from the problem of quantization of Poisson Lie groups. When the paper [KL] appeared Drinfeld asked whether the methods of [KL] could be useful for the problem of universal quantization of Lie bialgebras. This paper gives a positive answer to a number of Drinfeld's questions, using the methods and ideas of [KL]. In particular, we show the existence of a universal quantization. We plan to provide positive answers to most of the remaining questions in [Dr3] in the following papers of this series.

Introduction

The main result of this paper is a construction of a universal quantization for Lie bialgebras(see [Dr3] Section 1).

The paper consists of two parts. In the first part we construct the quantization of a finite dimensional Lie bialgebra. In the second part we generalize this result to the infinite-dimensional case. The construction in the first part consists of three steps.

1) Given a finite dimensional Lie bialgebra $\mathfrak{a}$ over a field k of characteristic zero, we construct the double $\mathfrak{g}$ of $\mathfrak{a}$. Our definition of the double coincides with the one in [Dr1]. We consider the category $\mathcal{M}$ whose objects are $\mathfrak{g}$-modules and $\mathrm{Hom}_{\mathcal{M}}(U,W) = \mathrm{Hom}_{\mathfrak{g}}(U,W)[[h]]$. For any associator Φ ([Dr2,Dr4]) we define a structure of a braided monoidal category on $\mathcal{M}$, as in [Dr2].

2) We construct Verma modules M_+, M_- over $\mathfrak{g}$, and use them to construct a fiber functor from $\mathcal{M}$ to the tensor category of topologically free $k[[h]]$ modules: $F(V) = \mathrm{Hom}_{\mathcal{M}}(M_+ \otimes M_-, V)$. According to the categorical yoga, the existence of such a functor implies the existence of a (topological) Hopf algebra H isomorphic to $U(\mathfrak{g})[[h]]$ such that the tensor category $\mathcal{M}$ is equivalent to the category of representations of H. We show that H is isomorphic, as a topological algebra, to $U(\mathfrak{g})[[h]]$, where $U(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$.

3) We construct Hopf subalgebras $H_\pm$ of H and show that H_+ is a quantization of $\mathfrak{a}$ and that the algebra H is the quantum double of the Hopf algebra H_+.

Remark. We do not expect the existence of a quantization of any Lie bialgebra $\mathfrak{a}$ which is isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra.

As an application of our techniques, we prove that any classical r-matrix r over an associative algebra A ($r \in A\otimes A$) can be quantized. In other words, there exists a quantum R-matrix $R \in A\otimes A[[h]]$ such that $R = 1 + hr$. We also show that R is unitary ($R^{21}R = 1$) if r is unitary ($r^{21} = -r$). This answers questions in Section 3 of [Dr3]. As another application, we show the existence of the quantization of a quasitriangular Lie bialgebra $\mathfrak{a}$ (not necessarily finite dimensional) such that the obtained quantized universal enveloping algebra has a quasitriangular structure and is isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra, which solves questions in Section 4 of [Dr3].

The construction of quantization given in Part I has two drawbacks. First, it does not work literally for infinite dimensional Lie bialgebras. Second, it does not allow to prove functoriality and universality of quantization. Therefore, in Part II we slightly modify the construction, which puts the results of the first part in a more general setting. Now we consider arbitrary Lie bialgebras, not necessarily finite dimensional. In this case the double $\mathfrak{g}$ of $\mathfrak{a}$ can also be constructed, but it carries a nontrivial topology if $\dim\mathfrak{a} = \infty$. Instead of the category of all $\mathfrak{g}$-modules, we now consider the category $\mathcal{M}^e$ whose objects are equicontinuous $\mathfrak{g}$-modules, which are topological $\mathfrak{g}$-modules satisfying certain conditions. On this category, we define a braided monoidal structure analogously to the finite-dimensional case.

We construct Verma modules M_+, M_- over $\mathfrak{g}$ analogously to the finite-dimensional case. The module M_- is equicontinuous. The module M_+, in general, is not equicontinuous, but the module M_+^*, dual to M_+ in an appropriate topology, is an equicontinuous $\mathfrak{g}$-module. Using M_- and M_+^*, we define a fiber functor from $\mathcal{M}^e$ to the category of topological $k[[h]]$-modules, by $F(V) = \mathrm{Hom}_{\mathcal{M}^e}(M_-, M_+^*\otimes V)$. Since the module M_+ is not always equicontinuous, this functor is not always representable in $\mathcal{M}^e$. We define a tensor structure on F similarly to the finite dimensional case, and show that if $\mathfrak{g}$ is finite dimensional, the functors obtained in the first and second parts of the paper are isomorphic as tensor functors.

Next, we consider the algebra $H = \mathrm{End}F$. It is a topological algebra over $k[[h]]$ with a "coproduct" Δ, which maps H into a completion of $H\otimes H$, but not necessarily in $H\otimes H$.

Finally, we construct a subalgebra H_+ of H such that $\Delta(H_+) \subset H_+\otimes H_+$. This is a quantized universal enveloping algebra which is a quantization of $\mathfrak{a}$. For finite dimensional $\mathfrak{a}$, this quantization is isomorphic to the one obtained in the first part.

At the end of the paper we settle Drinfeld's question of the existence of a universal quantization of Lie bialgebras by showing that the quantization obtained in the second part of the paper is universal. In Drinfeld's language this means that the product and coproduct in the quantized algebra express in terms of acyclic tensor calculus via the commutator and cocmmutator. This result implies that our quantization of Lie bialgebras is a functor from the category of Lie bialgebras to the category of topological Hopf algebras. It also shows that our quantizations of classical r-matrices, unitary r-matrices, and quansitriangular Lie bialgebras are universal and functorial. Thus we answer positively the corresponding questions of Drinfeld [Dr3].

Remarks. 1. The material of Part I does not seem sufficient for proving the universality and functoriality. In fact, during the computation of the h^2-term of multiplication in $U_h(\mathfrak{a})$, using the method of Part I, one gets non-acyclic expressions, which cancel at the end of computation. Thus, the generalization to the infinite-dimensional case is essential for the proof of functoriality, even for finite-dimensional Lie bialgebras.

2. Most of the results of the paper could be formulated and proved over the ring $k[h]/(h^N)$ rather than $k[[h]]$, and then the results over $k[[h]]$ could be obtained as a limit. The only problem arises with the notions of the dual quantized universal enveloping algebra and the quantum double, which collapse over $k[[h]]/(h^N)$. This is why we chose to work over $k[[h]]$.

In fact, it is easy to see that the main results of the paper hold in a more general setting than stated. Namely, one can take the Lie bialgebra $\mathfrak{a}$ to be "dependent on h", i.e. to be a Lie bialgebra over the ring $k[[h]]$, which is topologically free as a $k[[h]]$-module. The universal acyclic formulas for quantization whose existence is shown in Section 10 are well defined for this case, and define a functor $\mathfrak{a} \to U_h(\mathfrak{a})$, from the category of Lie bialgebras over $k[[h]]$ which are topologically free as $k[[h]]$-modules to the category of quantized universal enveloping algebras. In the second paper of this series we will show that this functor is in fact an equivalence of categories.

Moreover, it is easy to see that the acyclic formulas of Chapter 10 which express the product and coproduct in $U_h(\mathfrak{a})$ in terms of commutator $[,]$ and cocommutator δ of $\mathfrak{a}$, are in fact formal series in $[,]$ and $h\delta$ with coefficients independent of h. This allows to generalize the results even further. Namely, let K be any local Artinian or pro-Artinian algebra over $\mathbb{Q}$, and I be the maximal ideal in K. Let $k = K/I$ (it is a field of characteristic 0). Given a Lie bialgebra $\mathfrak{a}$ over K which is (topologically) free as a K-module and cocommutative modulo I. Then the quantization functor of Chapter 10 is defined and assigns to $\mathfrak{a}$ a quantized universal enveloping algebra $U_{quant}(\mathfrak{a})$ over K. In the second paper we will show that this is an equivalence of categories between the category of Lie bialgebras over K which are topologically free as K-modules, and the category of quantized universal enveloping algebras over K. In particular, if $K = k[[h]]$ then $U_{quant}(\mathfrak{a}) = U_h(\tilde{\mathfrak{a}})$, where $\tilde{\mathfrak{a}}$ is $\mathfrak{a}$ with the same commutator and cocommutator $\delta_{\tilde{\mathfrak{a}}} = h^{-1}\delta_{\mathfrak{a}}$.

The third paper of this series is not written yet. Therefore we will only indicate the topics which we are planning to present in this part. First of all, we plan to consider the case of graded bialgebras with finite-dimensional homogeneous components and to show that in this case our formal quantization defines a family of Hopf algebras H_h, depending on a parameter $h \in k$. Our second goal is to prove that for Kac-Moody bialgebras our quantization coincides with the quantum Kac-Moody algebra. As another application of our techniques we plan to show how to define a quantum analog of the Kac-Moody algebra for arbitrary symmetrizable Cartan matrix (not necessarily integral) and show that for generic values of q the "size" of the quantized algebra is the same as of the usual Kac-Moody algebra. This would settle the questions in Section 8 of [Dr3].

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Part I

1. Drinfeld category.

The definitions and statements of Sections 1.1, 1.2 can be found in [Dr1].

1.1. *Lie bialgebras.*

Throughout this paper, k denotes a field of characteristic zero. Let $\mathfrak{a}$ be a Lie algebra over k, and. We δ be a linear map $\delta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$.

Definition.. *One says that the map δ defines a Lie bialgebra structure on $\mathfrak{a}$ if it satisfies two conditions:*

(i) δ is a 1-cocycle of $\mathfrak{a}$ with coefficients in $\mathfrak{a} \otimes \mathfrak{a}$, i.e.

$$\delta([ab]) = [1 \otimes a + a \otimes 1, \delta(b)] + [\delta(a), 1 \otimes b + b \otimes 1];$$

(ii) The map $\delta^* : \mathfrak{a}^* \otimes \mathfrak{a}^* \to \mathfrak{a}^*$ dual to δ is a Lie bracket on $\mathfrak{a}^*$.

In this case δ is called the cocommutator of $\mathfrak{a}$.

If $\mathfrak{a}$ is a finite dimensional Lie bialgebra then $\mathfrak{a}^*$ is a Lie bialgebra as well. Namely, the commutator in $\mathfrak{a}^*$ is dual to the cocommutator in $\mathfrak{a}$, and the cocommutator in $\mathfrak{a}^*$ is dual to the commutator in $\mathfrak{a}$. If $\mathfrak{a}$ is infinite-dimensional, then $\mathfrak{a}^*$ is not in general a Lie bialgebra but is a topological Lie bialgebra. That is, $\mathfrak{a}^*$ is a Lie algebra in the usual sense, but the cocommutator maps $\mathfrak{a}^*$ into the completed tensor product $\mathfrak{a}^* \hat{\otimes} \mathfrak{a}^*$ and not necessarily into the usual tensor product $\mathfrak{a}^* \otimes \mathfrak{a}^*$.

For any Lie bialgebra $\mathfrak{a}$, the vector space $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$ has a natural structure of a Lie algebra. Namely, $\mathfrak{a}, \mathfrak{a}^*$ are Lie subalgebras in $\mathfrak{g}$ with bracket defined above, and commutator between elements of $\mathfrak{a}, \mathfrak{a}^*$ is given by

$$[a, b] = (\mathrm{ad}^* a)b - (1 \otimes b)(\delta(a)), a \in \mathfrak{a}, b \in \mathfrak{a}^*, \tag{1.1}$$

where ad^* denotes the coadjoint action. There is an invariant nondegenerate inner product on $\mathfrak{g}$ given by $\langle a + a', b + b' \rangle = a'(b) + b'(a)$, $a, b \in \mathfrak{a}, a', b' \in \mathfrak{a}^*$. It is easy to show that (1.1) is the unique extension of the commutator from $\mathfrak{a}, \mathfrak{a}^*$ to $\mathfrak{g}$ for which the inner product $<, >$ is ad-invariant.

1.2. *Manin triples.*

Definition. A triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, where $\mathfrak{g}$ is a finite dimensional Lie algebra with a nondegenerate invariant inner product $\langle , \rangle$, and $\mathfrak{g}_+$,$\mathfrak{g}_-$ are isotropic Lie subalgebras, such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space, is called a finite dimensional Manin triple. To every finite dimensional Lie bialgebra $\mathfrak{a}$ one can associate the corresponding Manin triple $(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*, \mathfrak{a}, \mathfrak{a}^*)$, where the Lie structure on $\mathfrak{g}$ is as above. Conversely, if $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a finite dimensional Manin triple then $\mathfrak{g}_+$ (and $\mathfrak{g}_-$) is naturally a Lie bialgebra. Namely the pairing $\langle , \rangle$ identifies $\mathfrak{g}_+$ with $\mathfrak{g}_-^*$, so we can define $\delta : \mathfrak{g}_+ \to \mathfrak{g}_+ \otimes \mathfrak{g}_+$ to be the dual map to the commutator of $\mathfrak{g}_-$. This map is a 1-cocycle of the Lie algebra $\mathfrak{g}_+$ with coefficients in the module $\mathfrak{g}_+ \otimes \mathfrak{g}_+$, so it defines a structure of a Lie bialgebra on $\mathfrak{g}_+$.

Thus, there is a one-to-one correspondence between finite dimensional Lie bialgebras and finite dimensional Manin triples.

If $\mathfrak{a}$ is a Lie bialgebra then the Lie algebra $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*$ also has a natural structure of a Lie bialgebra. Namely, the cocommutator on $\mathfrak{g}$ is $\delta_{\mathfrak{g}} = \delta_{\mathfrak{a}} \oplus (-\delta_{\mathfrak{a}^*})$, where $\delta_{\mathfrak{a}}$, $\delta_{\mathfrak{a}^*}$ are the cocommutators of $\mathfrak{a}$, $\mathfrak{a}^*$.

The 1-cocycle $\delta_{\mathfrak{g}}$ is the coboundary of an element in $\mathfrak{g}\otimes\mathfrak{g}$. Namely, if $r \in \mathfrak{a}\otimes\mathfrak{a}^* \subset \mathfrak{g}\otimes\mathfrak{g}$ is the canonical element corresponding to the identity operator $\mathfrak{a} \to \mathfrak{a}$, then $\delta_{\mathfrak{g}} = dr$, where r is regarded as a 0-cochain of $\mathfrak{g}$ with coefficients in $\mathfrak{g}\otimes\mathfrak{g}$, and d is the differential in the cochain complex; that is $\delta_{\mathfrak{g}}(x) = [x\otimes 1 + 1\otimes x, r]$.

The Lie bialgebra $\mathfrak{g}$ is called the double of $\mathfrak{a}$.

Let $\mathfrak{a}$ be a Lie algebra, and $r \in \mathfrak{a}\otimes\mathfrak{a}$. The equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \tag{1.2}$$

in $U(\mathfrak{a})^{\otimes 3}$ is called the classical Yang-Baxter equation. It is easy to check that the canonical element r satisfies this equation.

Definition. We say that a Lie bialgebra $\mathfrak{a}$ is quasitriangular if it is equipped with an element $r \in \mathfrak{a}\otimes\mathfrak{a}$ satisfying the classical Yang-Baxter equation, such that $\delta(a) = [a\otimes 1 + 1\otimes a, r]$ for any $a \in \mathfrak{a}$ (i.e. δ is a coboundary of r).

For example, the double $\mathfrak{g}$ of any finite dimensional Lie bialgebra $\mathfrak{a}$ equipped with the canonical element r is a quasitriangular Lie bialgebra.

1.3. *Associators.* Recall some notation and definitions from the theory of associators [Dr2,BN]. Let T_n be the algebra over k generated by elements t_{ij}, $1 \le i, j \le n$, $i \ne j$, with defining relations $t_{ij} = t_{ji}$, $[t_{ij}, t_{lm}] = 0$ if i, j, l, m are distinct, and $[t_{ij}, t_{ik} + t_{jk}] = 0$.

Let $P_1, ..., P_n$ be disjoint subsets of $\{1, ..., m\}$. There exists a unique homomorphism $\rho_{P_1...P_n} : T_n \to T_m$ defined by

$$\rho_{P_1...P_m}(t_{ij}) = \sum_{p\in P_i, q\in P_j} t_{pq}. \tag{1.3}$$

For any $X \in T_n$, we denote $\rho_{P_1...P_n}(X)$ by $X_{P_1,...,P_n}$.

Let $\Phi \in T_3$. The relation

$$\Phi_{1,2,34}\Phi_{12,3,4} = \Phi_{2,3,4}\Phi_{1,23,4}\Phi_{1,2,3} \tag{1.4}$$

in $T_4[[h]]$ (=relation (1.2) in [Dr2]) is called the pentagon relation.

Let $B = e^{ht_{12}/2} \in T_2[[h]]$. The relations

$$\begin{aligned} B_{12,3} &= \Phi_{3,1,2}B_{1,3}\Phi^{-1}_{1,3,2}B_{2,3}\Phi_{1,2,3}, \\ B_{1,23} &= \Phi^{-1}_{2,3,1}B_{1,3}\Phi_{2,1,3}B_{1,2}\Phi^{-1}_{1,2,3}. \end{aligned} \tag{1.5}$$

in $T_3[[h]]$ (=relations (3.9a),(3.9b) in [Dr2]) are called the hexagon relations.

The element Φ is called an associator if it satisfies the pentagon and hexagon relations.

For $k = \mathbb{C}$, an example of an associator is the Drinfeld associator Φ_{KZ} obtained from the KZ equations, as explained in [Dr2].

The following theorem about associators is due to Drinfeld ([Dr4], Theorem A').

Theorem 1.1. *There exists an associator defined over* $\mathbb{Q}$.

This theorem implies that there exists an associator defined over any field k of characteristic zero. From now on we will fix such an associator Φ.

1.4. *Drinfeld category.*

Let $\mathfrak{g}$ be a Lie algebra over k, and $\Omega \in S^2\mathfrak{g}$ be a $\mathfrak{g}$-invariant element.

We will be mostly interested in the case when $\mathfrak{g}$ belongs to a finite dimensional Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$, and $\Omega = \sum_i g_i \otimes g^i$, where $\{g_i\}$ is a basis of $\mathfrak{g}$, and $\{\mathfrak{g}^i\}$ is the dual basis to $\{\mathfrak{g}_i\}$ with respect to the invariant inner product on $\mathfrak{g}$. In this case the element Ω is called the Casimir element.

Let $\mathcal{M}$ denote the category whose objects are $\mathfrak{g}$-modules, and $\mathrm{Hom}_{\mathcal{M}}(U, W) = \mathrm{Hom}_{\mathfrak{g}}(U, W)[[h]]$. This is a $k[[h]]$-linear additive category. For brevity we will later write Hom for $\mathrm{Hom}_{\mathcal{M}}$.

Drinfeld [Dr2] defined a structure of a braided monoidal category on $\mathcal{M}$ as follows.

For any $V_1, V_2, V_3 \in \mathcal{M}$, consider a homomorphism $\theta : T_3[[h]] \to \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$ by $\theta(t_{ij}) = \Omega_{ij}$, and define $\Phi_{V_1V_2V_3} = \theta(\Phi)$.

For any $V_1, V_2 \in \mathcal{M}$, define $V_1 \otimes V_2 \in \mathcal{M}$ to be the usual tensor product of V_1, V_2 and the associativity morphism to be $\Phi_{V_1V_2V_3}$, regarded as an element of $\mathrm{Hom}((V_1 \otimes V_2) \otimes V_3, V_1 \otimes (V_2 \otimes V_3))$. For any $V_1, V_2 \in \mathcal{M}$, introduce the braiding $\beta_{V_1V_2} : V_1 \otimes V_2 \to V_2 \otimes V_1$ by the formula $\beta = s \circ e^{h\Omega/2}$, where s is the permutation. It follows from relations (1.4), (1.5) that the morphisms $\Phi_{V_1V_2V_3}$ and $\beta_{V_1V_2}$ define the structure of a braided monoidal category on $\mathcal{M}$ (see [Dr2]).

2. The fiber functor.

2.1. *The category of topologically free $k[[h]]$-modules.*

Let V be a vector space over k. Then the space $V[[h]]$ of formal power series in h with coefficients in V has a natural structure of a topological $k[[h]]$-module. We call a topological $k[[h]]$-module topologically free if it is isomorphic to $V[[h]]$ for some V.

Let $\mathcal{A}$ be the category of topologically free $k[[h]]$-modules, where morphisms are continuous $k[[h]]$-linear maps. It is an additive category. Define the tensor structure on $\mathcal{A}$ as follows: for $V, W \in A$ define $V \otimes W$ to be the projective limit of the $k[h]/h^n$-modules $(V/h^nV) \otimes_{k[h]/h^n} (W/h^nW)$ as $n \to \infty$.

Let Vect be the category of vector spaces. We have the functor of extension of scalars, $V \mapsto V[[h]]$, acting from Vect to $\mathcal{A}$. This functor respects the tensor product, i.e. $(V \otimes W)[[h]]$ is naturally isomorphic to $V[[h]] \otimes W[[h]]$. The category $\mathcal{A}$ equipped with the functor $\otimes$ is a symmetric monoidal category.

If $X \in \mathcal{A}$ then $X^* = \mathrm{Hom}_{\mathcal{A}}(X, k[[h]])$ is a topologically free $k[[h]]$-module. The assignment $X \to X^*$ is a contravariant functor from $\mathcal{A}$ to itself.

2.2. *The forgetful functor.*

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a finite dimensional Manin triple, $\Omega \in S^2\mathfrak{g}$ be the Casimir element associated to the inner product $\langle , \rangle$ on $\mathfrak{g}$, and $\mathcal{M}$ be the Drinfeld category associated to $\mathfrak{g}$.

Let $F : \mathcal{M} \to \mathcal{A}$ be the functor given by $F(M) = \mathrm{Hom}(U(\mathfrak{g}), M)$, where $U(\mathfrak{g})$ is regarded as a left $\mathfrak{g}$-module. This functor is naturally isomorphic to the "forgetful" functor which assigns to every $\mathfrak{g}$-module M the $k[[h]]$-module $M[[h]]$. The isomorphism between these two functors is given by the assignment $f \in F(M) \to f(1) \in M[[h]]$.

2.3. *The Verma modules.*

Consider the Verma modules $M_+ = \mathrm{Ind}_{\mathfrak{g}_+}^{\mathfrak{g}} \mathbf{1}$, $M_- = \mathrm{Ind}_{\mathfrak{g}_-}^{\mathfrak{g}} \mathbf{1}$ (here $\mathbf{1}$ denotes the trivial 1-dimensional representation). By the Poincare-Birkhoff-Witt theorem, the product in $U(\mathfrak{g})$ defines linear isomorphisms $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-) \to U(\mathfrak{g})$, and

$U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+) \to U(\mathfrak{g})$. This shows that the modules $M_\pm$ are freely generated over $U(\mathfrak{g}_\mp)$ by vectors $1_\pm$ such that $\mathfrak{g}_\pm 1_\pm = 0$, and are identified (as vector spaces) with $U(\mathfrak{g}_\mp)$ via $x1_\pm \to x$.

Since the vectors $1_\pm \otimes 1_\pm \in M_\pm \otimes M_\pm$ are $\mathfrak{g}_\pm$-invariant, there exist unique $\mathfrak{g}$-module morphisms $i_\pm : M_\pm \to M_\pm \otimes M_\pm$ such that $i_\pm(1_\pm) = 1_\pm \otimes 1_\pm$. These morphisms in the category $\mathcal{M}$ will play a crucial role in our constructions below.

Lemma 2.1. *The assignment* $1 \to 1_+ \otimes 1_-$ *extends to an isomorphism of* $\mathfrak{g}$*-modules* $\phi : U(\mathfrak{g}) \to M_+ \otimes M_-$.

Proof. Since $M_\pm$ has been identified with $U(\mathfrak{g}_\mp)$, we can regard the map ϕ as a linear map $U(\mathfrak{g}) \to U(\mathfrak{g}_-) \otimes U(\mathfrak{g}_+)$. It is clear that this map preserves the standard filtration, so it defines a map of the associated graded objects: $S\mathfrak{g} \to S\mathfrak{g}_- \otimes S\mathfrak{g}_+$. This map is the isomorphism induced by the isomorphism $\mathfrak{g} \to \mathfrak{g}_- \oplus \mathfrak{g}_+$. Therefore, ϕ is an isomorphism. $\square$

Lemma 2.1 implies that the functor F can be identified with the functor $V \to \mathrm{Hom}(M_+ \otimes M_-, V)$. This definition of F will be used from now on.

2.4. *Tensor structure on the functor* F.

Let $(\mathcal{C}, \otimes)$ be a monoidal category, Φ be the associativity constraint in $\mathcal{C}$, and $\mathbf{1}$ be the identity object in $\mathcal{C}$. For simplicity we assume that $\mathbf{1} \otimes X = X \otimes \mathbf{1} = X$ for any object $X \in \mathcal{C}$, and the functorial isomorphisms $X \otimes X \otimes \mathbf{1}$, $X \to \mathbf{1} \otimes X$ the are identity morphisms.

Let $F : \mathcal{C} \to \mathcal{A}$ be a functor such that $F(\mathbf{1}) = k[[h]]$.

what would be "an automorphism of F"?

Definition. By a tensor structure on the functor F one means a functorial isomorphism $J_{VW} : F(V) \otimes F(W) \to F(V \otimes W)$ satisfying the associativity identity $F(\Phi_{VWU}) J_{V \otimes W, U} \circ (J_{VW} \otimes 1) = J_{V, W \otimes U} \circ (1 \otimes J_{WU})$, such that for any object V $J_{V\mathbf{1}} = J_{\mathbf{1}V} = 1$. A functor equipped with a tensor structure is called a tensor functor.

Now we describe a tensor structure on the functor F constructed in Section 2.2.

For any $v \in F(V)$, $w \in F(W)$ define $J_{VW}(v \otimes w)$ to be the composition of morphisms:

$$
\begin{aligned}
M_+ \otimes M_- &\xrightarrow{i_+ \otimes i_-} (M_+ \otimes M_+) \otimes (M_- \otimes M_-) \xrightarrow{\text{associativity morphism}} \\
&(M_+ \otimes (M_+ \otimes M_-)) \otimes M_- \xrightarrow{(1 \otimes \beta_{23}) \otimes 1} \\
&(M_+ \otimes (M_- \otimes M_+)) \otimes M_- \xrightarrow{\text{associativity morphism}} \\
&(M_+ \otimes M_-) \otimes (M_+ \otimes M_-) \xrightarrow{v \otimes w} V \otimes W,
\end{aligned}
\tag{2.1}
$$

Is the sense of this xing ever used? Is all the same under $\beta_{23} \to \beta_{23}^{-1}$?

where β_{23} denotes the braiding β acting in the second and third components of the tensor product.

It is clear from this definition that all combinatorial complexity of the morphism J comes from the arrows "associativity morphism" which involve associators.

The arrows "associativity morphism" make the problem of checking various identities for J (for example, the associativity identity) rather tedious. To avoid this, we can use MacLane's theorem, which says that any monoidal category is equivalent to a strict one. Namely, when we check identities between morphisms in the category, we will assume that the category $\mathcal{M}$ is replaced with an equivalent strict monoidal category and ignore associativity morphisms. For example, the definition of J will look as follows:

$$J_{VW}(v \otimes w) = (v \otimes w) \circ (1 \otimes \beta_{23} \otimes 1) \circ (i_+ \otimes i_-).$$

However, when we do computations with vectors in modules from $\mathcal{M}$, it is important to pay attention to brackets, since different positions of brackets are related with each other by the associator.

Proposition 2.2. *The maps J_{VW} are isomorphisms and define a tensor structure on the functor F.*

Proof.. It is obvious that J_{VW} is an isomorphism since it is an isomorphism modulo h. It is also clear that $J_{V\mathbf{1}} = J_{\mathbf{1}V} = 1$. Thus the only thing we need to check is the associativity identity $J_{V\otimes W,U} \circ (J_{VW} \otimes 1) = J_{V,W\otimes U} \circ (1 \otimes J_{WU})$. To prove this equality, we need the following result.

Lemma 2.3 $(i_\pm \otimes 1) \circ i_\pm = (1 \otimes i_\pm) \circ i_\pm$ in $\mathrm{Hom}(M_\pm, M_\pm^{\otimes 3})$.

Proof. We prove the identity for i_+. The identity for i_- is proved in the same way.

We need to show that for any vector $x \in M_+$

$$\Phi \cdot (i_+ \otimes 1) i_+ x = (1 \otimes i_+) i_+ x. \tag{2.2}$$

Since comultiplication in $U(\mathfrak{g}_-)$ is coassociative, i.e. $(i_+ \otimes 1)i_+x = (1 \otimes i_+)i_+x$, it is sufficient to show that the associator Φ is the identity on the image of $(i_+ \otimes 1)i_+$. Because Φ is $\mathfrak{g}$-invarint, it is enough to show that $\Phi \cdot (i_+ \otimes 1)i_+1_+ = (i_+ \otimes 1)i_+1_+$, i.e.

$$\Phi \cdot (1_+ \otimes 1_+ \otimes 1_+) = 1_+ \otimes 1_+ \otimes 1_+. \tag{2.3}$$

Since the subalgebras $\mathfrak{g}_+, \mathfrak{g}_-$ are isotropic, the operators Ω_{12}, Ω_{23} annihilate the vector $1_+ \otimes 1_+ \otimes 1_+$. Thus, equation (2.3) follows from the definition of Φ.□

Now we can finish the proof of the proposition. Let $\psi_1, \psi_2 : M_+ \otimes M_- \to (M_+ \otimes M_-)^{\otimes 3}$ be the morphisms defined by

$$\psi_1 = (1 \otimes \beta_{23} \otimes 1 \otimes 1 \otimes 1) \circ (i_+ \otimes i_- \otimes 1 \otimes 1) \circ (1 \otimes \beta_{23} \otimes 1) \circ (i_+ \otimes i_-),$$
$$\psi_2 = (1 \otimes 1 \otimes 1 \otimes \beta_{45} \otimes 1) \circ (1 \otimes 1 \otimes i_+ \otimes i_-) \circ (1 \otimes \beta_{23} \otimes 1) \circ (i_+ \otimes i_-), \tag{2.4}$$

Then for any $v \in F(V), w \in F(W), u \in F(U)$ we have

$$J_{V\otimes W,U}(J_{VW} \otimes 1)(v \otimes w \otimes u) = (v \otimes w \otimes u) \circ \psi_1,$$
$$J_{V,W\otimes U}(1 \otimes J_{WU})(v \otimes w \otimes u) = (v \otimes w \otimes u) \circ \psi_2.$$

Therefore, to prove the proposition, it is sufficient to show that $\psi_1 = \psi_2$.

To prove this equality, we observe that the functoriality of the braiding implies the identities

$$(i_+ \otimes i_- \otimes 1 \otimes 1) \circ (1 \otimes \beta_{23} \otimes 1) = (1 \otimes \beta_{3,45} \otimes 1) \circ (i_+ \otimes 1 \otimes i_- \otimes 1),$$
$$(1 \otimes 1 \otimes i_+ \otimes i_-) \circ (1 \otimes \beta_{23} \otimes 1) = (1 \otimes \beta_{23,4} \otimes 1) \circ (1 \otimes i_+ \otimes 1 \otimes i_-) \tag{2.5}$$

(here $\beta_{3,45}$ means the braiding applied to the third factor and to the product of the fourth and the fifth factors). Using Lemma 2.3 and identities (2.5), we reduce the statement $\psi_1 = \psi_2$ to the identity

(2.6) $(1 \otimes \beta_{23} \otimes 1 \otimes 1 \otimes 1) \circ (1 \otimes \beta_{3,45} \otimes 1) = (1 \otimes 1 \otimes 1 \otimes \beta_{45} \otimes 1) \circ (1 \otimes \beta_{23,4} \otimes 1),$

which follows directly from the braiding axioms. □

We call the functor F equipped with the tensor structure J the fiber functor.

Why "fiber"?

3. Quantization of the double of a Lie bialgebra.

3.1. *Topological Hopf algebras.*

Let A be an algebra over $k[[h]]$ with unit. Let I be a proper two-sided ideal in A such that $h \in I$. This ideal gives rise to a translation invariant topology on A such that $\{I^n, n \geq 0\}$ is a basis of neighborhoods of 0. We will call A a topological algebra if A is complete in this topology, and $A/h^N A$ is a free $k[h]/(h^N)$-module for each $N \geq 1$.

In modules instead of algebras, not the same as free $k[[h]]$-modules, but is it the same as $\hat{\oplus} h^n B$ for a fixed B? Is $B = A/hA$?

Let A, B be two topological algebras, I, J be the the corresponding ideals. Define $A \hat{\otimes} B$ to be the projective limit of algebras $A/I^n \otimes_{k[h]/h^n} B/J^n$ as $n \to \infty$. Then $A \hat{\otimes} B$ is also a topological algebra, with topology defined by the ideal $I \hat{\otimes} B + A \hat{\otimes} J$.

We say that a topological algebra A is a topological Hopf algebra if it is equipped with comultiplication $\Delta : A \to A \hat{\otimes} A$, the counit $\varepsilon : A \to k[[h]]$, and the antipode $S : A \to A$, which are $k[[h]]$-linear, continuous, and satisfy the standard axioms of a Hopf algebra. Note that an infinite dimensional topological Hopf algebra may not be literally a Hopf algebra because the image of comultiplication may not belong to the algebraic tensor square of A.

Topological algebras and Hopf algebras over k are defined similarly.

If A is a topological algebra or Hopf algebra over $k[[h]]$ then $B = A/hA$ is a topological algebra, respectively Hopf algebra, over k. In such a case we say that A is a formal deformation of B. In particular, if $B = U(\mathfrak{g})$ with discrete topology, where $\mathfrak{g}$ is a Lie algebra, then A is called a quantized universal enveloping algebra [Dr1].

The following definition is due to Drinfeld [Dr1].

Definition. *Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. We say that a quantized universal enveloping algebra A is a quantization of $(\mathfrak{g}, \delta)$, or that $(\mathfrak{g}, \delta)$ is the quasiclassical limit of A, if*

(i) A/hA is isomorphic to $U(\mathfrak{g})$ as a Hopf algebra, and

(ii) For any $x_0 \in \mathfrak{g}$ and any $x \in A$ equal to x_0 mod h one has

$$h^{-1}(\Delta(x) - \Delta^{op}(x)) \equiv \delta(x_0) \text{ mod } h,$$

where Δ^{op} is the opposite comultiplication ($\Delta^{op} = s\Delta$).

Suppose I had the world's best behaving expansion for v-knots. Will it buy me a quantization of doubles? PS. I still don't understand twists, in vKTG-only language.

3.2. *The algebra of endomorphisms of the fiber functor.*

Let $H = \text{End}(F)$ be the algebra of endomorphisms of the functor F. This algebra is naturally isomorphic to $U(\mathfrak{g})[[h]]$. Namely, the map $\alpha : U(\mathfrak{g})[[h]] \to H$ is defined on $x \in U(\mathfrak{g})$ by the formula $(\alpha(x)f)(y) = f(yx)$, where $f \in \text{Hom}(U(\mathfrak{g}), M)$, and is extended by linearity and continuity to $U(\mathfrak{g})[[h]]$. This map is an isomorphism of algebras. From now on we will make no distinction between $U(\mathfrak{g})[[h]]$ and H, identifying them by α.

Let $F^2 : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ be the bifunctor defined by $F^2(V, W) = F(V) \otimes F(W)$. It is clear that $\mathrm{End}(F^2) = H \otimes H$.

The algebra H has a natural comultiplication $\Delta : H \to H \otimes H$ defined by $\Delta(a)_{V,W}(v \otimes w) = J_{VW}^{-1} a_{V \otimes W} J_{VW}(v \otimes w)$, $a \in H$, $v \in F(V), w \in F(W)$ where a_V denotes the action of a in $F(V)$. We can also define the counit on H by $\varepsilon(a) = a_{\mathbf{1}} \in k[[h]]$, where $\mathbf{1}$ is the neutral object.

For any $V \in \mathcal{M}$, let V^* be the dual space to V (regarded as an object of $\mathcal{M}$), and let $\sigma_V : V^* \otimes V \to \mathbf{1}$ be the canonical pairing. We have a functorial isomorphism $\xi_V : F(V^*) \to F(V)^*$ defined by $\xi_V(v^*)(v) = F(\sigma_V) J_{V^*V}(v^* \otimes v)$, $v \in F(V), v^* \in F(V^*)$. For any $a \in H$, let $\widetilde{S(a)}_V = (\xi_V^*)^{-1} a_{V^*}^* \xi_V^*$ be a morphism $F(V)^{**} \to F(V)^{**}$. It is easy to show that the subspace $F(V) \subset F(V)^{**}$ is invariant under this morphism. The antipode $S : H \to H$ is defined by $S(a)_V = \widetilde{S(a)}_V|_{F(V)}$.

Proposition 3.1. *The algebra H equipped with Δ, ε, S is a topological Hopf algebra.*

The proof is straightforward.

3.3. *Explicit representation of comultiplication and antipode.*

Let $\Delta_0 : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be the standard coproduct. For any $V, W \in \mathcal{M}$, let $J_{VW}^0 : F(V) \otimes F(W) \to F(V \otimes W)$ be the morphism defined by the formula $J_{VW}^0(v \otimes w)(x) = (v \otimes w)(\Delta_0(x))$, $x \in U(\mathfrak{g})$, $v \in F(V)$, $w \in F(W)$. It is clear that $J_{VW} \equiv J_{VW}^0 \mod h$.

Let $J \in U(\mathfrak{g})^{\otimes 2}[[h]]$ be defined by the formula

(3.1)

$$J = (\phi^{-1} \otimes \phi^{-1}) \Big(\Phi_{1,2,34}^{-1} (1 \otimes \Phi_{2,3,4}) s e^{h\Omega_{23}/2} (1 \otimes \Phi_{2,3,4}^{-1}) \Phi_{1,2,34} (1_+ \otimes 1_+ \otimes 1_- \otimes 1_-) \Big),$$

where ϕ is the isomorphism of Lemma 2.1.

Proposition 3.2. *For any $V, W \in \mathcal{A}$, $v \in F(V)$, $w \in F(W)$ one has $J_{VW}(v \otimes w) = J_{VW}^0 J(v \otimes w)$.*

Proof. The statement follows from the definition (2.1) of J_{VW}. □

Lemma 3.3. *Let $a \in H$. Then*

$$\Delta(a) = J^{-1} \Delta_0(a) J. \tag{3.2}$$

Proof. The lemma follows from Proposition 3.2 and the identities $\Delta_0(a)_{V,W} = (J_{VW}^0)^{-1} a_{V \otimes W} J_{VW}^0$, $\Delta(a)_{V,W} = J_{VW}^{-1} a_{V \otimes W} J_{VW}$, $a \in U(\mathfrak{g})$. □

Now consider the explicit expression for the antipode. For any $V \in \mathcal{M}$ define the morphism $\xi_V^0 : F(V^*) \to F(V)^*$ by $\xi_V^0(v^*)(v) = F(\sigma_V) J_{V^*V}^0(v^* \otimes v)$, $v \in V$, $v^* \in V^*$. It is clear that $\xi_V \equiv \xi_V^0 \mod h$.

Let $S_0 : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the usual antipode. Let $J = \sum_j x_j \otimes y_j$, $x_j, y_j \in U(\mathfrak{g})[[h]]$ (the sum is finite modulo h^n for any n). Define an element $Q \in U(\mathfrak{g})[[h]]$ by $Q = \sum_j S_0(x_j) y_j$.

Lemma 3.4. *Let $a \in H$. Then*

$$S(a) = Q^{-1} S_0(a) Q. \tag{3.3}$$

Proof. It follows from the definitions of ξ_V, ξ^0_V, and Q that $\xi_V = \xi^0_V S_0(Q)_{V^*}$. Thus the Lemma follows from the formulas $S(a)_V = (\xi^*_V)^{-1} a^*_{V^*} \xi^*_V|_{F(V)}$, $S_0(a)_V = (\xi^{0*}_V)^{-1} a^*_{V^*} \xi^{0*}_V|_{F(V)}$. □

Thus, we have proved the following result.

Corollary 3.5. *Introduce a new comultiplication and antipode on the topological Hopf algebra $U(\mathfrak{g})[[h]]$ by*

$$\Delta(x) = J^{-1}\Delta_0(x)J,\ S(x) = Q^{-1}S_0(x)Q, \tag{3.4}$$

where Δ_0, S_0 are the usual comultiplication and antipode. Then $(U(\mathfrak{g})[[h]], \Delta, S)$ is a topological Hopf algebra isomorphic to H.

We will denote the topological Hopf algebra $(U(\mathfrak{g})[[h]], \Delta, S)$ by $U_h(\mathfrak{g})$.

Remark. It is easy to see that according to the terminology of [Dr2], the element J^{-1} is a twist that realizes an equivalence between the quasi-Hopf algebra $(U(\mathfrak{g})[[h]], \Phi)$ and the Hopf algebra $U_h(\mathfrak{g})$.

3.4. *The quasiclassical limit of $U_h(\mathfrak{g})$.*

Proposition 3.6. *The topological Hopf algebra $U_h(\mathfrak{g})$ is a quantization of the Lie bialgebra $(\mathfrak{g}, \delta_\mathfrak{g})$.*

Proof. Take $a \in \mathfrak{g} \subset U_h(\mathfrak{g})$. Let $\delta(a) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be defined by the formula $\delta(a) = h^{-1}(\Delta(a) - \Delta^{op}(a))$ mod h. To prove the proposition, we need to show that for any $a \in \mathfrak{g}$ one has $\delta(a) = \delta_\mathfrak{g}(a)$, where $\delta_\mathfrak{g}(a)$ is defined in Chapter 1.

It is easy to check the following identities:

$$e^{h\Omega/2} \equiv 1 + h\Omega/2 \text{ mod } h^2, \Phi \equiv 1 \text{ mod } h^2. \tag{3.5}$$

Let $\{g^+_j\}$ be a basis of $\mathfrak{g}_+$, $\{g^-_j\}$ be the dual basis of $\mathfrak{g}_-$, and $r = \sum_j g^+_j \otimes g^-_j$. Identities (3.1) and (3.5) imply that

$$J \equiv 1 + hr/2 \text{ mod } h^2. \tag{3.6}$$

Therefore, by Lemma 3.3,

$$\Delta(a) \equiv \Delta_0(a) + \frac{h}{2}[\Delta_0(a), r] \text{ mod } h^2. \tag{3.7}$$

Thus,

$$\Delta(a) - \Delta^{op}(a) \equiv \frac{h}{2}[\Delta_0(a), r - sr] \text{ mod } h^2. \tag{3.8}$$

Since $r + sr$ $(= \Omega)$ is $\mathfrak{g}$-invariant, we obtain

$$\delta = dr = \delta_\mathfrak{g}, \tag{3.9}$$

Q.E.D. □

3.5. *The quaitriangular structure on $U_h(\mathfrak{g})$.*

Define the element

$$R = (J^{op})^{-1} e^{h\Omega/2} J \in U_h(\mathfrak{g})^{\otimes 2}, \tag{3.10}$$

where J^{op} is obtained from J by permuting components. We call this element the universal R-matrix of $U_h(\mathfrak{g})$.

Proposition 3.7. *R defines a quasitriangular structure on $U_h(\mathfrak{g})$. That is, R is invertible and*

$$R\Delta = \Delta^{op} R, \tag{3.11}$$

$$(\Delta \otimes 1)(R) = R_{13}R_{23}, (1 \otimes \Delta)(R) = R_{13}R_{12}. \tag{3.12}$$

Moreover, R is a quantization of r, i.e.

$$R \equiv 1 + hr \ mod \ h^2. \tag{3.13}$$

Proof. Identity (3.13) follows from (3.5),(3.6) and the definition of R. This identity implies that R is invertible.

One has

$$R\Delta(a) = (J^{op})^{-1}e^{h\Omega/2}J\Delta(a) = (J^{op})^{-1}e^{h\Omega/2}\Delta_0(a)J = (J^{op})^{-1}\Delta_0(a)e^{h\Omega/2}J =$$
$$\Delta^{op}(a)(J^{op})^{-1}e^{h\Omega/2}J = \Delta^{op}(a)R, \tag{3.14}$$

which proves (3.11).

Now let us prove the first identity of (3.12). The second identity is proved analogously.

According to the definition of R, for any $V, W \in \mathcal{M}$, $v \in F(V)$, $w \in F(W)$, one has $R(v \otimes w) = sJ_{WV}^{-1}F(\beta_{VW})J_{VW}$. Thus, for any $U \in \mathcal{M}$, $u \in F(U)$ one has

$$(\Delta \otimes 1)(R)(v \otimes w \otimes u) = (J_{VW}^{-1} \otimes 1)R(J_{VW} \otimes 1)(v \otimes w \otimes u) =$$
$$s_{12,3}(1 \otimes J_{VW}^{-1})J_{U,V\otimes W}^{-1}F(\beta_{V\otimes W,U})J_{V\otimes W,U}(J_{VW} \otimes 1)(v \otimes w \otimes u), \tag{3.15}$$

where $s_{12,3}$ is the permutation of the first two components with the third one. Using the braiding property $\beta_{V\otimes W,U} = (\beta_{VU} \otimes 1) \circ (1 \otimes \beta_{WU})$, the associativity of J_{VW}, and the obvious identities $J_{U\otimes V,W}^{-1}F(\beta_{VU} \otimes 1)J_{V\otimes U,W} = F(\beta_{VU}) \otimes 1$, $J_{V,U\otimes W}^{-1}F(1 \otimes \beta_{WU})J_{V,W\otimes U} = 1 \otimes F(\beta_{WU})$, one finds that the right hand side of (3.15) equals to $R_{13}R_{23}(v \otimes w \otimes u)$, as desired. □

4. Quantization of finite-dimensional Lie bialgebras.

Our purpose in this section is to represent the quasitriangular topological Hopf algebra $U_h(\mathfrak{g})$ as a quantum double of another topological Hopf algebra, $U_h(\mathfrak{g}_+)$. The topological Hopf algebra $U_h(\mathfrak{g}_+)$ will be a quantization of the Lie bialgebra $\mathfrak{g}_+$.

4.1. *The algebras $U_h(\mathfrak{g}_\pm)$.*

As we have seen, the fiber functor F which we used to construct the quantum group $U_h(\mathfrak{g})$, is represented by the object $M_+ \otimes M_-$ of $\mathcal{M}$. Therefore, we have a homomorphism $\theta : \text{End}(M_+ \otimes M_-) \to \text{End}(F) = U_h(\mathfrak{g})$ defined by $\theta(a)v = v \circ a$, $v \in F(V)$, $V \in \mathcal{M}$, $a \in \text{End}(M_+ \otimes M_-)$.

Lemma 4.1. *The map θ is an isomorphism.*

Proof. The Lemma follows from Lemma 2.1. □

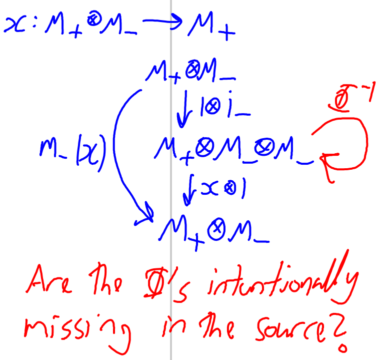
Thus, we can identify $\mathrm{End}(M_+ \otimes M_-)$ with $U_h(\mathfrak{g})$. From now on we make no distinction between them.

Now let us define the subalgebras $U_h(\mathfrak{g}_\pm) \subset U_h(\mathfrak{g})$.

Let $x \in F(M_+)$. Define the endomorphism $m_-(x)$ of $M_+ \otimes M_-$ to be the composition of the following morphisms in $\mathcal{M}$: $m_-(x) = (x \otimes 1) \circ (1 \otimes i_-)$. This defines a linear map $m_- : F(M_+) \to U_h(\mathfrak{g})$. Denote the image of this map by $U_h(\mathfrak{g}_-)$.

Let $m_-^0(x) \in U(\mathfrak{g}_-)$ be defined by the equation $x(1_+ \otimes 1_-) = m_-^0(x)1_+$. It is easy to show that $m_-(x) \equiv m_-^0(x) \bmod h$, which implies that m_- is an embedding.

A similar definition can be made for $x \in F(M_-)$. Define the endomorphism $m_+(x)$ of $M_+ \otimes M_-$ to be the composition of the following morphisms in $\mathcal{M}$: $m_+(x) = (1 \otimes x) \circ (i_+ \otimes 1)$. This defines an injective linear map $m_+ : F(M_-) \to U_h(\mathfrak{g})$. Denote the image of this map by $U_h(\mathfrak{g}_+)$.



Proposition 4.2. *$U_h(\mathfrak{g}_\pm)$ are subalgebras in $U_h(\mathfrak{g})$.*

Proof. Let us give a proof for $U_h(\mathfrak{g}_-)$. The proof for $U_h(\mathfrak{g}_+)$ is analogous.

Using Lemma 2.3, we obtain

$$\begin{aligned} m_-(x) \circ m_-(y) &= (x \otimes 1) \circ (1 \otimes i_-) \circ (y \otimes 1) \circ (1 \otimes i_-) = \\ &= (x \otimes 1) \circ (y \otimes 1 \otimes 1) \circ (1 \otimes 1 \otimes i_-) \circ (1 \otimes i_-) = \\ &= (x \otimes 1) \circ (y \otimes 1 \otimes 1) \circ (1 \otimes i_- \otimes 1) \circ (1 \otimes i_-) = (z \otimes 1) \circ (1 \otimes i_-), \end{aligned} \tag{4.1}$$

where $z = x \circ (y \otimes 1) \circ (1 \otimes i_-) \in F(M_+)$.

So by the definition we get $m_-(x) \circ m_-(y) = m_-(z)$. □

Note that the algebra $U_h(\mathfrak{g}_-)$ is a deformation of the algebra $U(\mathfrak{g}_-)$. Indeed, we can define a linear isomorphism $\mu : U(\mathfrak{g}_-)[[h]] \to U_h(\mathfrak{g}_-)$ by $\mu(a)(1_+ \otimes 1_-) = a1_+$. This isomorphism has the property $\mu(ab) = \mu(a) \circ \mu(b) \bmod h^2$, which follows from (3.5), but in general $\mu(ab) \neq \mu(a) \circ \mu(b)$.

The subalgebra $U_h(\mathfrak{g}_-)$ has a unit since it is a deformation of the algebra with unit $U(\mathfrak{g}_-)$. In fact, one can show that the unit equals to $\mu(1)$, $1 \in U(\mathfrak{g}_-)$.

Similar statements apply to the algebra $U(\mathfrak{g}_+)$.

Proposition 4.3. *The map $U_h(\mathfrak{g}_+) \otimes U_h(\mathfrak{g}_-) \to U_h(\mathfrak{g})$ given by $a \otimes b \to ab$ is an isomorphism.*

Proof. The statement is true because it holds modulo h. □

4.2. *Polarization of the R-matrix.*

Define the element $\tilde{R} \in U_h(\mathfrak{g}_+) \otimes U_h(\mathfrak{g}_-)$ by the identity

$$\tilde{R} \circ \beta^{-1} \circ (i_+ \otimes i_-) = \beta \tag{4.2}$$

in $\mathrm{Hom}(M_+ \otimes M_-, M_- \otimes M_+)$. It is obvious that such an element is unique. It can be computed as follows.

Let $\nu : M_\pm[[h]] \to U_h(\mathfrak{g}_\mp)$ be the linear isomorphism defined by the equation $\nu(x(1_+ \otimes 1_-)) = m_\mp(x)$ for any $x \in F(M_\pm)$. Let $K \in U(\mathfrak{g})^{\otimes 2}[[h]]$ be given by (4.3)

$$K = (\phi^{-1} \otimes \phi^{-1}) \Big(\Phi^{-1}_{1,2,34}(1 \otimes \Phi_{2,3,4}) s e^{-h\Omega_{23}/2} (1 \otimes \Phi^{-1}_{2,3,4}) \Phi_{1,2,34}(1_+ \otimes 1_+ \otimes 1_- \otimes 1_-) \Big).$$

Then it is easy to check, using (4.2), that

$$\tilde{R} = (\nu \otimes \nu)(K^{-1} e^{h\Omega/2}(1_- \otimes 1_+)). \tag{4.4}$$

Proposition 4.4. $\tilde{R} = R$.

Proof. According to (3.10), the R-matrix $R \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$ is defined by the condition that for any $V, W \in \mathcal{M}$ and $v \in F(V), w \in F(W)$ one has the equality

$$\begin{aligned} R^{op}(v \otimes w) \circ \beta_{23} \circ (i_+ \otimes i_-) = \\ \beta \circ (w \otimes v) \circ \beta_{23} \circ (i_+ \otimes i_-) \end{aligned} \tag{4.5}$$

in $\mathrm{Hom}(M_+ \otimes M_-, V \otimes W)$.

By the functoriality of the braiding, $R^{op}(v \otimes w) = \beta \circ R(w \otimes v) \circ \beta^{-1}_{12,34}$. Besides, $\beta_{12,34} = \beta_{23} \circ \beta_{12} \circ \beta_{34} \circ \beta_{23}$. Substituting this into (4.5) and taking into account that $\beta \circ i_\pm = i_\pm$, we get

$$\begin{aligned} R(w \otimes v) \circ \beta^{-1}_{23} \circ (i_+ \otimes i_-) = \\ (w \otimes v) \circ \beta_{23} \circ (i_+ \otimes i_-) \end{aligned} \tag{4.6}$$

in $\mathrm{Hom}(M_+ \otimes M_-, W \otimes V)$.

To show that $R = \tilde{R}$ we have to prove the identity

$$\begin{aligned} (1 \otimes \tilde{R} \otimes 1) \circ (i_+ \otimes 1 \otimes 1 \otimes i_-) \circ \beta^{-1}_{23} \circ (i_+ \otimes i_-) = \\ \beta_{23} \circ (i_+ \otimes i_-) \end{aligned} \tag{4.7}$$

in $\mathrm{Hom}(M_+ \otimes M_-, M_+ \otimes M_- \otimes M_+ \otimes M_-)$.

Interchanging the order of factors on the left hand side of (4.7) and using Lemma 2.3, we can rewrite (4.7) in the form:

$$\begin{aligned} (1 \otimes \tilde{R} \otimes 1) \circ \beta^{-1}_{34} \circ (1 \otimes i_+ \otimes i_- \otimes 1) \circ (i_+ \otimes i_-) = \\ \beta_{23} \circ (i_+ \otimes i_-) \end{aligned} \tag{4.8}$$

in $\mathrm{Hom}(M_+ \otimes M_-, M_+ \otimes M_- \otimes M_+ \otimes M_-)$.

It is obvious that identity (4.8) follows from is the definition of $\tilde{R}$. The proposition is proved. □

4.3. *Subalgebras $U_h(\mathfrak{g}_\pm)$ in terms of the R-matrix.*

Let $U_h(\mathfrak{g}_\pm)^* = \mathrm{Hom}_{\mathcal{A}}(U_h(\mathfrak{g}_\pm), k[[h]])$. Define $k[[h]]$-linear maps $\rho_\pm : U_h(\mathfrak{g}_\mp)^* \to U_h(\mathfrak{g}_\pm)$, by $\rho_+(f) = (1 \otimes f)(R)$, $\rho_-(f) = (f \otimes 1)(R)$. Let $U_\pm$ be the images of the maps $\rho_\pm$, and $\bar{U}_\pm$ be the closures of the $k[[h]]$-subalgebras generated by $U_\pm$.

Proposition 4.5.
$U_h(\mathfrak{g}_\pm) \otimes_{k[[h]]} k((h))$ is the h-adic completion of $\tilde{U}_\pm \otimes_{k[[h]]} k((h))$.

Proof. We prove the statement for $\tilde{U}_+$. The proof for $\tilde{U}_-$ is similar.
We start with the following statement.

Lemma 4.6. For any $x \in U(\mathfrak{g}_+)$ there exists an element $t_x \in \tilde{U}_+ \otimes k((h))$ such that $t_x = x + O(h)$. If x has degree $\leq m$ with respect to the standard filtration in $U(\mathfrak{g}_+)$, then t_x can be chosen in such a way that $h^m t_x \in \tilde{U}_+$.

Proof of the Lemma. It is clear that $1 \in \tilde{U}_+$ since $1 = \rho_+(\varepsilon)$. So we can set $t_1 = 1$.

Now consider the case $x \in \mathfrak{g}_+$. Let $f \in U_h(\mathfrak{g}_-)^*$ be any element such that $f(1) = 0$ and $f(\tilde{a}) = \langle x, a\rangle$ for any $a \in \mathfrak{g}_-$ and $\tilde{a} \in U_h(\mathfrak{g}_-)$ such that $\tilde{a} = a \mod h$. Then it follows from (3.13) that $\rho_+(f) = hx + O(h^2)$. So we can let $t_x = h^{-1}\rho_+(f)$. Thus, the Lemma is true for $x \in \mathfrak{g}_+$.

Since $\tilde{U}_+$ is an algebra, the validity of the Lemma for $x \in \mathfrak{g}_+$ implies its validity for any $x \in U(\mathfrak{g}_+)$. □

Now we can prove the proposition. Let $T_0 \in U_h(\mathfrak{g}_+)$. Let $x_0 \in U(\mathfrak{g}_+)$ be the reduction of $T_0 \mod h$. Then $T_0 - t_{x_0}$ is divisible by h, so we can consider $T_1 = h^{-1}(T_0 - t_{x_0})$ and repeat our procedure. This gives us a sequence $x_i \in U(\mathfrak{g}_+)$, and $T_0 = \sum_{m\geq 0} t_{x_m} h^m$. This shows that T_0 belongs to the h-adic completion of $\tilde{U}_+ \otimes k((h))$, as desired. □

Theorem 4.7. *The subalgebras $U_h(\mathfrak{g}_\pm)$ are Hopf subalgebras in $U_h(\mathfrak{g})$.*

Proof. The fact that $U_h(\mathfrak{g}_\pm)$ are closed under the comultiplication Δ follows from Proposition 4.5 and identities (3.12). The fact that $U_h(\mathfrak{g}_\pm)$ are closed under the antipode S follows from Proposition 4.5 and the identity $(S \otimes 1)(R) = R^{-1}$, which holds in any quasitriangular Hopf algebra. □

Remark. In fact, it is possible to prove the following explicit formula for coproduct in $U_h(\mathfrak{g}_\mp)$: for any $x \in F(M_\pm)$

$$\Delta(m_\mp(x)) = (m_\mp \otimes m_\mp)(J^{-1}_{M_\pm M_\pm}(i_\pm \circ x)). \tag{4.9}$$

The proof is a direct verification. A similar formula is contained in Proposition 9.3.

It is obvious that $U_h(\mathfrak{g}_+)/hU_h(\mathfrak{g}_+)$ is isomorphic to $U(\mathfrak{g}_+)$ as a Hopf algebra. Therefore, $U_h(\mathfrak{g}_+)$ is a quantized universal enveloping algebra. It follows from Proposition 3.6 that its quasiclassical limit is the Lie bialgebra $\mathfrak{g}_+$. Similar statements apply to $U_h(\mathfrak{g}_-)$.

4.4. *Duality of quantized universal enveloping algebras and the quantum double.*
The following general constructions can be found in [Dr1].

If A is a quantized universal enveloping algebra then the dual $A^* = \mathrm{Hom}_{\mathcal{A}}(A, k[[h]])$ carries a natural structure of a topological algebra. Namely, for any $x, y \in A$, $f, g \in A^*$ $fg(x) = (f \otimes g)(\Delta(x))$, and the unit is ε. It can be shown that A^* has a unique maximal ideal I^*, which is the kernel of the linear map $A \to k$ given by $f \to f(1) \mod h$. The topology on A^* is defined by the condition that $\{(I^*)^n, n \geq 0\}$ is a basis of neighborhoods of zero. This implies that the topological algebras $(A \otimes A)^*$ and $A^* \otimes A^*$ are isomorphic.

The algebra A^* has a natural structure of a topological Hopf algebra. Namely, the coproduct is defined by $\Delta(f)(x \otimes y) = f(xy)$, the counit is 1, and the antipode is

S^*. (The definition of coproduct makes sense since the algebra $A^* \otimes A^*$ is isomorphic to $(A \otimes A)^*$).

As a topological $k[[h]]$-module, A^* is isomorphic to $k[[X_1, ..., X_N]][[h]]$.

Let A be any quantized universal enveloping algebra. Let A^* be the dual algebra, and let I^* be the maximal ideal in A^*. Consider the h-adic completion $A^\vee$ of the subalgebra $\sum_{n\geq 0} h^{-n}(I^*)^n$ in the algebra $A^* \otimes_{k[[h]]} k((h))$. Then $A^\vee$ is a new quantized universal enveloping algebra [Dr1]. This algebra is called the dual quantized universal enveloping algebra to A.

The algebra A^* can be identified with a subalgebra in $A^\vee$ which is constructed as follows.

Let $\Delta^n : A \to A^{\otimes n}$ be the iterated coproduct maps: $\Delta^0(a) = \varepsilon(a)$, $\Delta^1(a) = a$, $\Delta^2(a) = \Delta(a)$, $\Delta^n(a) = (\Delta \otimes 1^{\otimes(n-2)})(\Delta^{n-1}(a))$, $n > 2$.

Let $\Sigma = \{i_1, ..., i_k\} \subset \{1, ..., n\}$, and $i_1 < ... < i_k$. Let $j_\Sigma : A^{\otimes k} \to A^{\otimes n}$ be the homomorphism defined by $j_\Sigma(a_1 \otimes ... \otimes a_k) = b_1 \otimes ... \otimes b_n$, $a_1, ..., a_k \in A$, where $b_i = 1$ if $i \notin \Sigma$, and $b_{i_m} = a_m$, $m = 1, ..., k$.

Let $\Delta_\Sigma(a) = j_\Sigma(\Delta^k(a))$, $a \in A$.

Define linear mappings $\delta_n : A \to A^{\otimes n}$ for all $n \geq 1$ by

$$\delta_n(a) = \sum_{\Sigma\subset\{1,...,n\}} (-1)^{n-|\Sigma|}\Delta_\Sigma(a)$$

and a Hopf subalgebra $A' = \{a \in A : \delta_n(a) \in h^n A^{\otimes n}\}$ in A.

It is easy to check that $A^* = (A^\vee)'$.

If A is any Hopf algebra, let A^{op} denote the Hopf algebra A with the comultiplication Δ replaced by Δ^{op}, and the antipode S replaced with S^{-1}. A^{op} is also a Hopf algebra.

Now we can define the notion of the quantum double. Let A be a quantized universal enveloping algebra. Consider the $k[[h]]$-module $D(A) = A \otimes (A^\vee)^{op}$. Let $R \in A \otimes A^* \subset A \otimes (A^\vee)^{op}$ be the canonical element. We can regard R as an element of $D(A) \otimes D(A)$ using the embedding $A \otimes (A^\vee)^{op} \to D(A) \otimes D(A)$ given by $x \otimes y \to x \otimes 1 \otimes 1 \otimes y$. Drinfeld [Dr1] showed that there exists a unique structure of a topological Hopf algebra on $D(A)$ such that

1) $A \otimes 1, 1 \otimes (A^\vee)^{op}$ are Hopf subalgebras in $D(A)$,

2) R defines a quasitriangular structure on $D(A)$, i.e. is invertible and satisfies (3.12), (3.13), and

3) The linear mapping $A \otimes (A^\vee)^{op} \to D(A)$ given by $a \otimes b \to ab$ is bijective.

$D(A)$, equipped with this structure, is a quasitriangular quantized universal enveloping algebra. It is called the quantum double of A.

4.5. *The quantum double of $U_h(\mathfrak{g}_+)$.*

Proposition 4.8. *ρ_+ is a homomorphism of topological Hopf algebras $(U_h(\mathfrak{g}_-)^{op})^* \to U_h(\mathfrak{g}_+)$. ρ_- is a homomorphism of topological Hopf algebras $U_h(\mathfrak{g}_-)^* \to U_h(\mathfrak{g}_+)^{op}$.*

Proof. We only prove the first statement. The second one is proved analogously.

It is clear that ρ_+ is continuous. Also, for any $f, g \in (U_h(\mathfrak{g}_-)^{op})^*$ one has

$$\rho_+(fg) = (1 \otimes fg)(R) = (1 \otimes f \otimes g)((1 \otimes \Delta^{op})(R)) = (1 \otimes f \otimes g)(R_{12}R_{13}) =$$
$$(1 \otimes f)(R) \cdot (1 \otimes g)(R) = \rho_+(f)\rho_+(g);$$
$$\Delta(\rho_+(f)) = \Delta((1 \otimes f)(R)) = (1 \otimes 1 \otimes f)((\Delta \otimes 1)(R)) =$$
$$(1 \otimes 1 \otimes f)(R_{13}R_{23}) = (1 \otimes 1 \otimes \Delta(f))(R_{13}R_{24}) = (\rho_+ \otimes \rho_+)(\Delta(f)).$$

It is obvious that $\rho_+(1) = 1$ and $\varepsilon(\rho_+(f)) = \varepsilon(f)$ for any f. Also, it is easy to check that $\rho_+((S^{-1})^* f) = S(\rho_+(f))$. The proposition is proved. □

Corollary 4.9. *$U_\pm$ are Hopf subalgebras in $U_h(\mathfrak{g}_\pm)$. In particular, $\tilde{U}_\pm = U_\pm$.*

Proof. The first statement is clear. The second statement follows from the first one and the fact that $U_\pm$ is closed in $U_h(\mathfrak{g}_\pm)$, which is easy to check. □

Proposition 4.10. *The maps ρ_+, ρ_- are injective.*

Proof. We show the injectivity of ρ_+ (the case of ρ_- is similar). Fix an element $f \in U_h(\mathfrak{g}_-)^*, f \neq 0$. We can always assume that $f \neq 0 \bmod h$. Let $x \in U(\mathfrak{g}_-)$ be such that $f(t_x) \neq 0$ mod h (where t_x was defined in Lemma 4.6), $n \geq 0$ be such that $h^n t_x \in U_-$, and $g \in U_h(\mathfrak{g}_+)^*$ be such that $\rho_-(g) = h^n t_x$. Such a g exists by the definition of n. Then $g(\rho_+(f)) = (g \otimes f)(R) = f(\rho_-(g)) = h^n f(t_x) \neq 0$. Therefore, $\rho_+(f) \neq 0$. □

Proposition 4.11. $U_\pm = U_h(\mathfrak{g}_\pm)'$.

Proof. We give the proof for U_+. The proof for U_- is similar.

First we need the following statement.

Lemma 4.12 Let $t \in U_h(\mathfrak{g}_+)'$ be an element such that $h^{-n}t \in U_h(\mathfrak{g}_+)$ and $h^{-n}t = x + O(h)$, $x \in U(\mathfrak{g}_+)$, $x \neq 0$. Then x has degree $\leq n$.

Proof of the Lemma. By the definition, $\delta_{n+1}(h^{-n}t)$ is divisible by h. On the other hand, $\delta_{n+1}(h^{-n}t) = \delta_{n+1}(x) + O(h)$. Thus, $\delta_{n+1}(x) = 0$, which implies that the degree of x is $\leq n$, since the kernel of δ_{n+1} on $U(\mathfrak{g}_+)$ is the set of all elements of $U(\mathfrak{g}_+)$ whose degree is $\leq n$. □

Now we can prove the proposition. By Lemma 4.6, for any $x \in U(\mathfrak{g}_+)$ of degree $\leq n$, an element t_x can be chosen in such a way that $h^n t_x \in U_+$. This implies the inclusion $U_+ \supset U_h(\mathfrak{g}_+)'$. Indeed, let $T_0 \in U_h(\mathfrak{g}_+)'$, and $T_0 \equiv h^m x_0 \bmod h^{m+1}$, where $x_0 \in U(\mathfrak{g}_+)$. Then, according to Lemma 4.12, the degree of x_0 is $\leq m$. Therefore, $h^m t_{x_0} \in U_+$. Thus, $T_1 = T_0 - h^m t_{x_0} \in U_+$ and is divisible by h^{m+1}, so we can repeat our procedure. This gives us a sequence of elements $x_i \in U(\mathfrak{g}_+)$ of degrees m_i ($m_0 = m$), such that $m_0 < m_1 < ... < m_i < ...$, and $T_0 = \sum_{i \geq 0} t_{x_i} h^{m_i}$. This shows that T_0 belongs to U_+, as desired.

To demonstrate the inclusion $U_+ \subset U_h(\mathfrak{g}_+)'$, observe that according to (3.12),

$$(\Delta^n \otimes 1)(R) = R_{1n+1} ... R_{nn+1}.$$

This implies that

$$(\delta_n \otimes 1)(R) = (R_{1n+1} - 1)...(R_{nn+1} - 1) = O(h^n).$$

Therefore, $\delta_n(\rho_+(f))$ is divisible by h^n for any $f \in U_h(\mathfrak{g}_-)^*$. □

Comparing our results with the definitions of the previous, section we see that we have obtained the following result.

Theorem 4.13. *Let $\mathfrak{g}_+$ be a finite dimensional Lie bialgebra and $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be the associated Manin triple. Then*

(i) There exist quantized universal enveloping algebras $U_h(\mathfrak{g})$ and $U_h(\mathfrak{g}_\pm) \subset U_h(\mathfrak{g})$, which are quantizations of the Lie bialgebras $\mathfrak{g}$, $\mathfrak{g}_\pm \subset \mathfrak{g}$, respectively;

(ii) The multiplication map $U_h(\mathfrak{g}_+) \otimes U_h(\mathfrak{g}_-) \to U_h(\mathfrak{g})$ is a linear isomorphism;

(iii) The algebras $U_h(\mathfrak{g}_+), U_h(\mathfrak{g}_-)^{op}$ are dual to each other as quantized universal enveloping algebras, in the sense of Drinfeld [Dr1]*;*
(iv) The factorization $U_h(\mathfrak{g}) = U_h(\mathfrak{g}_+)U_h(\mathfrak{g}_-)$ defines an isomorphism of $U_h(\mathfrak{g})$ with the quantum double of $U_h(\mathfrak{g}_+)$;
(v) $U_h(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$ as a topological algebra.

5. Quantization of solutions of the classical Yang-Baxter equation.

Let A be an associative algebra over k with unit, and $r \in A \otimes A$. The element r is called a classical r-matrix if it satisies the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{5.1}$$

We say that r is unitary if $r^{op} = -r$. An algebra A equipped with a classical r-matrix r is called a classical Yang-Baxter algebra. A is called unitary if r is unitary.

Let A be a topological algebra over $k[[h]]$. Let $R \in A \otimes A$. We say that R is a quantum R-matrix if it satisfies the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{5.2}$$

We say that R is unitary if $R^{op} = R^{-1}$. A topological algebra A equipped with a quantum R-matrix R is called a quantum Yang-Baxter algebra. A is called unitary if R is unitary.

The following theorem answers question 3.1 in [Dr3]. It shows that any classical Yang-Baxter algebra can be quantized.

Theorem 5.1. *Let A be an associative algebra with unit over k, and $r \in A \otimes A$ be a classical r-matrix. Then there exists a quantum R-matrix $R \in A \otimes A[[h]]$ such that $R = 1 + hr \bmod h^2$. If in addition r is unitary then R can also be chosen unitary.*

Proof.
We start with a construction of Reshetikhin and Semenov-Tian-Shansky [RS]. Let $\mathfrak{g}_+ = \{(1 \otimes f)(r), f \in A^*\}, \mathfrak{g}_- = \{(f \otimes 1)(r), f \in A^*\}$ be vector subspaces in A. It is clear that $\mathfrak{g}_+, \mathfrak{g}_-$ are finite-dimensional, $r \in \mathfrak{g}_+ \otimes \mathfrak{g}_-$, and the map $\chi_r : \mathfrak{g}_+^* \to \mathfrak{g}_-$ defined by $\chi_r(f) = (f \otimes 1)(r)$, is an isomorphism of vector spaces.

Remark. Note that the spaces $\mathfrak{g}_+$ and $\mathfrak{g}_-$ may intersect nontrivially and even coincide.

Lemma 5.2.. *$\mathfrak{g}_+, \mathfrak{g}_-$ are Lie subalgebras in A.*

Proof. Let $x, y \in \mathfrak{g}_+$, $x = (1 \otimes f)(r), y = (1 \otimes g)(r)$. Using (5.1), we have

$$[xy] = (1 \otimes f \otimes g)([r_{12}r_{13}]) = -(1 \otimes f \otimes g)([r_{12} + r_{13}, r_{23}]) = (1 \otimes h)(r), \tag{5.3}$$

where $h \in A^*$, $h(a) = (f \otimes g)([r, a \otimes 1 + 1 \otimes a])$. Thus, $[xy] \in \mathfrak{g}_+$, i.e. $\mathfrak{g}_+$ is a Lie algebra. The proof for $\mathfrak{g}_-$ is similar. □

Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a vector space Define the skew-symmetric bracket $[,] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ as follows. If $x, y \in \mathfrak{g}_+$ or $x, y \in \mathfrak{g}_-$ then the bracket $[xy]$ is the Lie bracket in $\mathfrak{g}_+$ or $\mathfrak{g}_-$, respectively. If $x \in \mathfrak{g}_+, y \in \mathfrak{g}_-$, then $[xy]$ is defined by

$$[xy] = (\mathrm{ad}^*x)y - (ad^*y)x. \tag{5.4}$$

Let $\pi : \mathfrak{g} \to A$ be the linear map whose restrictions to $\mathfrak{g}_+, \mathfrak{g}_-$ are the corresponding embeddings. The restrictions of π to $\mathfrak{g}_+$, $\mathfrak{g}_-$ are injective but in general π itself is not an embedding.

Lemma 5.3. $\pi([xy]) = [\pi(x), \pi(y)]$, $x, y \in \mathfrak{g}$.

Proof. The Lemma is a direct consequence of the classical Yang-Baxter equation. $\square$

Lemma 5.4. $(\mathfrak{g}, [,])$ *is a Lie algebra.*

Proof. We have to check the Jacobi identity in $\mathfrak{g}$. It is enough to check it for three elements a, x, y such that $a \in \mathfrak{g}_+$, $x, y \in \mathfrak{g}_-$. For brevity we write $a(x)$ for $(\mathrm{ad}^* a)x$. We have

$$\begin{aligned} [a[xy]] &= a([xy]) - [xy](a), \\ [y[ax]] &= [y, a(x) - x(a)] = [y, a(x)] - y(x(a)) + y(a(x)), \\ [x[ya]] &= [x, y(a) - a(y)] = -[x, a(y)] + x(y(a)) - x(a(y)). \end{aligned} \tag{5.5}$$

Adding these three identities, and using the fact that $[xy](a) = x(y(a)) - y(x(a))$, we get

$$[a[xy]] + [y[ax]] + [x[ya]] = a([xy]) + [y, a(x)] - [x, a(y)] + y(a(x)) - x(a(y)). \tag{5.6}$$

Denote the right hand side of (5.6) by X. Applying π to both sides of (5.6), and using Lemma 5.3 and the Jacobi identity in A, we get

$$\pi(X) = 0. \tag{5.7}$$

Since $X \in \mathfrak{g}_+$, and π is injective on $\mathfrak{g}_+$, we get $X = 0$, which implies the Jacobi identity in $\mathfrak{g}$. $\square$

Let $\langle , \rangle$ be the inner product on $\mathfrak{g}$ such that $\langle x_+ + x_-, y_+ + y_- \rangle = x_- \cdot y_+ + y_- \cdot x_+$, where $x_+, y_+ \in \mathfrak{g}_+$, $x_-, y_- \in \mathfrak{g}_-$, and the dot denotes the natural pairing $\mathfrak{g}_- \otimes \mathfrak{g}_+ \to k$ defined by the map χ_r. This inner product is ad-invariant. Thus, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple.

Now we can finish the proof of the theorem. Lemma 5.3 implies that $\pi : \mathfrak{g} \to A$ is a homomorphism of Lie algebras. Therefore, it extends to a homomorphism of associative algebras $\pi : U(\mathfrak{g}) \to A$. Furthermore, $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. The Lie bialgebra $\mathfrak{g}$ is quasitriangular, and its quasitriangular structure is defined by the classical r-matrix $\tilde{r} = \sum x_+^i \otimes x_-^i$, where x_+^i is a basis of $\mathfrak{g}_+$, and x_-^i is a dual basis of $\mathfrak{g}_-$. Note that $(\pi \otimes \pi)(\tilde{r}) = r$.

By Theorem 4.13, there exists a quasitriangular topological Hopf algebra $U_h(\mathfrak{g})$, with a quasitriangular structure $\tilde{R} \in U_h(\mathfrak{g}) \otimes U_h(\mathfrak{g})$. Moreover, the associative algebra $U_h(\mathfrak{g})$ is isomorphic to $U(\mathfrak{g})[[h]]$, and the isomorphism can be chosen to be the identity modulo h. Thus, we can assume that $\tilde{R} \in (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[h]]$.

Set $R = (\pi \otimes \pi)(\tilde{R})$. From what we said above it follows that R satisfies (5.2) and $R = 1 + hr$ modulo h^2.

Assume now that $r^{op} = -r$. Let $\tilde{\Omega} = \tilde{r} + \tilde{r}^{op}$. It follows immediately from the construction of $\tilde{R}$ that $\tilde{R}^{op}\tilde{R}$ is conjugate to $e^{h\tilde{\Omega}}$. But $(\pi \otimes \pi)(\tilde{\Omega}) = r + r^{op} = 0$. This implies that $R^{op}R = 1$, as desired.

The theorem is proved.$\square$

Let $\mathcal{R}$ be the ring of algebraic functions of a variable h with coefficients in k which are regular at $h = 0$.

Theorem 5.5. *Let A be a finite-dimensional associative algebra with unit over k, and $r \in A \otimes A$ be a classical r-matrix. Then there exists a family of quantum R-matrices $R(h) \in A \otimes A \otimes \mathcal{R}$ such that $R = 1 + hr + O(h^2)$, $h \to 0$. If in addition r is unitary then $R(h)$ can also be chosen unitary.*

Proof. The theorem follows immediately from Theorem 5.1 and the following result of M.Artin [Ar].

Theorem. Any system of polynomial equations in indeterminates $x_1, ..., x_n$ with coefficients in $k[h]$ which has solutions over $k[[h]]$ also has solutions over $\mathcal{R}$.

Indeed, let us write R in the form $R = 1 + hr + h^2 X(h)$, and look for a series $X(h)$ such that R satisfies the quantum Yang-Baxter equation, and the unitarity condition in the case when r is unitary. This is a system of polynomial equations on the components of $X(h)$ with coefficients in $k[h]$. By Theorem 5.1, it has solutions over $k[[h]]$. Therefore, by Artin's theorem, it has solutions over $\mathcal{R}$.

6. Quantization of quasitriangular Lie bialgebras.

6.1. *Quasitriangular quantization of quasitriangular Lie bialgebras.* In this section we give a recipe of quantization of a quasitriangular Lie bialgebra $\mathfrak{a}$ (not necessarily finite dimensional), which produces a quantized universal enveloping algebra isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra. This answers questions from Section 4 of [Dr3].

Let $\mathfrak{g}_+ = \{(1 \otimes f)(r), f \in \mathfrak{a}^*\}$, $\mathfrak{g}_- = \{(f \otimes 1)(r), f \in \mathfrak{a}^*\}$. be subspaces in $\mathfrak{a}$. By Lemma 5.2, applied to $A = U(\mathfrak{a})$, these subspaces are finite dimensional Lie subalgebras in $\mathfrak{a}$. Moreover, let $\mathfrak{g}$ be the vector space $\mathfrak{g}_+ \oplus \mathfrak{g}_-$. This space is a Lie algebra with bracket defined by (5.4) and an invariant inner product. By Lemma 5.3, we have a natural homomorphism of Lie algebras $\pi : \mathfrak{g} \to \mathfrak{a}$, and it is easy to see that this homomorphism is a morphism of quasitriangular Lie bialgebras.

Let $\mathcal{M}_\mathfrak{a}$ be the category whose objects are $\mathfrak{a}$-modules, and morphisms are defined by $\mathrm{Hom}_{\mathcal{M}_\mathfrak{a}}(V, W) = \mathrm{Hom}_\mathfrak{a}(V, W)[[h]]$. Let $\mathcal{M}_\mathfrak{g}$ be the Drinfeld category associated to $\mathfrak{g}$. We have the pullback functor $\pi^* : \mathcal{M}_\mathfrak{a} \to \mathcal{M}_\mathfrak{g}$. Define the braided monoidal structure on $\mathcal{M}_\mathfrak{a}$ to be the pullback of the braided monoidal structure on $\mathcal{M}_\mathfrak{g}$. This definition makes sense, since the element $\Omega = r + r^{op} \in \mathfrak{g} \otimes \mathfrak{g}$ is $\mathfrak{g}$-invariant by the definition of a quasitriangular Lie bialgebra.

Let M_+, M_- be the Verma modules over $\mathfrak{g}$. Define a functor $F : \mathcal{M}_\mathfrak{a} \to \mathcal{A}$ by $F(V) = \mathrm{Hom}_{\mathcal{M}_\mathfrak{g}}(M_+ \otimes M_-, \pi^*(V))$. The tensor structure on F is introduced in the same way as in Section 1.8. Let $H = \mathrm{End}F$. Since the functor F is isomorphic to the "forgetful" functor $V \to$ " the $k[[h]]$ module $V[[h]]$", the algebra H is isomorphic to $U(\mathfrak{a})[[h]]$ as a topological algebra over $k[[h]]$. On the other hand, H has a natural coproduct and antipode defined analogously to Section 3.2, and a quasitriangular structure $R \in H \otimes H$ defined analogously to Section 3.5. It is easy to check that the quasiclassical limit of H is the Lie bialgebra $\mathfrak{a}$, and $R = 1 + hr + O(h^2)$, so r is the quasiclassical limit of R.

Furthermore, suppose that the original Lie bialgebra r is triangular, i.e. r is a unitary r-matrix. Then $\Omega = r + r^{op} = 0$, and hence $R^{op}R = J^{-1}e^{h\Omega}J = 1$, so the Hopf algebra H is triangular, too.

Thus, we have the following theorem.

Theorem 6.1. *Any quasitriangular Lie bialgebra $\mathfrak{a}$ admits a quantization $U_h(\mathfrak{a})$ which is a quasitriangular quantized universal enveloping algebra isomorphic to*

$U(\mathfrak{a})[[h]]$ *as a topological algebra. If* $\mathfrak{a}$ *is triangular, so is* $U_h(\mathfrak{a})$.

6.2. *Identification of two quantizations of a quasitriangular Lie bialgebra.* Let $\mathfrak{a}$ be a finite dimensional quasitriangular Lie bialgebra. Let $U_h(\mathfrak{a})$ be the quantization of $\mathfrak{a}$ constructed in Section 4, and $U_h^{qt}(\mathfrak{a})$ be the quasitriangular quantization of $\mathfrak{a}$ constructed in Section 6.1.

Theorem 6.2. *The quantized universal enveloping algebras* $U_h(\mathfrak{a})$, $U_h^{qt}(\mathfrak{a})$ *are isomorphic.*

The proof of this theorem is given below and uses the functoriality of quantization, which is proved in Chapter 10.

Corollary 6.3. *The quantization of the double* $\mathfrak{g}$ *of a finite dimensional Lie bialgebra* $\mathfrak{a}$ *constructed in Chapter 3 is isomorphic to the quantization of* $\mathfrak{g}$ *as a Lie bialgebra, constructed in Chapter 4.*

To prove Theorem 6.2, we first need the following result, which appears (in somewhat different form) in [RS].

Lemma 6.4. *Let* $\mathfrak{a}$ *be a quasitriangular Lie bialgebra, and* $\mathfrak{g}$ *be the double of* $\mathfrak{a}$. *Then the linear map* $\tau : \mathfrak{g} \to \mathfrak{a}$ *defined by*

$$\tau(x+f) = x + (f \otimes 1)(r), x \in \mathfrak{a}, f \in \mathfrak{a}^*, \tag{6.1}$$

is a homomorphism of quasitriangular Lie bialgebras.

Proof. First we show that τ is a homomorphism of Lie algebras, i.e. $\tau([g_1 g_2]) = [\tau(g_1)\tau(g_2)]$. This is obvious when $g_1, g_2 \in \mathfrak{a}$. Assume that $f, g \in \mathfrak{a}^*$. Then, using the classical Yang-Baxter equation, we get

$$\begin{aligned} &\tau([fg]) = ([fg] \otimes 1)(r) = (f \otimes g \otimes 1)((\delta \otimes 1)(r)) = \\ &(f \otimes g \otimes 1)([r_{13} + r_{23}, r_{12}]) = (f \otimes g \otimes 1)([r_{13}, r_{23}]) = [\tau(f)\tau(g)]. \end{aligned} \tag{6.2}$$

Now assume that $x \in \mathfrak{a}, f \in \mathfrak{a}^*$. Then

$$\begin{aligned} &\tau([xf]) = \tau(\mathrm{ad}^* x(f)) - \tau(\mathrm{ad}^* f(x)) = \\ &\tau((f \otimes 1)([r, x \otimes 1])) + \tau((f \otimes 1)([x \otimes 1 + 1 \otimes x, r])) = \\ &\tau((f \otimes 1)([1 \otimes x, r])) = [\tau(x)\tau(f)]. \end{aligned} \tag{6.3}$$

Now we check that τ is a homomorphism of quasitriangular Lie bialgebras. Let $\tilde{r}$ be the quasitriangular structure on $\mathfrak{g}$. If x_i is a basis of $\mathfrak{a}$, and f_i is the dual basis of $\mathfrak{a}^*$, then $\tilde{r}$ is given by the formula $\tilde{r} = \sum_i x_i \otimes f_i$. Thus we have

$$(\tau \otimes \tau)(\tilde{r}) = \sum_i \tau(x_i) \otimes \tau(f_i) = \sum_i x_i \otimes (f_i \otimes 1)(r) = r. \tag{6.4}$$

The Lemma is proved. □

Proof of Theorem 6.2. Lemma 6.4 claims that there exists a morphism of quasitriangular Lie bialgebras $\tau : \mathfrak{g} \to \mathfrak{a}$ which is the identity on $\mathfrak{a}$. Theorem 10.6 below states that quasitriangular quantization of Section 6.1 is a functor from the category of quasitriangular Lie bialgebras to the category of quasitriangular topological Hopf algebras over $k[[h]]$. Thus, τ defines a morphism $\hat{\tau} : U_h^{qt}(\mathfrak{g}) \to U_h^{qt}(\mathfrak{a})$. On the other hand, $U_h(\mathfrak{a})$ was constructed as a subalgebra in $U_h^{qt}(\mathfrak{g})$, so we have an embedding $\eta : U_h(\mathfrak{a}) \to U_h^{qt}(\mathfrak{g})$. Consider the morphism $\tau \circ \eta : U_h(\mathfrak{a}) \to U_h^{qt}(\mathfrak{a})$. This morphism is an isomorphism since it equals to 1 modulo h. The theorem is proved. □

Remark. An analogous theorem holds for infinite dimensional Lie bialgebras. Namely, the "usual" quantization of $\mathfrak{a}$ defined in Section 9 is isomorphic to its quasitriangular quantization. The proof is analogous to the finite dimensional case.

6.3. *Representations of $U_h(\mathfrak{g})$.*

Let $\mathfrak{a}$ be a quasitriangular Lie bialgebra (not necessarily finite dimensional). By a representation of $U_h(\mathfrak{a})$ we mean a topologically free $k[[h]]$-module V together with a homomorphism $\pi_V : U_h(\mathfrak{a}) \to \mathrm{End}_{k[[h]]}V$. Representations of $U_h(\mathfrak{g})$ form a braided tensor category, with the trivial associativity morphism and braiding defined by the R-matrix. Denote this category by $\mathcal{R}$.

The functor $F : \mathcal{M}_\mathfrak{a} \to \mathcal{A}$ can be regarded as a functor from $\mathcal{M}_\mathfrak{a}$ to $\mathcal{R}$, since for any $W \in \mathcal{M}_\mathfrak{a}$ the $k[[h]]$-module $F(W)$ is equipped with a natural action of $U_h(\mathfrak{g})$. We denote this new functor also by F. This functor inherits the tensor structure defined by the maps J_{VW}.

Theorem 6.5. *The functor F defines an equivalence of braided tensor categories $\mathcal{M}_\mathfrak{a} \to \mathcal{R}$.*

Proof. The theorem follows from the definition of the functor F, the algebra $U_h(\mathfrak{g})$ and the R-matrix R. □

Part II

7. Drinfeld category for an arbitrary Lie bialgebra.

7.1. *Topological vector spaces.* Recall the definition of the product topology. Let S be a set, T a topological space, and T^S the space of functions from S to T. This space has a natural weak topology, which is the weakest of the topologies in which all the evaluation maps $T^S \to T$, $f \to f(s)$, are continuous. Namely, let B be a basis of the topology on T. For any integer $n \geq 1$, elements $s_1, ..., s_n \in S$, and open sets $U_1, ..., U_n \in B$, define $V(s_1, ..., s_n, U_1, ..., U_n) = \{f \in T^S : f(s_i) \in U_i, i = 1, ..., n\}$. Let $\mathcal{B}$ be the collection of all such sets V. This is a basis of a topology on T^S which is called the weak topology. The obtained topological space is the product of copies of T corresponding to elements of S. If X is any subset in T^S, the weak topology on T^S induces a topology on X. We will call it the weak topology as well.

Let k be a field of characteristic zero with the discrete topology. Let V be a topological vector space over k. The topology on V is called linear if open subspaces of V form a basis of neighborhoods of 0.

Remark. It is clear that in any topological vector space, an open subspace is also closed.

Let V be a topological vector space over k with linear topology. V is called separated if the map $V \to \varprojlim(V/U)$ is a monomorphism, where U runs over open subspaces of V.