A long-aborted project.

TOWARDS AN ELEMENTARY THEORY OF FINITE TYPE INVARIANTS OF INTEGRAL HOMOLOGY SPHERES

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This is a pre-preprint. Your comments are welcome.

Abstract. Following Ohtsuki, Garoufalidis, and Habegger we provide an elementary introduction to finite type invariants of integral homology spheres, culminating with a proof of the upper bound for the magnitude of the space $\mathcal{I}$ of such invariants in terms of the space $\mathcal{A}(\emptyset)$ of oriented trivalent graphs modulo the so-called AS and IHX relations. We raise the issue of "the fundamental theorem" for finite type invariants of integral homology spheres, which says that $\mathcal{A}(\emptyset)$ is also a lower bound for $\mathcal{I}$ (and hence, up to the difference between a filtered space and a graded space, the two are equal). There are several constructive but transcendental proofs of the fundamental theorem, and we underline a few problems whose solution may yield a direct topological proof of that theorem.

Contents

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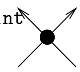
1. Introduction

1.1. **Stories.** Recent years saw an explosion of literature (see [B-N6]) on so-called "finite-type invariants". The basic idea behind those invariants is simple. Suppose in a certain class $\mathcal{O}$ of objects (whose invariants we seek) there is a natural notion of "a small modification" of an object in the class. A good example to keep in mind, and the first of that type that was considered, is the class $\mathcal{O} = \mathcal{K}$ of knots, whose "small modifications" are the operations of flipping a crossing from an undercrossing to an overcrossing:

flip  (a common notation for this small modification is 

)

Using these small modifications, one can "differentiate" an invariant I of objects in the class $\mathcal{O}$, by declaring the value of the derivative $I^{(1)}$ on some pair *(object, small modification)* to be the difference of the values of I on the given object before and after the modification. Assuming some further luck (which certainly occurs in the case of knots), one can talk about several "sites" on an object in $\mathcal{O}$, and one can carry small modifications of the object in each site independently, allowing by iteration for the definition of multiple derivatives $I^{(n)}$, defined on pairs *(an object, n small modifications occurring in n different sites).* By analogy with the case of knots, where singular knots are used to represent such pairs, these pairs are often called "n-singular objects".

Finite type invariants are now the straight-forward analogues of polynomials on a vector space; namely, we say that an invariant I is of type n if its $(n+1)$-st derivative is identically 0. I.e., if $I^{(n+1)} \equiv 0$. The analogy with multi-variable calculus persists a bit more: one of the main ways of studying a type n invariant is through its "weight system" — it's nth derivative $W = I^{(n)}$. The point is that if $I^{(n+1)} \equiv 0$, then W is a "constant". More precisely, it is oblivious to small modifications made to its argument, an n-singular object, and thus it can be regarded as a function on n-singular object modulo small modifications. Typically objects become simpler when regarded modulo small modifications, often turning from relatively complicated topological objects into easy-to-enumerate combinatorial objects in some class $\bar{\mathcal{O}}_n$ of "n-symbols". (In the case of knots, $\bar{\mathcal{K}}_n$ is the space of "n-chord diagrams").

Clearly, if two type n invariants have the same weight system, then their difference is a type $n-1$ invariant, and hence, modulo lower invariants, the enumeration of finite type invariants reduces to the enumeration of fully integrable weight systems: functionals on $\bar{\mathcal{O}}_n$ that are "integrable n times"; namely, that are the nth derivatives of an invariant. In most cases treated so far, the latter problem was solved in two steps:

- **Constraining:** First and more easily, one finds some constraints that functionals on $\bar{\mathcal{O}}_n$ have to satisfy to be integrable once or twice, hence bounding from above the

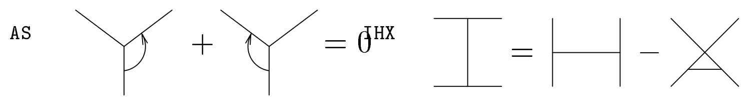


Figure 1. The Anti-Symmetry (AS) and IHX relations. fig:ASIHX

magnitude of the space $\mathcal{I}_n = \mathcal{I}_n(\mathcal{O})$ of type n invariants by the space $\mathcal{A}_n = \mathcal{A}_n(\mathcal{O}) = \bar{\mathcal{O}}_n/(\text{constraints found})$.

- **Constructing:** And then with much more effort and often using techniques which are transcendental to the question at hand and with some loss of generality [BS], one constructs an appropriate "universal finite type invariant" $Z = Z^{\mathcal{O}}$ with values in $\mathcal{A} = \mathcal{A}(\mathcal{O}) = \bigoplus_n \mathcal{A}_n(\mathcal{O})$. In the cases treated so far, it was easy to show that Z is surjective, and hence the magnitude of $\mathcal{I}_n$ is also bounded from below by the same space $\mathcal{A}_n$.

As already noted, the first class of objects on which those ideas were tried (in fact, before the generality of the ideas was appreciated), was the class $\mathcal{K}$ of knots. This has been a fruitful experience, as it turned out that the resulting class of invariants, also called "Vassiliev invariants" [B-N3, Bi, BL, Go1, Go2, Ko1, Vas1, Vas2], is rich, interesting, and well connected to other parts of mathematics. The study of other classes of objects (links, braids [St2, B-N5, Hu], tangles [LM1, LM2, B-N4], plane curves, etc.) quickly followed, enriching the theory even further.

Perhaps the first link between Vassiliev invariants and faraway parts of mathematics that was observed was its link with perturbative Chern-Simons theory [B-N1, B-N2]. Roughly speaking, the Feynman diagrams that arise in the perturbative expansion of the Chern-Simons path integral (in the presence of a knot, a "Wilson loop") are practically the same diagrams as those in $\bar{\mathcal{K}}_n$. This observation even leads to one of the known constructions of a "universal Vassiliev invariant" as discussed above [Th, AF].

There was an intriguing part to the story, though. Chern-Simons theory in the absence of Wilson loops predicts a similar-looking diagram valued invariant of integral homology spheres, and in a series of papers [AS1, AS2] on perturbative Chern-Simons theory Axelrod and Singer did practically all that was necessary for its construction (though they have packaged their results in a slightly different wrapping). This suggested that there ought to be a theory of finite type invariants of integral homology spheres, whose associated space of weight systems should have a description in terms of the diagram-combinations that appear in perturbative Chern-Simons theory — formal linear combinations of trivalent graphs with oriented vertices modulo the AS and IHX relations of Figure 1. At the time (circa 1992), such a theory did not exist.

Things started to look better around 1995, when in a short and beautiful paper [Oh] Tomotada Ohtsuki proposed a notion of "small modification" of an integral homology sphere (roughly, it is a surgery along a single component link), and hence a theory of finite type invariants of integral homology spheres. In his paper, Ohtsuki also demonstrated a relationship between his class of invariants and a certain class of diagrams, though the diagrams were not quite the same as those expected from perturbative Chern-Simons theory, and the AS and IHX relations were not present. Later on, in a joint paper of his and S. Garoufalidis [GO], the "constraining" step (as above) was completed, finding that the space

$\mathcal{A}(\mathcal{M})$ is precisely the one predicted by perturbative Chern-Simons theory, including the AS and IHX relations. Somewhat later, the "constructing" step was also completed, by T.Q.T. Le, J. Murakami, and T. Ohtsuki [LMO, Le] (see also a more conceptual construction in [BGRT1, BGRT2]).

The purpose of this paper is twofold:

- To increase the accessibility of the "constraining" part of the theory; as it is, it is spread over several papers [Oh, Ga, GL, GO], that combine to a rather formidable reading challenge. The hardest bit in this challenge is the proof of the IHX relation. This proof was simplified greatly by N. Habegger [Ha], but we feel that our version (which is merely an improvement of his) is even simpler.
- To call for the construction of a Hutchings-style integration theory in the context of finite type invariants of integral homology spheres.

Let me briefly explain what the latter point means. The "constructing" part of the theory, as described above, is somewhat unsatisfactory. While the constructions in [LMO, Le] and in [BGRT1, BGRT2] are clean and elegant, they use deep mathematics that goes way beyond the relatively limited-scope mathematics of the "constraining" part. One hopes that there should be a simpler, downward-inductive, way of "integrating" suitable functionals on $\bar{\mathcal{O}}_n$, by first integrating them to functionals on $\mathcal{O}_{n-1}$, and then integrating again to get functionals on $\mathcal{O}_{n-2}$, and on until we get functionals on $\mathcal{O}_0 = \mathcal{O}$, namely, finite type invariants of the original class of objects.

This downward-inductive approach is no new; in the case of knots it goes back to the earliest papers on Vassiliev invariants [Vas1, Vas2, BL] and some further progress was made in [St1, DD, Wi]. A big step forward was then carried by Michael Hutchings [Hu], who discovered that the obstruction for the success of this downward inductive process lies in some combinatorialy-defined homology group, akin to H^1 of a variant of Kontsevich's graph cohomology [Ko2]. Hutchings also proved that this homology group vanishes in the case of braids, raising hopes that it would also vanish in the case of other finite-type theories.

While the form of Hutchings' theory is clear in the case of knots (and for the sake of completeness we review this case in Section 2), much is missing in the case of integral homology spheres. There are good reasons to believe that the resulting obstruction group would be H^1 of Kontsevich's graph cohomology, but many of the necessary steps along the way are not yet understood. Thus throughout this paper we try to use notation comapatible with a future construction of a Hutchings-style integration theory, and towards its end we disscuss the remaining problems.

1.2. **Plan of the paper.** We conclude this introducion with the definition of the class $\mathcal{M}_n$ of "n-singular integral homology spheres", and of finite type invariants of integral homology spheres. Even an introduction should have some contents beyond story-telling. In Section 2 we review the better understood case of finite type invariants of knots and especially we review Hutchings' integration theory in the case of knots. Some readers may find that section interesting on its own right; unfortunately, Hutchings' approach didn't receive as much exposure as it deserves. Thus our review is a bit imbalanced — we quickly run through some of the material, but spend considerable time reviewing the less-appreciated parts. In Section 3 we recall some of the background we need from surgery theory and about the triple linking numbers. In Section 4 we study $\bar{\mathcal{M}}_n$, the space of "n-symbols" of the present context and the parallel of the space of n-chord diagrams from the knot context. In Section 5 we complete the "constraining" step by studying the conditions that functionals on $\bar{\mathcal{M}}_n$ have to satisfy to be integrable once; namely, to be the derivatives of functionals on $\mathcal{M}_{n-1}$. In the final Section 6 we disscuss the ingredients still missing for a construction of a Hutchings-style integration theory.

1.3. **Finite type invariants of integral homology spheres, the definition.**

Definition 1.1. An n-singular integral homology sphere is a pair (M, L) where M is an integral homology sphere and $L = \bigcup_{i=1}^n L_i$ is a unit-framed algebraically split ordered n-component link in M. Namely, the components L_i of M are numbered 1 to n ("ordered"), framed with ± 1 framing ("unit framed"), and the pairwise linking numbers between the different components of L are 0 ("algebraically split"). We think of L as marking n sites for performing small modifications of M, each modification being the surgery on one of the components of L. Let us temporarily define $\mathcal{M}_n$ to be the $\mathbb{Z}$-module of all formal $\mathbb{Z}$-linear combinations of n-singular integral homology spheres. A correction to the definition of $\mathcal{M}_n$ will be given in Definition 1.2 below. Notice that $\mathcal{M}_0$, which we often simply denote by $\mathcal{M}$, is simply the space of all $\mathbb{Z}$-linear combinations of integral homology spheres.

If $L = L^1 \cup L^2$ is a framed link (presented as a union of two sublinks L^1 and L^2) in some 3-manifold M, we denote by $(M, L^1)_{L^2}$ the result of surgery[1] of (M, L^1) along L^2. Namely, $(M, L^1)_{L^2}$ is a pair $(M', L^{1'})$, in which M' is the result of surgery of M along L^2, and $L^{1'}$ is the image in M' of L^1. Notice that if (M, L) is an $(n+1)$-singular integral homology sphere, then $(M, L - L_i)_{L_i}$ is again an n-singular integral homology sphere for any component L_i of L.

We now wish to define the co-derivative map $\delta : \mathcal{M}_{n+1} \to \mathcal{M}_n$, whose adjoint will be the differentiation map for invariants:

Definition def:codiff **1.2.** Define δ_i on generators by $\delta_i(M, L) = (M, L - L_i) - (M, L - L_i)_{L_i}$, and extend it to be a $\mathbb{Z}$-linear map $\mathcal{M}_{n+1} \to \mathcal{M}_n$. For later convinience, we want to set $\delta = \delta_i$ for any i, but the different i's may give different answers. We resolve this by redefining $\mathcal{M}_n$. Set

$$\text{eq:codiff}\quad \mathcal{M}_n = (\text{old } \mathcal{M}_n) \Big/ \left(\begin{array}{c} \delta_i(M, L) = \delta_j(M, L) \\ \text{for all } 1 \leq i, j \leq n+1 \text{ and all} \\ (n+1)\text{-singular integral homology spheres} \end{array} \right). \tag{1}$$

We can now set (in the new $\mathcal{M}_n$)

$$\delta(M, L) = (M, L - L_i) - (M, L - L_i)_{L_i} \quad \text{for } \textit{any } i.$$

The relations in equation (1) are called "the co-differentiability relations".

We can finally differentiate invariants using the adjoint $\partial = \delta^\star : \mathcal{M}_n^\star \to \mathcal{M}_{n+1}^\star$. That is, if $I \in \mathcal{M}_n^\star$ is a differentiable invariant of n-singular integral homology spheres (namely, which vanishes on the co-differentiability relations), let its derivative $I' \in \mathcal{M}_{n+1}^\star$ be $\partial I = I \circ \delta$. Iteratively, one can define multiple derivatives such us $I^{(k)}$ for any $k \geq 0$.

Definition 1.3. (Ohtsuki [Oh] We say that an invariant I of integral homology spheres if of type n if $I^{(n+1)} \equiv 0$. We say that it is of finite type if it is of type n for some natural number n.

[1] We recall some basic facts about surgery in Section 3.1.

Unravelling the definitions, we find that I is of type n precisely when for all integral homology spheres M and all unit-framed algebraically split links L in M,

$$\text{eq:altsum} \sum_{L' \subset L} (-1)^{|L'|} I\left(M_{L'}\right) = 0, \tag{2}$$

where the sum runs on all sublinks L' of L (including the empty and full sublinks), $|L'|$ is the number of components of L', and $M_{L'}$ is the result of surgery of M along L'. We will not use equation (2) in this paper.

1.4. **Acknowledgement.** I thank all.

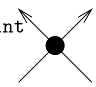
2. The case of knots

sec:Knots

2.1. **Singular knots, the co-differential δ, and finite type invariants.** As we have already indicated in the introduction, the finite-type theory for knots (Vassiliev theory) is built around the notions of n-singular knots, and differences between overcrossings and undercrossings. Let us make those notions precise:

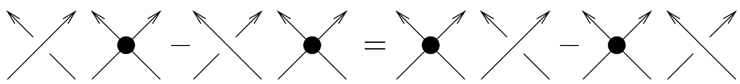
def:nSingularKnot
Definition 2.1. An n-singular knot is an oriented knot in an oriented $\mathbb{R}^3$, which is allowed to have n singular points that locally look like the picture on the right. For simplicity in the later parts of this section, we only consider framed (singular or not) knots, and always use blackboard framing when a knot projection or a part of a knot projection is drawn.

DoublePoint



def:Kcodiff
Definition 2.2. Let $\mathcal{K}_n$ be the $\mathbb{Z}$-module freely generated by all n-singular knots, modulo the following "co-differentiability relation":

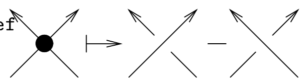
codiff



Notice that $\mathcal{K}_0 = \mathcal{K}$ is simply the free $\mathbb{Z}$-module generated by all (framed) knots.

Definition 2.3. Let $\delta : \mathcal{K}_{n+1} \to \mathcal{K}_n$ be defined by "resolving" any one of the singular points in an $(n+1)$-singular knot in $\mathcal{K}_{n+1}$:

(3)



Note that thanks to the co-differentiability relation, δ is well defined. It is called "the co-derivative". We denote the adjoint of δ by ∂ and call it "the derivative". It is a map $\partial : \mathcal{K}_n^\star \to \mathcal{K}_{n+1}^\star$.

The name "derivative" is justified by the fact that $(\partial V)(K)$ for some $V \in \mathcal{K}_n^\star$ and $K \in \mathcal{K}_{n+1}$ is by definition the difference of the values of V on two "neighboring" n-singular knots, in harmony with the usual definition of derivative for functions on $\mathbb{R}^d$.

Definition 2.4. An invariant of knots V (equivalently, a $\mathbb{Z}$-linear functional on $\mathcal{K}$) is said to be of finite type n if its $(n+1)$-st derivative vanishes, that is, if $\partial^{n+1} V \equiv 0$. (This definition is the analog of one of the standard definitions of polynomials on $\mathbb{R}^d$).

When thinking about finite type invariants, it is convenient to have in mind the following ladders of spaces and their duals, printed here with the names of some specific elements that we will use later:

eq:ladders

$$
\begin{array}{ccccccccccccc}
\ldots \longrightarrow & \mathcal{K}_{n+1} & \xrightarrow{\delta} & \mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \longrightarrow & \ldots & \longrightarrow & \mathcal{K}_1 & \xrightarrow{\delta} & \mathcal{K}_0 = \mathcal{K} \\
\ldots \longleftarrow & \mathcal{K}^\star_{n+1} & \xleftarrow{\delta} & \mathcal{K}^\star_n & \xleftarrow{\delta} & \mathcal{K}^\star_{n-1} & \longleftarrow & \ldots & \longleftarrow & \mathcal{K}^\star_1 & \xleftarrow{\delta} & \mathcal{K}^\star_0 = \mathcal{K}^\star \\
& \Cup & & \Cup & & & & & & & & \Cup \\
& \partial^{n+1}V \equiv 0 & & \partial^n V = W & & & & & & & & V
\end{array}
\tag{4}
$$

One may take the definition of a general "theory of finite type invariants" to be the data in (4), with arbitrary "n-singular objects" replacing the n-singular knots. Much of what we will say below depends only on the existance of the ladders (4), or on the existance of certain natural extensions thereof, and is therefore quite general.

2.2. **Constancy conditions, $\mathcal{K}_n/\delta\mathcal{K}_{n+1}$, and chord diagrams.** As promised in the introduction, we study invariants of type n by studying their nth derivatives. Clearly, if V is of type n and $W = \partial^n V$, then $\partial W = 0$ ("W is a constant"). Glancing at (4), we see that W descends to a linear functional, also called W, on $\mathcal{K}_n/\delta\mathcal{K}_{n+1}$:

Definition 2.5. We call $\bar{\mathcal{K}}_n := \mathcal{K}_n/\delta\mathcal{K}_{n+1}$ the space of "n-symbols" associated with the ladders in (4). (The name is inspired by the theory of differential operators, where the "symbol" of an operator is essentially its equivalence class modulo lower order operators. The symbol is responsible for many of the properties of the original operator, and for many purposes, two operators that have the same symbol are "the same".) We denote the projection mapping $\mathcal{K}_n \to \bar{\mathcal{K}}_n$ that maps every singular knot to its symbol by π.

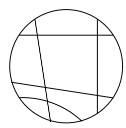
The following classical proposition (see e.g. [B-N3, Bi, BL, Go1, Go2, Ko1, Vas1, Vas2] identifies the space of n-symbols in our case:

prop:Ksymbols

Proposition 2.6. *The space $\bar{\mathcal{K}}_n$ of n-symbols for* (4) *is canonically isomorphic to the space $\mathcal{D}_n$ of n-chord diagrams, defined below.* □

Definition 2.7. An n-chord diagram is a choice of n pairs of distinct points on an oriented circle, considered up to orientation preserving homeomorphisms of the circle. Usually an n-chord diagram is simply drawn as a circle with n chords (whose ends are the n pairs), as in the 5-chord example on the right. The space $\mathcal{D}_n$ is the space of all formal $\mathbb{Z}$-linear combinations of n-chord diagrams.

5Chords



2.3. **Integrability conditions,** $\ker\delta$**, lassoing singular points, and four-term relations.** Next, we wish to find conditions that a "potential top derivative" has to satisfy in order to actually be a top derivative. More precisely, we wish to find conditions that a functional $W \in \bar{\mathcal{K}}^\star_n$ has to satisfy in order to be $\partial^n V$ for some invariant V. A first condition is that W must be "integrable once"; namely, there has to be some $W^1 \in \mathcal{K}^\star_{n-1}$ with $W = \partial W^1$. Another quick glance at (4), and we see that W is integrable once iff it vanishes on $\ker\delta$, which is the same as requiring that W descends to $\mathcal{A}_n = \mathcal{A}_n(\mathcal{K}) := \bar{\mathcal{K}}_n/\pi(\ker\delta) = \mathcal{K}_n/(\operatorname{im}\delta + \ker\delta)$ (there should be no confusion regarding the identities of the δ's involved). Often elements

T4T

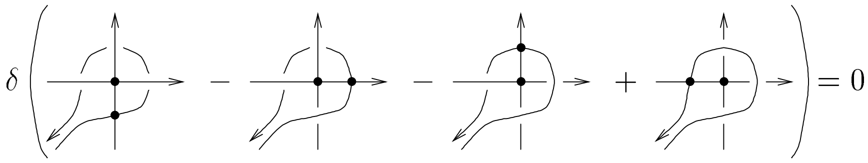


Figure 2. A Topological 4-Term ($T4T$) relation. Each of the four graphics in the picture represents a part of an n-singular knot (so there are $n-2$ additional singular points not shown), and, as usual in knot theory, the 4 singular knots in the equation are the same outside the region shown.

fig:T4T

Lasso

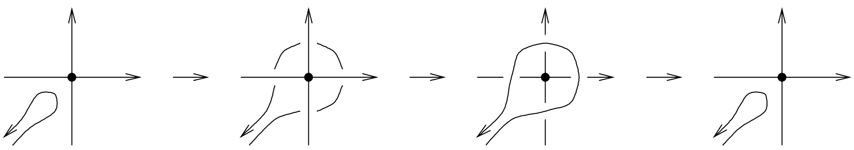


Figure 3. Lassoing a singular point: Each of the graphics represents an $(n-1)$-singular knot, but only one of the singularities is explicitly displayed. Start from the left-most graphic, pull the "lasso" under the displayed singular point, "lasso" the singular point by crossing each of the four arcs emenating from it one at a time, and pull the lasso back out, returning to the initial position. Each time an arc is crossed, the difference between "before" and "after" is δ of an n-singular knot (up to signs). The four n-singular knot thus obtained are the ones making the Topological 4-Term relation, and δ of their signed sum is the difference between the first and the last $(n-1)$-singular knot shown in this figure; namely, it is 0.

fig:Lasso

of $\mathcal{A}_n^\star$ are refered to as "weight systems". A more accurate name would be "once-integrable weight systems".

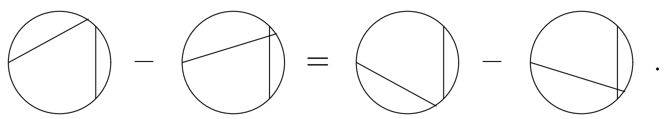
We see that it is necessary to understand $\ker\delta$. In Figure 2 we show a family of members of $\ker\delta$, the "Topological 4-Term" ($T4T$) relations. Figure 3 explains how they arise from "lassoing a singular point". The following theorem says that this is all:

thm:Stanford

Theorem 1. *(Stanford* [St1]*) The $T4T$ relations of Figure 2 span* $\ker\delta$. ☐

Pushing the $T4T$ relations down to the level of symbols, we get the well-known $4T$ relations, which span $\pi(\ker\delta)$: (see e.g. [B-N3])

$4T$: 4T



We thus find that $\mathcal{A}_n$ = (chord diagrams)/($4T$ relations), as usual in the theory of finite type invariants of knots.

2.4. **Hutchings' theory of integration.** We have so far found that if V is a type-n invariant, then $W=\partial^n V$ is a linear functional on $\mathcal{A}_n$. A question arises whether every linear functional on $\mathcal{A}_n$ arises in this way. At least if the ground ring is extended to $\mathbb{Q}$, the answer is positive:

thm:Fundamental
Theorem 2. *(The Fundamental Theorem of Finite-Type Invariants, Kontsevich* [Ko1]*) Over $\mathbb{Q}$, for every $W \in \mathcal{A}_n^\star$ there exists a type n invariant V with $W = \partial^n V$. In other words, every once-integrable weight system is fully integrable.*

The problem with the Fundamental Theorem is that all the proofs we have for it are somehow "transcendental", using notions from realms outside the present one, and none of the known proofs settles the question over the integers (see [BS]). In this section we describe what appears to be the most natural and oldest approach to the proof, having been mentioned already in [Vas1, BL]. Presently, we are stuck and the so-called "topological" approach does not lead to a proof. But it seems to me that it's worth studying further; when something natural fails, there ought to be a natural reason for that, and it would be nice to know what it is.

The idea of the topological approach is simple: To get from W to V, we need to "integrate" n times. Let's do this one integral at a time. By the definition of $\mathcal{A}_n$, we know that we can integrate once and find $W^1 \in \mathcal{K}_{n-1}^\star$ so that $\partial W^1 = W$. Can we work a bit harder, and find a "good" W^1, so that there would be a $W^2 \in \mathcal{K}_{n-2}^\star$ with $\partial W^2 = W^1$? Proceeding like that and assuming that all goes well along the way, we would end with a $V = W^n \in \mathcal{K}_0^\star$ with $\partial^n V = W$, as required. Thus we are naturally lead to the following conjecture, which implies the Fundamental Theorem by the backward-inductive argument just sketched:

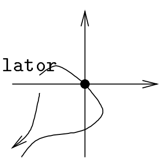
conj:InductionStep
Conjecture 1. *Every once-integrable invariant of n-singular knots also twice integrable. Glancing at* (4)*, we see that this is the same as saying that* $(\ker\delta^2)/(\ker\delta) = 0$.

This conjecture is somewhat stronger than Theorem 2. Indeed, Theorem 2 is equivalent to Conjecture 1 restricted to the case when the given invariant has some (possibly high) derivative identically equal to 0 (exercise!). But it is hard to imagine a topological proof of the restricted form of Conjecture 1 that would not prove it in full.

The difficulty in Conjecture 1 is that it's hard to say much about $\ker\delta^2$. In [Hu], Michael Hutchings was able to translate the statement $(\ker\delta^2)/(\ker\delta) = 0$ to an easier-looking combinatorial-topological statement, which is implied by and perhaps equivalent to an even simpler fully combinatorial statement. Furthermore, Hutchings proved the fully combinatorial statement in the analogous case of finite-type braid invariants, thus proving Conjecture 1 and Theorem 2 (over $\mathbb{Z}$) in that case, and thus proving the viability of his technique.

Hutchings' first step was to write a chain of isomorphisms reducing $(\ker\delta^2)/(\ker\delta)$ to something more manageable. Our next step will be to introduce all the spaces participating in Hutchings' chain. First, let us consider the space of all $T4T$ relations:

def:Rn
Definition 2.8. Let $\mathcal{K}_n^1$ be the $\mathbb{Z}$-module generated by all (framed) knots having $n-2$ singularities as in Definition 2.1, and plus one additional "Topological Relator" singularity that locally looks like the picture on the right, modulo the same co-differentiability relations as in Definition 2.2. Define $\delta : \mathcal{K}_{n+1}^1 \to \mathcal{K}_n^1$ in the same way as for knots, using equation (3). Finally, define $b : \mathcal{K}_n^1 \to \mathcal{K}_n$ by mapping the topological relator to the topological 4-term relation, the 4-term alternating sum inside the paranthesis in Figure 2.



The spaces $\mathcal{K}_n^1$ form a ladder similar to the one in (4), and, in fact, they combine with the ladder in (4) to a single commutative diagram:

$$
\text{eq:RKladders}\qquad
\begin{array}{ccccccccc}
\dots \xrightarrow{\delta} & \mathcal{K}^1_{n+1} & \xrightarrow{\delta} & \mathcal{K}^1_n & \xrightarrow{\delta} & \mathcal{K}^1_{n-1} & \xrightarrow{\delta} \dots \\
 & \downarrow b & & \downarrow b & & \downarrow b & \\
\dots \xrightarrow{\delta} & \mathcal{K}_{n+1} & \xrightarrow{\delta} & \mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \xrightarrow{\delta} \dots ,
\end{array}
\tag{5}
$$

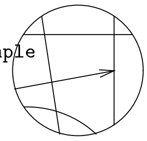
In this language, Stanford's theorem (Theorem 1) says that all L shapes in the above diagram (compositions $\delta \circ b$ of "down" followed by "right") are exact.

Just like singular knots had symbols which were simplar combinatorial objects (chord diagrams), so do toplogical relators have combinatorial symbols:

Definition 2.9. Let $\bar{\mathcal{K}}^1_n := \mathcal{K}^1_n / \delta\mathcal{K}^1_{n+1}$, and let $\pi : \mathcal{K}^1_n \to \bar{\mathcal{K}}^1_n$ be the projection map.

The following proposition is proved along the same lines as the standard proof of Proposition 2.6.

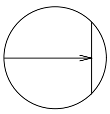
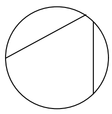
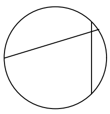
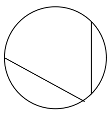
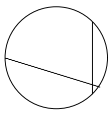
Proposition 2.10. *$\bar{\mathcal{K}}^1_n$ is canonically isomorphic to the space spanned by all "relator symbols", chord diagrams with $n-2$ chords and one $\rightarrow\!\!|$ piece corresponding to the special singularity of Definition 2.8. An example appears on the right.* RelatorExample



We need to display one additional commutative diagram before we can come to Hutchings' chain of isomorphisms:

$$
\text{eq:RKsymbols}\qquad
\begin{array}{ccccccc}
\mathcal{K}^1_n & \xrightarrow{\delta} & \mathcal{K}^1_{n-1} & \xrightarrow{\pi} & \bar{\mathcal{K}}^1_{n-1} & \longrightarrow & 0 \\
\downarrow b & & \downarrow b & & \downarrow \bar{b} & & \\
\mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \xrightarrow{\pi} & \bar{\mathcal{K}}_{n-1} & \longrightarrow & 0
\end{array}
\qquad \text{(exact rows).}
\tag{6}
$$

In this diagram, $\bar{b}$ is the "symbol level" version of b, and is induced by $b : \mathcal{K}^1_{n-1} \to \mathcal{K}_{n-1}$ in the usual manner. It can be described combinatorially by

barb $\bar{b}$:  $\mapsto$  $-$  $-$  $+$  .

Hutchings' chain of isomorphisms is the following chain of equalities and maps: (here the symbol $\cup$ means that the space below is a subspace of the space above, and the symbol $\uplus$ means that the space below is a sub-quotient of the space above)

$$
\begin{array}{ccccccccccccc}
\mathcal{K}_n & & \mathcal{K}_{n-1} & & \mathcal{K}_{n-1} & & \mathcal{K}^1_{n-1} & & \mathcal{K}^1_{n-1} & & \mathcal{K}^1_{n-1} & & \bar{\mathcal{K}}^1_{n-1} \\
\uplus & & \cup & & \cup & & \uplus & & \uplus & & \uplus & & \uplus \\
\dfrac{\ker\delta^2}{\ker\delta} & \xrightarrow{\delta} & \ker\delta\cap\operatorname{im}\delta & = & \operatorname{im} b\cap\ker\pi & \xleftarrow{b} & \dfrac{\ker\pi\circ b}{\ker b} & = & \dfrac{\ker\bar{b}\circ\pi}{\ker b} & = & \dfrac{\pi^{-1}(\ker\bar{b})}{\ker b} & \xrightarrow{\pi} & \dfrac{\ker\bar{b}}{\pi(\ker b)}.
\end{array}
$$

Theorem 3. *(Hutchings* [Hu]*) All maps in the above chain are isomorphisms. In particular,* $(\ker\delta^2)/(\ker\delta) \simeq (\ker\bar{b})/(\pi(\ker b))$.

Proof. Immediate from diagrams (5) and (6). □

It doesn't look like we've achieved much, but in fact we did, as it seems that $(\ker\bar{b})/(\pi(\ker b))$ is easier to digest than the original space of interest, $(\ker\delta^2)/(\ker\delta)$. The point is that $\ker\bar{b}$ lives fully in the combinatorial realm, being essentially the space of all relations between $4T$

relations at the symbol level. Similarly, $\pi(\ker b)$ is the space of projections to the symbol level of relation between $4T$ relations, and hence we have shown

Corollary 2.11. *Conjecture 1 is equivalent to the statement "every relation between $4T$ relations at the symbol level has a lift to the topological level".*

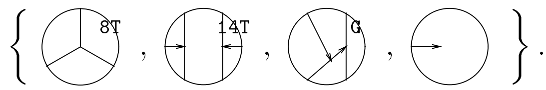
An obvious approach to proving Conjecture 1 thus emerges:

- Combinatorial step: Find all relations between $4T$ relations at the symbols level; that is, find a generating set for $\ker \bar{b}$.
- For every relation found in the combinatorial step, show that it lifts to the topological level.

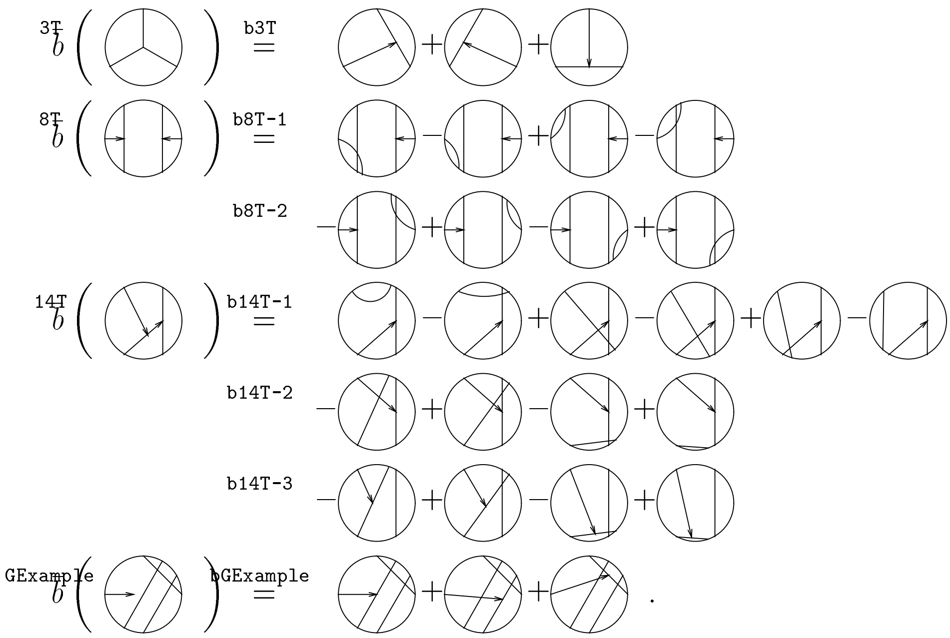
So far, the problem with this approach appears to be in the combinatorial step. There is a conjectural generating set $\bar{\mathcal{K}}^2_{n-1}$ for $\ker \bar{b}$. Every element in $\bar{\mathcal{K}}^2_{n-1}$ indeed has a lifting to $\ker b$, but we still don't know if $\bar{\mathcal{K}}^2_{n-1}$ indeed generates $\ker \bar{b}$. We state these facts very briefly; more information can be found in [Hu] and in [BS].

Definition 2.12. Define $\bar{\mathcal{K}}^2_{n-1}$ by

$$\bar{\mathcal{K}}^2_{n-1} = \operatorname{span}$$



As usual, each graphic in the above formula represents a large number of elements of $\bar{\mathcal{K}}^2_{n-1}$, obtained from the graphic by the addition of $n-3$ chords (first graphic), or $n-5$ chords (second graphic), or $n-4$ chords (third graphic), or $n-2$ chords (fourth graphic). Define also $\bar{b} : \bar{\mathcal{K}}^2_{n-1} \to \bar{\mathcal{K}}^1_{n-1}$ by



conj:K2

Conjecture 2. *The sequence $\bar{\mathcal{K}}^2_{n-1} \xrightarrow{\bar{b}} \bar{\mathcal{K}}^1_{n-1} \xrightarrow{\bar{b}} \bar{\mathcal{K}}_{n-1}$ is exact.*

A parallel of Conjecture 2 for braids was proven by Hutchings in [Hu].

LTDefinition

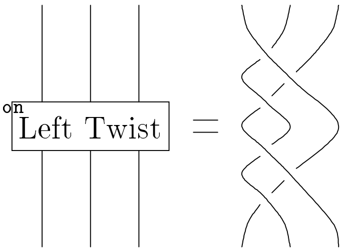


Figure 4. The Left Twist (LT).

Exercise 2.13. Find a space $\mathcal{K}_n^2$ and maps $\delta : \mathcal{K}_n^2 \to \mathcal{K}_{n-1}^2$ and $b : \mathcal{K}_n^2 \to \mathcal{K}_n^1$ that fit into a commutative diagram,

$$\begin{array}{ccccccc} \mathcal{K}_n^2 & \xrightarrow{\delta} & \mathcal{K}_{n-1}^2 & \xrightarrow{\pi} & \bar{\mathcal{K}}_{n-1}^2 & \longrightarrow & 0 \\ \downarrow b & & \downarrow b & & \downarrow \bar{b} & & \\ \mathcal{K}_n^1 & \xrightarrow{\delta} & \mathcal{K}_{n-1}^1 & \xrightarrow{\pi} & \bar{\mathcal{K}}_{n-1}^1 & \longrightarrow & 0 \end{array} \qquad \text{(exact rows)},$$

and hence show that the relations in $\ker \bar{b}$ all lift to $\ker b$.

Question 1. Is the sequence $\bar{\mathcal{K}}_{n-1}^2 \xrightarrow{\bar{b}} \bar{\mathcal{K}}_{n-1}^1 \xrightarrow{\bar{b}} \bar{\mathcal{K}}_{n-1}$ related to Kontsevich's graph cohomology [Ko2]?

3. Preliminaries

sec:prels

3.1. Surgery and the Kirby calculus.

subsec:surgery

3.2. The Borromean rings.

3.3. The triple linking numbers μ_{ijk}.

subsec:mu

4. Constancy conditions or $\mathcal{M}_n/\delta\mathcal{M}_{n+1}$

sec:const

4.1. Statement of the result.

Definition 4.1. Let $\mathcal{Y}_n$ be the unital commutative algebra over $\mathbb{Z}$ generated by symbols Y_{ijk} for distinct indices $1 \leq i, j, k \leq n$, modulo the anti-cyclicity relations $Y_{ijk} = Y_{jik}^{-1} = Y_{jki}$.

Warning 4.2. Below we will mostly regard $\mathcal{Y}_n$ as an $\mathbb{Z}$-module, and not as an algebra. Thus we will only use the product of $\mathcal{Y}_n$ as a convenient way of writing certain elements and linear combinations of elements. The subspaces of $\mathcal{Y}_n$ that we will consider will be subspaces in the linear sense, but not ideals or subalgebras, and similarly for quotients and maps from or to $\mathcal{Y}_n$.

It is easy to define a map $\mu : \mathcal{M}_n/\delta\mathcal{M}_{n+1} \to \mathcal{Y}_n$. For an n-link L set

$$\mu(L) = \prod_{1 \leq i < j < k \leq n} Y_{ijk}^{\mu_{ijk}(L)}.$$

It follows from Section 3.3 that this definition descends to the quotient of $\mathcal{M}_n$ by the co-derivatives of $(n+1)$-links.

MaskDefinition

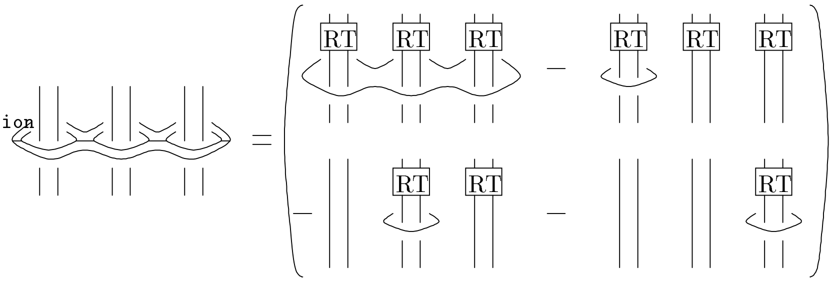


Figure 5. A 3-mask.

deltaMask

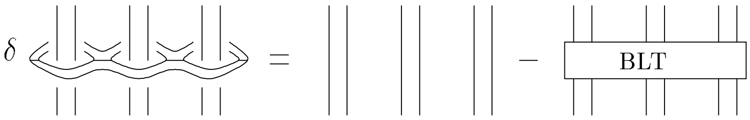


Figure 6. The co-derivative of a 3-mask.

BLTDefinition

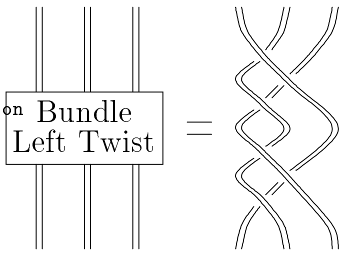


Figure 7. The Bundle Left Twist (BLT) is the same as the Left Twist, only that the strands within each "bundle" are not twisted internally.

Theorem 4. *The thus defined map* $\mu : \mathcal{M}_n/\delta\mathcal{M}_{n+1} \to \mathcal{Y}_n$ *is an isomorphism.*

4.2. **On a connected space, polynomials are determined by their values at any given point.**

4.3. **Homotopy invariance and pure braids.**

4.4. **The mask and the interchange move.**

4.5. **Reducing third commutators.**

5. Integrability conditions or $\ker\delta$

sec:integ

5.1. **$+1$ and -1 surgeries are opposites.**

5.2. **A total twist is a composition of many little ones.**

5.3. **The two ways of building an interchange.**

5.4. **Lassoing a Borromean link and the IHX relation.**

UndoingBLT

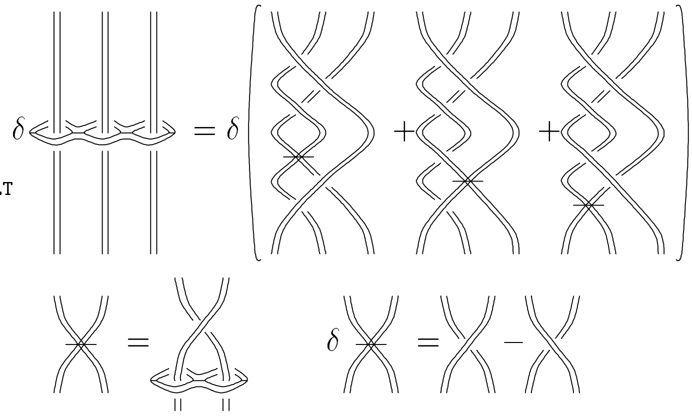


Figure 8. Undoing a Bundle Left Twist one crossing at a time.

TotalTwist

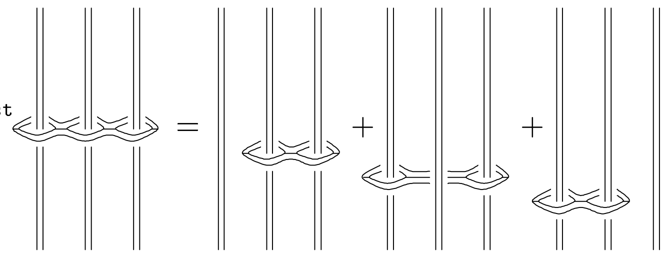


Figure 9. The Total Twist Relation (TTR).

TTR

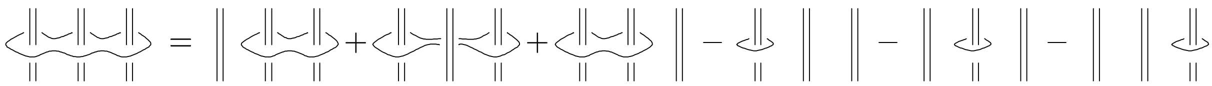


Figure 10. The Total Twist Relation (TTR).

Monster

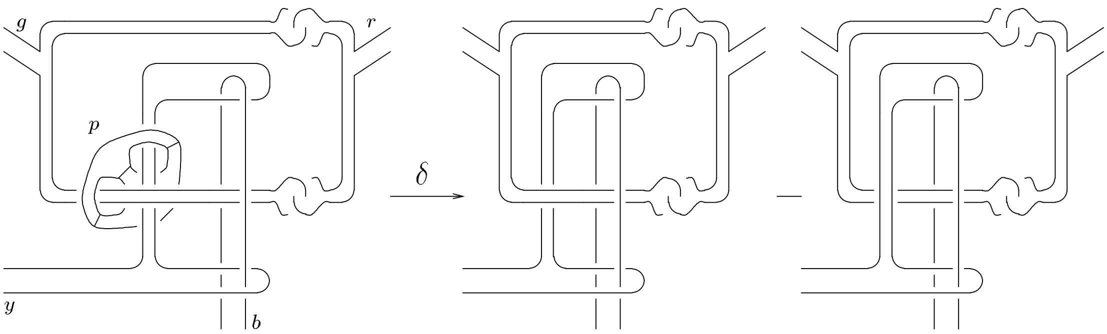


Figure 11. The Monster

fig:Monster

$$\tilde{Y}_{rab}Y_{rgb}(Y_{rgp}Y_{pyb} - Y_{rgp} - Y_{pyb}) = \tilde{Y}_{rab}Y_{rgb}(Y_{rgp}\tilde{Y}_{pyb} - Y_{pyb})$$

$$= Y_{rab}Y_{rgb}Y_{rgp}\tilde{Y}_{pyb} - Y_{rab}Y_{rgb}Y_{pyb} - Y_{rgb}Y_{rgp}\tilde{Y}_{pyb} + Y_{rgb}Y_{pyb}$$

LassoingBorromean

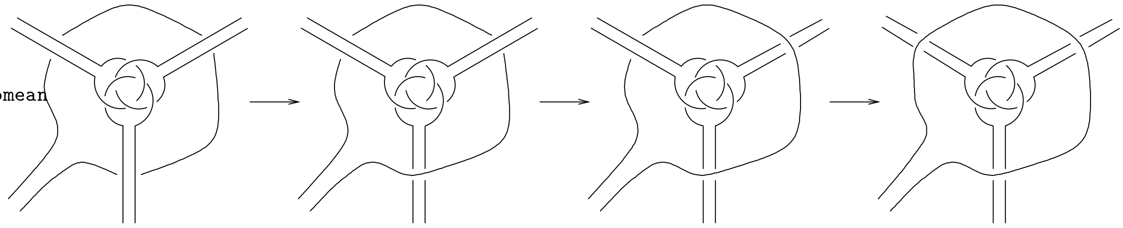


Figure 12. Lassoing a Borromean link.

$$= Y_{rab}Y_{rgb}Y_{rgp}\tilde{Y}_{pyb} - Y_{rab}Y_{rgb}\tilde{Y}_{pyb} - Y_{rgb}Y_{rgp}\tilde{Y}_{pyb} + Y_{rgb}\tilde{Y}_{pyb}$$

(The last equality holds because in the two error terms, $Y_{rab}Y_{rgb}$ and Y_{rgb}, the component p is unknotted). Now reduce the component r using the total twist relation. Only the first term is affected, and 3 of the 6 terms that are produced from its reduction cancel against the 3 remaining terms of the above equation. The result is:

$$= (Y_{rab}Y_{rgp} - Y_{rab} - Y_{rgp})\tilde{Y}_{pyb} = \tilde{Y}_{rab}\tilde{Y}_{rgp}\tilde{Y}_{pyb} - \tilde{Y}_{pyb}.$$

The last term here drops out because in it the component r is unknotted, and so the end result is $\tilde{Y}_{rab}\tilde{Y}_{rgp}\tilde{Y}_{pyb}$. In graphical terms, this is precisely the graph I! Cyclically permuting the roles of r, g, and b, we find that we have proven the IHX relation.

6. Towards a Hutchings' theory of integration in the case of integral homology spheres

sec:Hutchings

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