THE VIRTUAL AND UNIVERSAL BRAIDS

VALERIJ G. BARDAKOV

Abstract. We study the structure of the virtual braid group. It is shown that the virtual braid group is a semi–direct product of the virtual pure braid group and the symmetric group. Also, it is shown that the virtual pure braid group is a semi–direct product of free groups. From these results we obtain a normal form of words in the virtual braid group. We introduce the concept of a universal braid group. This group contains the classical braid group and has as its quotient groups the singular braid group, virtual braid group, welded braid group, and classical braid group.

Is there a technical definition to "normal form"?

Recently some generalizations of classical knots and links were defined and studied: singular links [1, 2], virtual links [3, 4] and welded links [5].

One of the ways to study classical links is to study the braid group. Singular braids [6, 2], virtual braids [3, 7], welded braids [5] were defined similar to the classical braid group. Theorem of A. A. Markov [8, Ch. 2.2] reduces the problem of classification of links to some algebraic problems of the theory of braid groups. These problems include the word problem and the conjugacy problem. There are generalizations of Markov theorem for singular links [9], virtual links, and welded links [10].

There are some different ways to solve the word problem for the singular braid monoid and singular braid group [11, 12, 13]. The solution of the word problem for the welded braid group follows from the fact that this group is a subgroup of the automorphism group of the free group [5]. A normal form of words in the welded braid group was constructed in [14].

In this paper we study the structure of the virtual braid group VB_n. Similar to the classical braid group B_n and welded braid group WB_n,

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[14] M. Gutiérrez, S. Krstić, Normal forms for basis-conjugating automorphisms of a free group. Int. J. Algebra Comput., 8, No. 6 (1998), 631-669.

the group VB_n contains the normal subgroup VP_n which is called *virtual pure braid group*. The quotient group VB_n/VP_n is isomorphic to the symmetric group S_n. In the article we find generators and defining relations of VP_n. Since VB_n is a semi–direct product of VP_n and S_n, we should study the structure of VP_n. It will be proved that VP_n is representable as the following semi–direct product

$$VP_n = V_{n-1}^* \leftthreetimes VP_{n-1} = V_{n-1}^* \leftthreetimes (V_{n-2}^* \leftthreetimes (\ldots \leftthreetimes (V_2^* \leftthreetimes V_1^*)) \ldots),$$

where V_i^* is some (in general infinitely generated for $i > 1$) free subgroup of VP_n. From this result it follows that there exists a normal form of words in VB_n.

In the last section we define the universal braid group UB_n which contains the braid group B_n and has as its quotient groups the singular braid group SG_n, virtual braid group VB_n, welded braid group WB_n, and braid group B_n. It is known [5] that VB_n has as its quotient the group WB_n. It will be proved that the quotient homomorphism maps VP_n into the welded pure braid group WP_n. This homomorphism agrees with the decomposition of this group into the semi-direct product given by Theorem 2 and by [15, 16].

[5]=[FRK]

[15, 16] by Bardakov.

By Artin theorem, B_n is embedded into the automorphism group $\mathrm{Aut}(F_n)$ of the free group F_n. In [5] it was proved that WB_n is also embedded into $\mathrm{Aut}(F_n)$. It is not known if it is true that SG_n and VB_n are embedded into $\mathrm{Aut}(F_n)$.

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1. Different classes of braids and their properties

In this section we remind (see references from the introduction) some known facts about braid groups, singular braid monoids, virtual braid groups and welded braid groups.

1.1. **The braid group and the group of conjugating automorphisms.** The braid group B_n, $n \geq 2$, on n strings can be defined as a group generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ (see Fig. 1)

FIGURE 1. Geometric braids representing σ_i and σ_i^{-1}

with the defining relations

$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1}, \ \ i = 1, 2, \dots, n-2, \tag{1}$$

$$\sigma_i \, \sigma_j = \sigma_j \, \sigma_i, \ \ |i - j| \geq 2. \tag{2}$$

There exists a homomorphism of B_n onto the symmetric group S_n on n letters. This homomorphism maps σ_i to the transposition $(i, i+1)$, $i = 1, 2, \dots, n-1$. The kernel of this homomorphism is called *pure braid group* and denoted by P_n. The group P_n is generated by a_{ij}, $1 \leq i < j \leq n$ (see Fig. 2). These generators can be expressed by the generators of B_n as follows

$$a_{i,i+1} = \sigma_i^2,$$
$$a_{ij} = \sigma_{j-1} \, \sigma_{j-2} \dots \sigma_{i+1} \, \sigma_i^2 \, \sigma_{i+1}^{-1} \dots \sigma_{j-2}^{-1} \, \sigma_{j-1}^{-1}, \ \ i+1 < j \leq n.$$

The group P_n is the semi–direct product of the normal subgroup U_n which is a free group with free generators $a_{1n}, a_{2n}, \dots, a_{n-1,n}$, and

Figure 2. The geometric braid a_{ij}

P_{n-1}. Similarly, P_{n-1} is the semi–direct product of the free group U_{n-1} with free generators $a_{1,n-1}, a_{2,n-1}, \ldots, a_{n-2,n-1}$ and P_{n-2}, and so on. Therefore, P_n is decomposable (see [17]) into the following semi–direct product

$$P_n = U_n \leftthreetimes (U_{n-1} \leftthreetimes (\ldots \leftthreetimes (U_3 \leftthreetimes U_2)) \ldots), \; U_i \simeq F_{i-1}, \; i = 2, 3, \ldots, n.$$

The group B_n has a faithful representation as a group of automorphisms of $\mathrm{Aut}(F_n)$ of the free group $F_n = \langle x_1, x_2, \ldots, x_n \rangle$. In this case the generator σ_i, $i = 1, 2, \ldots, n-1$, defines the automorphism

$$\sigma_i : \begin{cases} x_i \longmapsto x_i \, x_{i+1} \, x_i^{-1}, & \\ x_{i+1} \longmapsto x_i, & \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases}$$

By theorem of Artin [8, Theorem 1.9], an automorphism β from $\mathrm{Aut}(F_n)$ lies in B_n if and only if β satisfies to the following conditions:

$$1) \; \beta(x_i) = a_i^{-1} \, x_{\pi(i)} \, a_i, \; 1 \leq i \leq n,$$

$$2) \; \beta(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n,$$

where π is a permutation from S_n and $a_i \in F_n$.

An automorphism is called a *conjugating automorphism* (or a permutation–conjugating automorphism according to the terminology from [5]) if it satisfies to condition 1). The group of conjugating automorphisms C_n is generated by σ_i and automorphisms α_i, $i = 1, 2, \ldots, n-1$, where

$$\alpha_i : \begin{cases} x_i \longmapsto x_{i+1}, & \\ x_{i+1} \longmapsto x_i, & \\ x_l \longmapsto x_l, & l \neq i, i+1. \end{cases}$$

It is not hard to see that the automorphisms α_i generate the symmetric group S_n and, hence, satisfy the following relations

$$\alpha_i \, \alpha_{i+1} \, \alpha_i = \alpha_{i+1} \, \alpha_i \, \alpha_{i+1}, \ \ i = 1, 2, \ldots, n-2, \tag{3}$$

$$\alpha_i \, \alpha_j = \alpha_j \, \alpha_i, \ \ |i-j| \geq 2, \tag{4}$$

$$\alpha_i^2 = 1, \ \ i = 1, 2, \ldots, n-1. \tag{5}$$

The group C_n is defined by relations (1)–(2) of B_n, relations (3)–(5) of S_n, and the mixed relations (see [5, 18])

$$\alpha_i \, \sigma_j = \sigma_j \, \alpha_i, \ \ |i-j| \geq 2, \tag{6}$$

$$\sigma_i \, \alpha_{i+1} \, \alpha_i = \alpha_{i+1} \, \alpha_i \, \sigma_{i+1}, \ \ i = 1, 2, \ldots, n-2, \tag{7}$$

$$\sigma_{i+1} \, \sigma_i \, \alpha_{i+1} = \alpha_i \, \sigma_{i+1} \, \sigma_i, \ \ i = 1, 2, \ldots, n-2. \tag{8}$$

If we consider the group generated by automorphisms ε_{ij}, $1 \leq i \neq j \leq n$, where

$$\varepsilon_{ij} : \begin{cases} x_i \longmapsto x_j^{-1} \, x_i \, x_j, & i \neq j, \\ x_l \longmapsto x_l, & l \neq i, \end{cases}$$

then we get the group of *basis–conjugating automorphisms* Cb_n. The elements of Cb_n satisfy condition 1) for the identical permutation π, i. e., map each generator x_i to the conjugating element. J. McCool [19] proved that Cb_n is defined by the relations (from here different letters stand for different indices)

$$\varepsilon_{ij} \, \varepsilon_{kl} = \varepsilon_{kl} \, \varepsilon_{ij}, \tag{9}$$

$$\varepsilon_{ij} \, \varepsilon_{kj} = \varepsilon_{kj} \, \varepsilon_{ij}, \tag{10}$$

$$(\varepsilon_{ij} \, \varepsilon_{kj}) \, \varepsilon_{ik} = \varepsilon_{ik} \, (\varepsilon_{ij} \, \varepsilon_{kj}). \tag{11}$$

The group C_n is representable as the semi–direct product: $C_n = Cb_n \leftthreetimes S_n$, where S_n is generated by the automorphisms $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. The following equalities are true (see [18]):

$$\varepsilon_{i,i+1} = \alpha_i \, \sigma_i^{-1}, \ \ \varepsilon_{i+1,i} = \sigma_i^{-1} \, \alpha_i,$$

$$\varepsilon_{ij} = \alpha_{j-1} \, \alpha_{j-2} \ldots \alpha_{i+1} \, \varepsilon_{i,i+1} \, \alpha_{i+1} \ldots \alpha_{j-2} \, \alpha_{j-1} \ \ i < j,$$

$$\varepsilon_{ji} = \alpha_{j-1} \, \alpha_{j-2} \ldots \alpha_{i+1} \, \alpha_i \, \varepsilon_{i,i+1} \, \alpha_i \, \alpha_{i+1} \ldots \alpha_{j-2} \, \alpha_{j-1} \ \ i < j.$$

The structure of Cb_n was studied in [15, 16]. There it was proved that Cb_n, $n \geq 2$, is decomposable into the semi–direct product

$$Cb_n = D_{n-1} \leftthreetimes (D_{n-2} \leftthreetimes (\ldots \leftthreetimes (D_2 \leftthreetimes D_1)) \ldots),$$

of subgroups D_i, $i = 1, 2, \ldots, n-1$, generated by $\varepsilon_{i+1,1}, \varepsilon_{i+1,2}, \ldots, \varepsilon_{i+1,i}$, $\varepsilon_{1,i+1}, \ \varepsilon_{2,i+1}, \ \ldots, \varepsilon_{i,i+1}$. The elements $\varepsilon_{i+1,1}, \ \varepsilon_{i+1,2}, \ \ldots, \varepsilon_{i+1,i}$ generate a free group of rank i, elements $\varepsilon_{1,i+1}, \varepsilon_{2,i+1}, \ldots, \varepsilon_{i,i+1}$ generate a free abelian group of rank i.

The pure braid group P_n is contained in Cb_n and the generators of P_n can be written in the form

$$a_{i,i+1} = \varepsilon_{i,i+1}^{-1} \varepsilon_{i+1,i}^{-1}, \; i = 1, 2, \ldots, n-1,$$

$$a_{ij} = \varepsilon_{j-1,i} \varepsilon_{j-2,i} \ldots \varepsilon_{i+1,i} (\varepsilon_{ij}^{-1} \varepsilon_{ji}^{-1}) \varepsilon_{i+1,i}^{-1} \ldots \varepsilon_{j-2,i}^{-1} \varepsilon_{j-1,i}^{-1} =$$

$$= \varepsilon_{j-1,j}^{-1} \varepsilon_{j-2,j}^{-1} \ldots \varepsilon_{i+1,j}^{-1} (\varepsilon_{ij}^{-1} \varepsilon_{ji}^{-1}) \varepsilon_{i+1,j} \ldots \varepsilon_{j-2,j} \varepsilon_{j-1,j}, \; 2 \leq i+1 < j \leq n.$$

1.2. **The singular braid monoid.** *The Baez–Birman monoid* [6, 2] or *the singular braid monoid* SB_n is generated (as monoid) by elements σ_i, σ_i^{-1}, τ_i, $i = 1, 2, \ldots, n-1$. The elements σ_i, σ_i^{-1} generate the braid group B_n. The generators τ_i satisfy the defining relations

$$\tau_i \tau_j = \tau_j \tau_i, \; |i-j| \geq 1, \tag{12}$$

other relations are mixed:

$$\tau_i \sigma_j = \sigma_j \tau_i, \; |i-j| \geq 1, \tag{13}$$

$$\tau_i \sigma_i = \sigma_i \tau_i, \; i = 1, 2, \ldots, n-1, \tag{14}$$

$$\sigma_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \sigma_{i+1}, \; i = 1, 2, \ldots, n-2, \tag{15}$$

$$\sigma_{i+1} \sigma_i \tau_{i+1} = \tau_i \sigma_{i+1} \sigma_i, \; i = 1, 2, \ldots, n-2. \tag{16}$$

In the work [20] it was proved that the singular braid monoid SB_n is embedded into the group SG_n which is called the *singular braid group* and has the same defining relations as SB_n.

1.3. **The virtual braid group and welded braid group.** The virtual braid group VB_n was introduced in [3]. In [7] it was found more short system of defining relations (see below). The group VB_n is generated by σ_i, ρ_i, $i = 1, 2, \ldots, n-1$ (see Fig. 3).

The elements σ_i generate the braid group B_n with defining relations (1)–(2) and the elements ρ_i generate the symmetric group S_n which is defined by the relations

$$\rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1}, \; i = 1, 2, \ldots, n-2, \tag{17}$$

$$\rho_i \rho_j = \rho_j \rho_i, \; |i-j| \geq 1, \tag{18}$$

$$\rho_i^2 = 1 \; i = 1, 2, \ldots, n-1. \tag{19}$$

Other relations are mixed:

$$\sigma_i \rho_j = \rho_j \sigma_i, \; |i-j| \geq 1, \tag{20}$$

FIGURE 3. The geometric virtual braid ρ_i

$$\rho_i\,\rho_{i+1}\,\sigma_i = \sigma_{i+1}\,\rho_i\,\rho_{i+1},\ \ i = 1, 2, \ldots, n-2. \tag{21}$$

Note that the last relation is equivalent to the following relation:

$$\rho_{i+1}\,\rho_i\,\sigma_{i+1} = \sigma_i\,\rho_{i+1}\,\rho_i.$$

In the work [4] it was proved that the relations

$$\rho_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\rho_{i+1},\ \ \rho_{i+1}\,\sigma_i\,\sigma_{i+1} = \rho_i\,\sigma_{i+1}\,\sigma_i.$$

are not fulfilled in VB_n.

In the work [5] it was introduced the welded braid group WB_n. This group is generated by σ_i, α_i, $i = 1, 2, \ldots, n-1$. The elements σ_i generate the braid group B_n. The elements α_i generate the symmetric group S_n and the following mixed relations hold

$$\alpha_i\,\sigma_j = \sigma_j\,\alpha_i,\ \ |i-j| \geq 1, \tag{22}$$

$$\sigma_i\,\alpha_{i+1}\,\alpha_i = \alpha_{i+1}\,\alpha_i\,\sigma_{i+1},\ \ i = 1, 2, \ldots, n-2, \tag{23}$$

$$\sigma_{i+1}\,\sigma_i\,\alpha_{i+1} = \alpha_i\,\sigma_{i+1}\,\sigma_i,\ \ i = 1, 2, \ldots, n-2. \tag{24}$$

In the work [5] it was proved that WB_n is isomorphic to the group of conjugating automorphisms C_n.

Comparing the defining relations of VB_n with the defining relations of WB_n, we see that WB_n can be obtained from VB_n by adding some new relation. Therefore, there exists a homomorphism

$$\varphi_{VW} : VB_n \longrightarrow WB_n,$$

taking σ_i to σ_i and ρ_i to α_i for all i. Hence, WB_n is the homomorphic image of VB_n.

In [5] it was proved that the following relation (symmetric to (23))

$$\sigma_{i+1}\,\alpha_i\,\alpha_{i+1} = \alpha_i\,\alpha_{i+1}\,\sigma_i,$$

is true in WB_n. But the following relation is not fulfilled

$$\alpha_{i+1}\,\sigma_i\,\sigma_{i+1} = \sigma_i\,\sigma_{i+1}\,\alpha_i.$$

In [7] it was constructed the linear representations of VB_n and WB_n by matrices from $\mathrm{GL}_n(\mathbb{Z}[t,t^{-1}])$ which continue the well known Burau representation. The linear representation of $C_n \simeq WB_n$ it was constructed in [16]. This representation continue (with some conditions on parameters) the known Lawrence–Krammer representation.

2. Generators and defining relations of the virtual pure braid group

In this section we introduce a virtual pure braid group and find its generators and defining relations.

Define the map

$$\nu : VB_n \longrightarrow S_n$$

of VB_n onto the symmetric group S_n by actions on generators

$$\nu(\sigma_i) = \nu(\rho_i) = \rho_i,\ i = 1, 2, \ldots, n-1,$$

where S_n is the group generated by ρ_i. The kernel $\ker(\nu)$ of this map is called the *virtual pure braid group* and denoted by VP_n. It is clear that VP_n is a normal subgroup of index $n!$ of VB_n. Moreover, since $VP_n \bigcap S_n = e$ and $VB_n = VP_n \cdot S_n$, then $VB_n = VP_n \leftthreetimes S_n$, i. e., the virtual pure braid group is the semi–direct product of VP_n and S_n.

Define the following elements

$$\lambda_{i,i+1} = \rho_i\,\sigma_i^{-1},\ \lambda_{i+1,i} = \rho_i\,\lambda_{i,i+1}\,\rho_i = \sigma_i^{-1}\,\rho_i,\ i = 1, 2, \ldots, n-1,$$

$$\lambda_{ij} = \rho_{j-1}\,\rho_{j-2}\ldots\rho_{i+1}\,\lambda_{i,i+1}\,\rho_{i+1}\ldots\rho_{j-2}\,\rho_{j-1},$$

$$\lambda_{ji} = \rho_{j-1}\,\rho_{j-2}\ldots\rho_{i+1}\,\lambda_{i+1,i}\,\rho_{i+1}\ldots\rho_{j-2}\,\rho_{j-1},\ 1 \le i < j-1 \le n-1.$$

Obviously, all these elements belong to VP_n and have the following geometric interpretation (Fig. 4, 5)

The next lemma hold

Lemma 1. *Let $1 \le i < j \le n$. The following conjugating rules are fulfilled in VB_n:*
1) for $k < i-1$ or $i < k < j-1$ or $k > j$

$$\rho_k\,\lambda_{ij}\,\rho_k = \lambda_{ij},\ \rho_k\,\lambda_{ji}\,\rho_k = \lambda_{ji};$$

2) $\rho_{i-1}\,\lambda_{ij}\,\rho_{i-1} = \lambda_{i-1,j},\ \rho_{i-1}\,\lambda_{ji}\,\rho_{i-1} = \lambda_{j,i-1}$;
3) for $i < j-1$

FIGURE 4. The geometric virtual braid λ_{ij} $(1 \leq i < j \leq n)$

FIGURE 5. The geometric virtual braid λ_{ji} $(1 \leq i < j \leq n)$

$$\rho_i \, \lambda_{i,i+1} \, \rho_i = \lambda_{i+1,i}, \quad \rho_i \, \lambda_{ij} \, \rho_i = \lambda_{i+1,j},$$
$$\rho_i \, \lambda_{i+1,i} \, \rho_i = \lambda_{i,i+1}, \quad \rho_i \, \lambda_{ji} \, \rho_i = \lambda_{j,i+1};$$

4) for $i + 1 < j$

$$\rho_{j-1} \, \lambda_{ij} \, \rho_{j-1} = \lambda_{i,j-1}, \;\; \rho_{j-1} \, \lambda_{ji} \, \rho_{j-1} = \lambda_{j-1,i};$$

5) $\rho_j \, \lambda_{ij} \, \rho_j = \lambda_{i,j+1}, \;\; \rho_j \, \lambda_{ji} \, \rho_j = \lambda_{j+1,i}.$

Proof. We consider only the rules containing λ_{ij} for $i < j$ (the remaining rules can be considered analogously). Recall that

$$\lambda_{ij} = \rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1}.$$

If $k < i - 1$ or $k > j$ then ρ_k is permutable with $\rho_i, \rho_{i+1}, \ldots, \rho_{j-1}$ in view of relation (18) and with σ_i in view of relation (20). Hence, ρ_k is permutable with λ_{ij}.

Let $i < k < j-1$. Then

$$\rho_k \, \lambda_{ij} \, \rho_k = \rho_k \, (\rho_{j-1} \ldots \rho_{k+2} \, \rho_{k+1} \, \rho_k \ldots \rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1} \ldots \rho_k \, \rho_{k+1} \, \rho_{k+2} \ldots \rho_{j-1}) \, \rho_k.$$

Permuting ρ_k to $\lambda_{i,i+1}$ while it is possible, we get

$$\rho_{j-1} \, \ldots \rho_{k+2} \, (\rho_k \, \rho_{k+1} \, \rho_k) \ldots \rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1} \ldots (\rho_k \, \rho_{k+1} \, \rho_k) \, \rho_{k+2} \ldots \rho_{j-1}.$$

Using the relation $\rho_k \, \rho_{k+1} \, \rho_k = \rho_{k+1} \, \rho_k \, \rho_{k+1}$, rewrite the last formula as follows:

$$\rho_{j-1} \, \ldots \rho_{k+2} \, \rho_{k+1} \, \rho_k \, (\rho_{k+1} \, \rho_{k-1} \ldots \rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1} \ldots \rho_{k-1} \, \rho_{k+1}) \times$$
$$\times \rho_k \, \rho_{k+1} \, \rho_{k+2} \ldots \rho_{j-1} = \rho_{j-1} \ldots \rho_k \, (\rho_{k+1} \, \lambda_{i,k} \, \rho_{k+1}) \, \rho_k \ldots \rho_{j-1}.$$

In view of the case considered earlier, we have

$$\rho_{k+1} \, \lambda_{ik} \, \rho_{k+1} = \lambda_{ik}$$

and, hence,

$$\rho_{j-1} \ldots \rho_k \, (\rho_{k+1} \, \lambda_{ik} \, \rho_{k+1}) \, \rho_k \ldots \rho_{j-1} = \lambda_{ij}.$$

Thus, the first rule from 1) is proven.

2) Consider

$$\rho_{i-1} \, \lambda_{ij} \, \rho_{i-1} = \rho_{i-1} \, (\rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1}) \, \rho_{i-1}.$$

Using relation (18), let as permute ρ_{i-1} to $\lambda_{i,i+1}$ as long as it is possible. We get

$$\rho_{i-1} \, \lambda_{ij} \, \rho_{i-1} = \rho_{j-1} \ldots \rho_{i+2} \, \rho_{i+1} (\rho_{i-1} \, \lambda_{i,i+1} \, \rho_{i-1}) \, \rho_{i+1} \rho_{i+2} \ldots \rho_{j-2}. \tag{25}$$

The expression in the brackets can be rewritten in the following form

$$\rho_{i-1} \, \lambda_{i,i+1} \, \rho_{i-1} = \rho_{i-1} \, \rho_i \, \sigma_i^{-1} \, \rho_{i-1} = \rho_{i-1} \, \rho_i \, \sigma_i^{-1} \, \rho_{i-1} \rho_i \, \rho_i.$$

Using the relation $\sigma_i^{-1} \, \rho_{i-1} \, \rho_i = \rho_{i-1} \, \rho_i \, \sigma_{i-1}^{-1}$ (it follows from (21)) and (18), (19), we obtain

$$\rho_{i-1} \, \rho_i \, (\sigma_i^{-1} \, \rho_{i-1} \, \rho_i) \, \rho_i = \rho_{i-1} \, (\rho_i \, \rho_{i-1} \, \rho_i) \, \sigma_{i-1}^{-1} \, \rho_i =$$
$$= (\rho_{i-1} \, \rho_{i-1}) \, \rho_i \, \rho_{i-1} \, \sigma_{i-1}^{-1} \, \rho_i = \rho_i \, \lambda_{i-1,i} \, \rho_i.$$

Then from (25) we obtain

$$\rho_{i-1} \, \lambda_{ij} \, \rho_{i-1} = \lambda_{i-1,j}.$$

Therefore, the desired relations are proven.

3) The first formula follows from the definitions of $\lambda_{i,i+1}$ and $\lambda_{i+1,i}$. Let us consider

$$\rho_i \, \lambda_{ij} \, \rho_i = \rho_i \, (\rho_{j-1} \, \rho_{j-2} \ldots \rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1} \ldots \rho_{j-2} \, \rho_{j-1}) \, \rho_i.$$

Permuting ρ_i to $\lambda_{i,i+1}$ while it is possible, we obtain

$$\rho_i \, \lambda_{ij} \, \rho_i = \rho_{j-1} \ldots \rho_{i+2} \, (\rho_i \, \rho_{i+1} \, \lambda_{i,i+1} \, \rho_{i+1} \, \rho_i) \, \rho_{i+2} \, \ldots \rho_{j-1}.$$

Rewrite the expression in the brackets as follows

$$\rho_i\,\rho_{i+1}\,\lambda_{i,i+1}\,\rho_{i+1}\,\rho_i = \rho_i\,\rho_{i+1}\,\rho_i\,(\sigma_i^{-1}\,\rho_{i+1}\,\rho_i) = \rho_i\,\rho_{i+1}\,(\rho_i\,\rho_{i+1}\,\rho_i)\,\sigma_{i+1}^{-1} =$$

$$= \rho_i\,\rho_{i+1}\,\rho_{i+1}\,\rho_i\,\rho_{i+1}\,\sigma_{i+1}^{-1} = \rho_{i+1}\,\sigma_{i+1}^{-1}.$$

Hence,

$$\rho_i\,\lambda_{ij}\,\rho_i = \rho_{j-1}\ldots\rho_{i+2}\,(\rho_{i+1}\,\sigma_{i+1}^{-1})\,\rho_{i+2}\ldots\rho_{j-1} = \lambda_{i+1,j}.$$

Therefore, the desired relations are proven.

4) follows from the relation $\rho_{j-1}^2 = e$ and the definition of λ_{ij}.

5) is an immediate consequence of the definition of λ_{ij}. □

Corollary 1. *The group S_n acts by conjugation on the set $\{\lambda_{kl}\ |1 \leq k \neq l \leq n\}$. This action is transitive.*

In view of Lemma 1, the subgroup $\langle \lambda_{kl} \mid 1 \leq k \neq l \leq n\rangle$ of VP_n is normal in VB_n. Let us prove that this group coincides with VP_n and let us find its generators and defining relations. For this purpose we use the Reidemeister–Schreier method (see, for example, [21, Ch. 2.2]).

Let $m_{kl} = \rho_{k-1}\,\rho_{k-2}\ldots\rho_l$ for $l < k$ and $m_{kl} = 1$ in other cases. Then the set

$$\Lambda_n = \left\{ \prod_{k=2}^{n} m_{k,j_k} | 1 \leq j_k \leq k \right\}$$

is a Schreier set of coset representatives of VP_n in VB_n.

Theorem 1. *The group VP_n admits a presentation with the generators λ_{kl}, $1 \leq k \neq l \leq n$, and the defining relations:*

$$\lambda_{ij}\,\lambda_{kl} = \lambda_{kl}\,\lambda_{ij}, \tag{26}$$

$$\lambda_{ki}\,(\lambda_{kj}\,\lambda_{ij}) = (\lambda_{ij}\,\lambda_{kj})\,\lambda_{ki}, \tag{27}$$

where distinct letters stand for distinct indices.

Proof. Define the map $^{-} : VB_n \longrightarrow \Lambda_n$ which takes an element $w \in VB_n$ into the representative $\overline{w}$ from Λ_n. In this case the element $w\overline{w}^{-1}$ belong to VP_n. By Theorem 2.7 from [21] the group VP_n is generated by

$$s_{\lambda,a} = \lambda a \cdot (\overline{\lambda a})^{-1},$$

where λ run the set Λ_n and a run the set of generators of VB_n.

It is easy to establish that $s_{\lambda,\rho_i} = e$ for all representatives λ and generators ρ_i. Consider the generators

$$s_{\lambda,\sigma_i} = \lambda\sigma_i \cdot (\overline{\lambda\rho_i})^{-1}.$$

For $\lambda = e$ we get $s_{e,\sigma_i} = \sigma_i\rho_i = \lambda_{i,i+1}^{-1}$. Note that $\lambda\rho_i$ is equal to $\overline{\lambda\rho_i}$ in S_n. Therefore,

$$s_{\lambda,\sigma_i} = \lambda(\sigma_i\rho_i)\lambda^{-1}.$$

From Lemma 1 it follows that each generator s_{λ,σ_i} is equal to some λ_{kl}, $1 \leq k \neq l \leq n$. By Corollary 1, the inverse statement is also true, i. e., each element λ_{kl} is equal to some generator s_{λ,σ_i}. The first part of the theorem is proven.

To find defining relations of VP_n we define a rewriting process τ. It allows us to rewrite a word which is written in the generators of VB_n and presents an element in VP_n as a word in the generators of VP_n. Let us associate to the reduced word

$$u = a_1^{\varepsilon_1}\, a_2^{\varepsilon_2} \dots a_\nu^{\varepsilon_\nu},\ \varepsilon_l = \pm 1,\ a_l \in \{\sigma_1, \sigma_2, \dots, \sigma_{n-1}, \rho_1, \rho_2, \dots, \rho_{n-1}\},$$

the word

$$\tau(u) = s_{k_1,a_1}^{\varepsilon_1}\, s_{k_2,a_2}^{\varepsilon_2} \dots s_{k_\nu,a_\nu}^{\varepsilon_\nu}$$

in the generators of VP_n, where k_j is a representative of the $(j-1)$th initial segment of the word u if $\varepsilon_j = 1$ and k_j is a representative of the jth initial segment of the word u if $\varepsilon_j = -1$.

By [21, Theorem 2.9], the group VP_n is defined by relations

$$r_{\mu,\lambda} = \tau(\lambda\, r_\mu\, \lambda^{-1}),\ \lambda \in \Lambda_n,$$

where r_μ is the defining relation of VB_n.

Denote by

$$r_1 = \sigma_i\, \sigma_{i+1}\, \sigma_i\, \sigma_{i+1}^{-1}\, \sigma_i^{-1}\, \sigma_{i+1}^{-1}$$

the first relation of VB_n. Then

$$r_{1,e} = \tau(r_1) = s_{e,\sigma_i}\, s_{\overline{\sigma_i},\sigma_{i+1}}\, s_{\overline{\sigma_i\sigma_{i+1}},\sigma_i}\, s^{-1}_{\overline{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}},\sigma_{i+1}}\, s^{-1}_{\overline{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}},\sigma_i}\, s^{-1}_{\overline{r_1},\sigma_{i+1}} =$$

$$= \lambda_{i,i+1}^{-1}\, (\rho_i\, \lambda_{i+1,i+2}^{-1}\, \rho_i)\, (\rho_i\, \rho_{i+1}\, \lambda_{i,i+1}^{-1}\, \rho_{i+1}\, \rho_i) \times$$
$$\times (\rho_{i+1}\, \rho_i\, \lambda_{i+1,i+2}\, \rho_i\, \rho_{i+1})\, (\rho_{i+1}\, \lambda_{i,i+1}\, \rho_{i+1})\, \lambda_{i+1,i+2}.$$

Using the conjugating rules from Lemma 1, we get

$$r_{1,e} = \lambda_{i,i+1}^{-1}\, \lambda_{i,i+2}^{-1}\, \lambda_{i+1,i+2}^{-1}\, \lambda_{i,i+1}\, \lambda_{i,i+2}\, \lambda_{i+1,i+2}.$$

Therefore, the following relation

$$\lambda_{i,i+1}\, (\lambda_{i,i+2}\, \lambda_{i+1,i+2}) = (\lambda_{i+1,i+2}\, \lambda_{i,i+2})\, \lambda_{i,i+1}$$

is fulfilled in VP_n. The Remaining relations $r_{1,\lambda}$, $\lambda \in \Lambda_n$, can be obtained from this relation using conjugation by λ^{-1}. Bu the formulas from Lemma 1, we obtain relations (27).

Let us consider the next relation of VB_n:

$$r_2 = \sigma_i\, \sigma_j\, \sigma_i^{-1}\, \sigma_j^{-1},\ |i - j| > 1.$$

For it we have

$$r_{2,e} = \tau(r_2) = s_{e,\sigma_i}\, s_{\overline{\sigma_i},\sigma_j}\, s^{-1}_{\overline{\sigma_i \sigma_j \sigma_i^{-1}},\sigma_i}\, s^{-1}_{\overline{r_2},\sigma_j} =$$

$$= \lambda_{i,i+1}^{-1}\, \lambda_{j,j+1}^{-1}\, \lambda_{i,i+1}\, \lambda_{j,j+1}.$$

Hence, the relation

$$\lambda_{i,i+1}\, \lambda_{j,j+1} = \lambda_{j,j+1}\, \lambda_{i,i+1}, \ |i-j| > 1$$

holds in VP_n. Conjugating this relation by all representatives from Λ_n, we obtain relations (26).

Let us prove that only trivial relations follow from all other relations of VB_n. It is evident for relations (17)–(19) defining the group S_n because $s_{\lambda,\rho_i} = e$ for all $\lambda \in \Lambda_n$ and ρ_i.

Consider the mixed relation (21) (relation (20) can be considered similarly):

$$r_3 = \sigma_{i+1}\, \rho_i\, \rho_{i+1}\, \sigma_i^{-1}\, \rho_{i+1}\, \rho_i.$$

Using the rewriting process, we get

$$r_{3,e} = \tau(r_3) = s_{e,\sigma_{i+1}}\, s^{-1}_{\overline{\sigma_{i+1}\rho_i\rho_{i+1}\sigma_i^{-1}},\sigma_i} =$$

$$= \lambda_{i+1,i+2}^{-1}\, (\rho_i\, \rho_{i+1}\, \lambda_{i,i+1}\, \rho_{i+1}\, \rho_i) = \lambda_{i+1,i+2}^{-1}\, \lambda_{i+1,i+2} = e.$$

Therefore, VP_n is defined by relations (26) –(27). ☐

3. The structure of the virtual braid group

From the definition of VP_n and Lemma 1 it follows that $VB_n = VP_n \leftthreetimes S_n$, i. e., VB_n is the splittable extension of the group VP_n by S_n. Consequently, we have to study the structure of the virtual pure braid group VP_n. Let us define the subgroups

$$V_i = \langle \lambda_{1,i+1}, \lambda_{2,i+1}, \ldots, \lambda_{i,i+1}; \lambda_{i+1,1}, \lambda_{i+1,2}, \ldots, \lambda_{i+1,i} \rangle, \ i = 1, 2, \ldots, n-1,$$

of VP_n. Each V_i is a subgroup of VP_{i+1}. Let V_i^* be the normal closure of V_i in VP_{i+1}. The following theorem is the main result of this section.

Theorem 2. *The group VP_n, $n \geq 2$, is representable as the semi–direct product*

$$VP_n = V_{n-1}^* \leftthreetimes VP_{n-1} = V_{n-1}^* \leftthreetimes (V_{n-2}^* \leftthreetimes (\ldots \leftthreetimes (V_2^* \leftthreetimes V_1^*)) \ldots),$$

where V_1^* is a free group of rank 2 and V_i^*, $i = 2, 3, \ldots, n-1$, are free infinitely generated subgroups.

Let us prove the theorem by induction on n. For $n = 2$, we have

$$VP_2 = V_1 = V_1^*$$

and, by Theorem 1, the group V_1 is free generated by λ_{12} and λ_{21}.

To make the general case more clear consider the case $n = 3$.

3.1. **The structure of VP_3.** By Theorem 1, the group VP_3 is generated by subgroups V_1, V_2 and defined by the relations

$$\lambda_{12}\,(\lambda_{13}\,\lambda_{23}) = (\lambda_{23}\,\lambda_{13})\,\lambda_{12},\ \ \lambda_{21}\,(\lambda_{23}\,\lambda_{13}) = (\lambda_{13}\,\lambda_{23})\,\lambda_{21},$$
$$\lambda_{13}\,(\lambda_{12}\,\lambda_{32}) = (\lambda_{32}\,\lambda_{12})\,\lambda_{13},\ \ \lambda_{31}\,(\lambda_{32}\,\lambda_{12}) = (\lambda_{12}\,\lambda_{32})\,\lambda_{31},$$
$$\lambda_{23}\,(\lambda_{21}\,\lambda_{31}) = (\lambda_{31}\,\lambda_{21})\,\lambda_{23},\ \ \lambda_{32}\,(\lambda_{31}\,\lambda_{21}) = (\lambda_{21}\,\lambda_{31})\,\lambda_{32}.$$

From these relations we obtain the next lemma.

Lemma 2. *In VP_3 the following equalities hold:*
1)

$$\lambda_{13}^{\lambda_{12}} = \lambda_{32}^{\lambda_{12}}\,\lambda_{13}\,\lambda_{32}^{-1},\quad \lambda_{31}^{\lambda_{12}} = \lambda_{32}\,\lambda_{31}\,\lambda_{32}^{-\lambda_{12}},\quad \lambda_{23}^{\lambda_{12}} = \lambda_{13}\,\lambda_{23}\,\lambda_{32}\,\lambda_{13}^{-1}\,\lambda_{32}^{-\lambda_{12}},$$

$$\lambda_{13}^{\lambda_{12}^{-1}} = \lambda_{32}^{-1}\,\lambda_{13}\,\lambda_{32}^{\lambda_{12}^{-1}},\quad \lambda_{31}^{\lambda_{12}^{-1}} = \lambda_{32}^{-\lambda_{12}^{-1}}\,\lambda_{31}\lambda_{32},\quad \lambda_{23}^{\lambda_{12}^{-1}} = \lambda_{32}^{-\lambda_{12}^{-1}}\,\lambda_{13}^{-1}\,\lambda_{32}\,\lambda_{23}\,\lambda_{13},$$

2)

$$\lambda_{23}^{\lambda_{21}} = \lambda_{31}^{\lambda_{21}}\,\lambda_{23}\,\lambda_{31}^{-1},\quad \lambda_{32}^{\lambda_{21}} = \lambda_{31}\,\lambda_{32}\,\lambda_{31}^{-\lambda_{21}},\quad \lambda_{13}^{\lambda_{21}} = \lambda_{23}\,\lambda_{13}\,\lambda_{31}\,\lambda_{23}^{-1}\,\lambda_{31}^{-\lambda_{21}},$$

$$\lambda_{23}^{\lambda_{21}^{-1}} = \lambda_{31}^{-1}\,\lambda_{23}\,\lambda_{31}^{\lambda_{21}^{-1}},\quad \lambda_{32}^{\lambda_{21}^{-1}} = \lambda_{31}^{-\lambda_{21}^{-1}}\,\lambda_{32}\,\lambda_{31},\quad \lambda_{13}^{\lambda_{21}^{-1}} = \lambda_{31}^{-\lambda_{21}^{-1}}\,\lambda_{23}^{-1}\,\lambda_{31}\,\lambda_{13}\,\lambda_{23},$$

where a^b stand for $b^{-1}ab$.

Proof. The first and second relations from 1) immediately follow from the third and forth relations of VP_3 (see the relations before the lemma). Similarly, the first and second relations from 2) immediately follow from the fifth and sixth relations of VP_3.

Further, from the first and second relations of VP_3 we obtain

$$\lambda_{23}^{\lambda_{1,2}} = \lambda_{13}\,\lambda_{23}\,\lambda_{13}^{-\lambda_{12}},\ \ \lambda_{13}^{\lambda_{21}} = \lambda_{23}\,\lambda_{13}\,\lambda_{23}^{-\lambda_{21}}.$$

Using the proved formulas for $\lambda_{13}^{\lambda_{12}}$ and $\lambda_{23}^{\lambda_{21}}$, we get the third formulas from 1) and 2) respectively.

The formulas for conjugation by λ_{12}^{-1} and λ_{21}^{-1} can be obtained analogously. □

Note that there exists an epimorphism

$$\varphi_3 : VP_3 \longrightarrow VP_2,$$

which takes the generators of $V_2 = \langle \lambda_{13}, \lambda_{23}, \lambda_{31}, \lambda_{32}\rangle$ into the unit and fixes the generators of $V_1 = \langle \lambda_{12}, \lambda_{21}\rangle$. The kernel of this epimorphism is the normal closure of V_2 in VP_3, i. e., $\ker(\varphi_3) = V_2^*$.

Let u be the empty word or a reduced word beginning with non-zero power of λ_{12} and representing an element from V_1. Let $\lambda_{32}(u) = \lambda_{32}^u = u^{-1}\,\lambda_{32}\,u$. We call this element *the reduced power of the generator* λ_{32} with the power u. Analogously, if v is the empty word or a reduced word beginning with non-zero power of λ_{21} and representing an element from V_1, then we put $\lambda_{31}(v) = \lambda_{13}^v$ and call this element *the reduced power of generator* λ_{31} with the power v.

Lemma 3. *The group V_2^* is a free group with generators λ_{13}, λ_{23} and all reduced powers of λ_{31} and λ_{32}.*

Proof. To prove the lemma we can use the Reidemeister–Shreier method, but it is simpler to use the definitions of normal closure and semi-direct product. Evidently, the group V_2^* is generated by the elements

$$\lambda_{13}^w,\ \lambda_{23}^w,\ \lambda_{31}^w,\ \lambda_{32}^w,\ w \in V_1.$$

In view of Lemma 2, it is sufficient to take from these elements only λ_{13}, λ_{23} and all reduced powers of the generators λ_{31} and λ_{32}.

The freedom of V_2^* follows from the representation of VP_3 as the semi–direct product. Indeed, since $V_1 \bigcap V_2^* = e$, $V_1V_2^* = VP_3$, then $VP_3 = V_2^* \leftthreetimes V_1$. In this case the defining relations of VP_3 are equivalent to the conjugating rules from Lemma 2. Therefore, all relations define the action of the group V_1 on the group V_2^*. Since there are not other relations, this means that V_1 and V_2^* are free groups. □

As a consequence of this Lemma, we obtain the normal form of words in VP_3. Any element w from VP_3 can be written in the form $w = w_1w_2$, where w_1 is a reduced word over the alphabet $\{\lambda_{12}^{\pm 1}, \lambda_{21}^{\pm 1}\}$ and w_2 is a reduced word over the alphabet $\{\lambda_{13}^{\pm 1}, \lambda_{23}^{\pm 1}, \lambda_{31}(u)^{\pm 1}, \lambda_{32}(v)^{\pm 1}\}$, where $\lambda_{31}(u)$, $\lambda_{32}(v)$ are reduced powers of the generators λ_{31} and λ_{32} respectively.

3.2. **The proof of Theorem 2.** Now, we introduce the following notation. By λ_{ij}^* denote any λ_{ij} or λ_{ji} from VP_n.

Lemma 4. *For every $n \geq 2$ there exists a homomorphism*

$$\varphi : VP_n \longrightarrow VP_{n-1}$$

which takes the generators λ_{ij}^*, $i = 1, 2, \ldots, n-1$, to the unit and fixes other generators.

Proof. It is sufficient to prove that all defining relations turn to the defining relations by such defined map. For the defining relations of VP_{n-1} it is evident. If the relation of commutativity (see relation (26)) contains some generator of V_{n-1} then by acting with φ_n it turns to the trivial relation. Let us consider the left hand side of relation (27). We see that it contains every index two times. Hence, if this part includes some generator of V_{n-1} (i. e., one of the indices is equal to n) then some other generator contains the index n. Therefore, there are two generators of V_{n-1} in the left hand side of the relation. Since the right hand side contains all generators from the left hand side, then by acting with φ_n this relation turns to the trivial relation. □

Lemma 5. *The following formulas are fulfilled in the group VP_n:*

1) $\lambda_{kl}^{\lambda_{ij}^{\varepsilon}} = \lambda_{kl}, \ max\{i,j\} < max\{k,l\}, \ \varepsilon = \pm 1;$

2) $\lambda_{ik}^{\lambda_{ij}} = \lambda_{kj}^{\lambda_{ij}} \lambda_{ik} \lambda_{kj}^{-1}, \ \lambda_{ik}^{\lambda_{ij}^{-1}} = \lambda_{kj}^{-1} \lambda_{ik} \lambda_{kj}^{\lambda_{ij}^{-1}}, \ i < j < k \ or \ j < i < k;$

3) $\lambda_{ki}^{\lambda_{ij}} = \lambda_{kj} \lambda_{ki} \lambda_{kj}^{-\lambda_{ij}}, \ \lambda_{ki}^{\lambda_{ij}^{-1}} = \lambda_{kj}^{-\lambda_{ij}^{-1}} \lambda_{ki} \lambda_{kj}, \ i < j < k \ or \ j < i < k;$

4) $\lambda_{jk}^{\lambda_{ij}} = \lambda_{ik} \lambda_{jk} \lambda_{kj} \lambda_{ik}^{-1} \lambda_{kj}^{-\lambda_{ij}}, \ \lambda_{jk}^{\lambda_{ij}^{-1}} = \lambda_{jk}^{-\lambda_{ik}^{-1}} \lambda_{ij}^{-1} \lambda_{jk} \lambda_{kj} \lambda_{ij}, \ i < j < k \ or \ j < i < k,$

where, as usual, different letters stand for different indices.

Proof. The formula 1) immediately follows from the first relation of Theorem 1.

Consider relation (27) from Theorem 1:

$$\lambda_{ki} \, (\lambda_{kj} \, \lambda_{ij}) = (\lambda_{ij} \, \lambda_{kj}) \, \lambda_{ki}.$$

Note that the indices of generators are connected by one of the inequalities:

$$a) \ k < j < i, \ b) \ j < k < i, \ c) \ i < j < k,$$
$$d) \ j < i < k, \ e) \ k < i < j, \ f) \ i < k < j.$$

If the indices are connected by inequality $a)$ or $b)$ then from (27) we obtain

$$\lambda_{ki}^{\lambda_{kj}} = \lambda_{ij}^{\lambda_{kj}} \, \lambda_{ki} \, \lambda_{ij}^{-1},$$

and it is the first formula from 2).

If the indices in relation (27) are connected by inequality c) or d) we obtain

$$\lambda_{ki}^{\lambda_{ij}} = \lambda_{kj} \, \lambda_{ki} \, \lambda_{kj}^{-\lambda_{ij}},$$

and it is the first formula from 3).

If indices in relation (27) are connected by inequality e) or f) then

$$\lambda_{ij}^{\lambda_{ki}} = \lambda_{kj} \, \lambda_{ij} \, \lambda_{kj}^{-\lambda_{ki}}.$$

Using the formula from 2), we obtain

$$\lambda_{ij}^{\lambda_{ki}} = \lambda_{kj} \, \lambda_{ij} \, \lambda_{ji} \, \lambda_{kj}^{-1} \, \lambda_{ji}^{-\lambda_{ki}},$$

and it is the first formula from 4).

The formulas of conjugations by elements λ_{ij}^{-1} can be established similarly.

□

Assume that the theorem is proven for the group VP_{n-1}. Hence, any element $w \in VP_{n-1}$ can be written in the form

$$w = w_1 w_2 \dots w_{n-2}, \; w_i \in V_i^*,$$

where each word w_i is a reduced word over the alphabet consisting if generators $\lambda_{ki}^{\pm 1}$, $1 \leq k \leq i-1$, and reduced powers of generators λ_{ki}, $1 \leq k \leq i-1$, and their inverse. Let us define reduced powers of generators in the group V_{n-1}^*. We say that the element $\lambda_{nk}(w) = \lambda_{nk}^w$ is *the reduced power of the generator* λ_{nk} if w is the empty word or a word written in the normal form and begin with a reduced power of some generator λ_{lk} or its inverse.

The statement about decomposition in to the semi–direct product $VP_n = V_n^* \rtimes VP_{n-1}$ is quite evident. It remains to find generators of V_n^* and prove its freedom.

Lemma 6. *The group V_{n-1}^* is a free group. It is generated by $\lambda_{1n}, \lambda_{2n}, \dots, \lambda_{n-1,n}$ and all reduced powers of the generators $\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{n,n-1}$.*

Proof. The proof is similar to that of Lemma 3. From Lemma 5 it follows that this set is the set of generators of V_{n-1}^*. Further, since the set of defining relations of VP_n is equivalent to the set of conjugating formulas defining the action of VP_{n-1} on V_{n-1}^*, only trivial relations are fulfilled in V_{n-1}^*. □

Theorem 2 follows from these results.

As a consequence of this theorem we obtain the normal form of words in VB_n.

Corollary 2. *Every element from VB_n can be written uniquely in the form*

$$w = w_1 w_2 \dots w_{n-1} \lambda, \; \lambda \in \Lambda_n, \; w_i \in V_i^*,$$

where w_i is a reduced word in generators, reduced powers of generators and their inverse.

The defined above homomorphism of the virtual braid group onto the welded braid group agrees with the decomposition from Theorem 2 and with the decomposition of $C_n \simeq WB_n$ described in the first section.

Corollary 3. *The homomorphism $\varphi_{VW} : VB_n \longrightarrow WB_n$ agrees with the decomposition of these groups, i. e., it maps the group VP_n onto $Cb_n \simeq WP_n$ and the factors V_i^* onto the factors D_i, $i = 1, 2, \dots, n-1$.*

4. The universal braid group

Let us define *the universal braid group* UB_n as the group with generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$, $c_1, c_2, \dots, c_{n-1}$, defining relations (1)–(2), the relations:

$$c_i\, c_j = c_j\, c_i, \;\; |i-j| \geq 2,$$

and the mixed relations:

$$c_i\, \sigma_j = \sigma_j\, c_i \;\; |i-j| \geq 2.$$

Recall (see [22]) that Artin's group of the type I is called the group A_I with generators a_i, $i \in I$, and the defining relations

$$a_i\, a_j\, a_i \dots = a_j\, a_i\, a_j \dots, \;\; i, j \in I,$$

where words from the left and right hand sides consist of m_{ij} alternating letters a_i and a_j.

Proposition 1. *1) The group UB_n has as a subgroup the braid group B_n.*

2) There exist homomorphisms

$$\varphi_{US} : UB_n \longrightarrow SG_n, \;\; \varphi_{UV} : UB_n \longrightarrow VB_n, \;\; \varphi_{UB} : UB_n \longrightarrow B_n.$$

3) The group UB_n is Artin's group.

Proof. 1) Evidently, there exists a homomorphism $B_n \longrightarrow UB_n$. On the other hand, assuming $\psi(\sigma_i) = \sigma_i$, $\psi(c_i) = e$, $i = 1, 2, \dots, n-1$, we obtain the retraction ψ of UB_n onto B_n. Therefore, the subgroup $\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} \rangle$ of UB_n is isomorphic to the braid group B_n.

2) Let us define the map φ_{US} as follows

$$\varphi_{US}(\sigma_i) = \sigma_i, \;\; \varphi_{US}(c_i) = \tau_i, \;\; i = 1, 2, \dots, n-1.$$

Comparing the defining relations of UB_n and SG_n, we see that this map is a homomorphism. Analogously, we can show that the map

$$\sigma_i \longmapsto \sigma_i, \;\; c_i \longmapsto \rho_i,$$

is extendable to the homomorphism φ_{UV} and the map

$$\sigma_i \longmapsto \sigma_i, \;\; c_i \longmapsto e,$$

is extendable to the homomorphism φ_{UB}.

3) immediately follows from the defining relations of UB_n and the definition of Artin's group. □

It should be noted that none of the groups SG_n, VB_n, WB_n (in the natural presentations) is not Artin's group.

The following questions naturally arise in the context of the results obtained above.

Problems. 1) Solve the word and conjugacy problems in UB_n, $n > 2$.
2) Is it possible to give some geometric interpretation for elements of UB_n similar to the geometric interpretation for elements of the braid groups B_n, SG_n, VB_n, UB_n?

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Sobolev Institute of Mathematics, Novosibirsk 630090, Russia
E-mail address: `bardakov@math.nsc.ru`