Etingof-Kazhdan QLB-II

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"The art of writing just enough so that anyone who'd understand you would know you've done it, yet so as to make it maximally difficult to actually understand you, perfected."

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QUANTIZATION OF LIE BIALGEBRAS, II

Pavel Etingof and David Kazhdan

Department of Mathematics
Harvard University
Cambridge, MA 02138, USA
e-mail: etingof@math.harvard.edu
kazhdan@math.harvard.edu

Part III
Classification of quantized universal enveloping algebras

Abstract

This paper is a continuation of [EK]. We show that the quantization procedure of [EK] is given by universal acyclic formulas and defines a functor from the category of Lie bialgebras to the category of quantized universal enveloping algebras. We also show that this functor defines an equivalence between the category of Lie bialgebras over $k[[h]]$ and the category quantized universal enveloping (QUE) algebras.

Acknowledgements

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1. Universality and functoriality of the quantization of Lie bialgebras.

In this paper we will use the notation of [EK].

1.1. *Linear algebraic structures.*

Here we introduce the notion of a linear algebraic structure which is borrowed from [La]. [LA]:= Lawvere, 1963

Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal $\mathbb{Q}$-linear category whose objects are non-negative integers, such that $[n] = [1]^{\otimes n}$ (the unit object is $[0]$). We will call such a category a cyclic category (by analogy with a cyclic group).

Let $\mathcal{C}$ be a cyclic category. Let $S = \cup_{m,n\ge 0} S_{mn}$ be a set of morphisms of $\mathcal{C}$, $S_{mn} \subset \mathrm{Hom}_{\mathcal{C}}([m],[n])$, $m, n \in \mathbb{Z}_+$. We say that $\mathcal{C}$ is generated by S if any morphism of $\mathcal{C}$ can be obtained from the morphisms in S_{mn} and permutation morphisms in $\mathrm{Hom}_{\mathcal{C}}(m,m)$ by iterating three elementary operations:

1. Is there a "multi-coloured" version of this?
2. Are there "ideals" here?

1) composition of morphisms;
2) tensor product of morphisms;
3) linear combinations of morphisms over $\mathbb{Q}$.

Let $Z = \cup Z_{mn}$, $m, n \in \mathbb{Z}_+$, be a bigraded set.

Definition. *The free cyclic category $\mathcal{F}_Z$ is a cyclic category equipped with a collection of maps $\mu_{mn} : Z_{mn} \to Hom_{\mathcal{F}_Z}(m, n)$ such that*

(i) $\mathcal{F}_Z$ is generated by $\cup_{m,n\geq 0}\mu_{mn}(Z_{mn})$, and

(ii) for any cyclic category $\mathcal{C}$ equipped with a collection of maps $\phi_{mn} : Z_{mn} \to Hom_{\mathcal{C}}([m],[n])$ there exists a unique symmetric tensor functor (ACU functor, see [DM]*) $F : \mathcal{F}_Z \to \mathcal{C}$ such that $F([1]_{\mathcal{F}_Z}) = [1]_{\mathcal{C}}$, and $F(\mu_{mn}(z)) = \phi_{mn}(z)$, $z \in Z_{mn}$.*

[DM]:= Deligne-Milne "Tannakian Categories..."

The pair $(\mathcal{F}_Z, \{\mu_{mn}\})$ with such properties exists and is unique up to symmetric tensor equivalence.

Remark. Roughly, the free cyclic category is a cyclic category generated by a set of morphisms without any nontrivial relations.

Let $(\mathcal{C}, \otimes)$ be a $\mathbb{Q}$-linear monoidal category, and $\mathcal{I} = \{\mathcal{I}_{X,Y} \subset \mathrm{Hom}_{\mathcal{C}}(X, Y), X, Y \in \mathcal{C}\}$ be a collection of subspaces. We say that a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is $\mathcal{I}$-negligible if $\phi \in \mathcal{I}_{X,Y}$. We say that $\mathcal{I}$ is a tensor ideal in $\mathcal{C}$ if the composition in any order of any morphism with an $\mathcal{I}$-negligible morphism is $\mathcal{I}$-negligible, and the tensor product of any morphism with an $\mathcal{I}$-negligible morphism is $\mathcal{I}$-negligible.

These are ideals in the structure on an alg. structure, not in an algebra over it.

Given a tensor ideal $\mathcal{I}$ in $\mathcal{C}$. Define the quotient category $\mathcal{D} = \mathcal{C}/\mathcal{I}$ as follows. The objects of $\mathcal{D}$ are the same as those of $\mathcal{C}$, and $\mathrm{Hom}_{\mathcal{D}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathcal{I}_{X,Y}$. It is easy to show that $\mathcal{D}$ carries a natural structure of a $\mathbb{Q}$-linear monoidal category. We will say that an identity between morphisms holds in $\mathcal{C}$ modulo $\mathcal{I}$ if it holds in $\mathcal{D}$.

Let $J = \{J_{X,Y} \subset \mathrm{Hom}_{\mathcal{C}}(X, Y), X, Y \in \mathcal{C}\}$, be a collection of subsets. We denote by $\langle J\rangle$ the smallest tensor ideal in $\mathcal{C}$ such that $J_{X,Y} \subset \langle J\rangle_{X,Y}$ for any objects X, Y, and say that $\langle J\rangle$ is the tensor ideal generated by J.

Proposition 1.1. *Let $\mathcal{C}$ be any cyclic category generated by a set S of morphisms. Then $\mathcal{C}$ has the form $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is a tensor ideal in $\mathcal{F}_S$.*

Proof. Easy.□

Let $\mathcal{C}$ be a cyclic category, $\mathcal{I}$ be a tensor ideal in $\mathcal{C}$. For any positive integer n, define the power $\mathcal{I}^n$ to be the tensor ideal in $\mathcal{C}$ generated by the elements $\phi_1 \circ ... \circ \phi_n$, where $\phi_1, ..., \phi_n \in \mathcal{I}$.

Now we define the notion of completion of a cyclic category with respect to a tensor ideal. Let $\mathcal{C}$ be a cyclic category, $\mathcal{I}$ be a tensor ideal in $\mathcal{C}$. Then the spaces of morphisms of the cyclic categories $\mathcal{C}/\mathcal{I}^n$ form a projective system. Let $\mathcal{C}_{\mathcal{I}}$ be the cyclic category whose objects are the same as those of $\mathcal{C}$, and $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) = \varprojlim \mathrm{Hom}_{\mathcal{C}/\mathcal{I}^n}(X, Y)$, $X, Y \in \mathcal{C}$. This category is called the completion of $\mathcal{C}$ with respect to $\mathcal{I}$.

Define a topological cyclic category to be a cyclic category in which the sets of morpisms are topological vector spaces over $\mathbb{Q}$, and composition of morphisms is continuous. The category $\mathcal{C}_{\mathcal{I}}$ has a natural structure of a topological cyclic category, where the topology is defined by the ideal $\mathcal{I}$.

Throughout this Chapter, let $\mathcal{N}$ be a symmetric monoidal $\mathbb{Q}$-linear category, and X be an object in $\mathcal{N}$.

Let $\mathcal{C}$ be a cyclic category.

Definition. *A linear algebraic structure of type $\mathcal{C}$ on X is a symmetric tensor functor $G : \mathcal{C} \to \mathcal{N}$ such that $G([1]) = X$.*

Thus, a linear algebraic structure of type $\mathcal{C}$ on X is a collection of morphisms between tensor powers of X which satisfy certain consistency relations. If an object

X equipped with such morphisms has been fixed, the functor G corresponding to it is denoted by G_X.

The same definition applies to the case when $\mathcal{C}$ and $\mathcal{N}$ are topological categories.

Denote by $\mathcal{G}(\mathcal{C}, \mathcal{N})$ the category of symmetric tensor fucntors from $\mathcal{C}$ to $\mathcal{N}$, i.e. the category of linear algebraic structures of type $\mathcal{C}$ on objects of $\mathcal{N}$.

Let $\mathcal{C}$ be a (topological) cyclic category. We say that $\mathcal{C}$ is nondegenerate if the natural map $\mathbb{Q}[S_n] \to \mathrm{Hom}([n],[n])$ is injective. From now on, we consider only nondegenerate categories.

Let $\hat{\mathcal{C}}$ be the category obtained by formal addition to $\mathcal{C}$ of the kernels of all the idempotents $P \in \mathbb{Q}[S_n]$ acting on the objects $[n]$, $n \geq 1$. The objects of $\hat{\mathcal{C}}$ are pairs $([n], P)$, where $n \geq 0$, and $P \in \mathbb{Q}[S_n] \to \mathrm{Hom}_{\mathcal{C}}([n],[n])$ is an idempotent, where morphisms are defined by $\mathrm{Hom}_{\hat{\mathcal{C}}}(([n], P), ([m], Q)) = \{f \in \mathrm{Hom}_{\mathcal{C}}([n],[m]) : f \circ P = Q \circ f = f\}$. Let $\bar{\mathcal{C}}$ be the closure of $\hat{\mathcal{C}}$ under inductive limits. In this category, every object is isomorphic to a direct sum of indecomposable ones, and indecomposable objects correspond to irreducible representations of S_n, $n \geq 1$. In particular, we have objects $S^n[1] = ([n], \mathrm{Sym}_n)$, where Sym_n is the symmetrizer in $\mathbb{Q}[S_n]$.

Let $\mathcal{N}$ be closed under inductive limits. Then any linear algebraic structure G of type $\mathcal{C}$ on X extends to an additive symmetric tensor functor $G : \bar{\mathcal{C}} \to \mathcal{N}$.

1.2. *Examples of linear algebraic structures.*

Some common examples of linear algebraic structures are:

A. Associative algebras with unit. In this case the set S consists of an element of bidegree (2,1) ("the universal product"), and an element of bidegree (0,1) ("the unit"). The category $\mathcal{C} = \mathrm{AA}$ is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by the the associativity identity for the product and the unit axiom. An associative algebra in $\mathcal{N}$ is an object with a linear algebraic structure of type AA.

B. Lie algebras. In this case the set S consists of one element of bidegree (2,1) ("the universal commutator"), and the category $\mathcal{C} = \mathrm{LA}$ is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by two relations – skew-symmetry and the Jacobi identity for the commutator. A Lie algebra in $\mathcal{N}$ is an object with a linear algebraic structure of type LA.

We will deal with the following examples of linear algebraic structures.

1. Lie bialgebras. In this case the set S consists of two elements of bidegrees (2,1) and (1,2) ("the universal commutator and cocommutator"), and the category $\mathcal{C} = \mathrm{LBA}$ is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by five relations – skew-symmetry and the Jacobi identity for the commutator and cocommutator and the condition that cocommutator is a 1-cocycle. A Lie bialgebra in $\mathcal{N}$ is an object with a linear algebraic structure of type LBA.

2. Quasitriangular Lie bialgebras. In this case the set S consists of two elements of bidegrees (2,1) and (0,2) ("the universal commutator and classical r-matrix"), and the category $\mathcal{C} = \mathrm{QTLBA}$ is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by four relations – skew-symmetry and the Jacobi identity for the commutator, invariance of $r + r^{op}$, and the classical Yang-Baxter equation. A quasitriangular Lie bialgebra in $\mathcal{N}$ is an object with a linear algebraic structure of type QTLBA.

3. Hopf algebras. In this case the set S consists of six elements of bidegrees (2,1),(1,2),(0,1),(1,0),(1,1),(1,1) ("the universal product, coproduct, unit, counit, antipode,inverse antipode"), and the category $\mathcal{C} = \mathrm{HA}$ is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by the relations coming from the axioms of a Hopf algebra. A Hopf algebra in $\mathcal{N}$ is an object with a linear algebraic structure of type HA.

4. Quasitriangular Hopf algebras. In this case the set S consists of eight elements

of bidegrees (2,1),(1,2),(0,1),(1,0),(1,1),(1,1),(0,2),(0,2) ("the universal product, coproduct, unit, counit, antipode, inverse antipode, R-matrix, inverse R-matrix"), and the category $\mathcal{C} =$ QTHA is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by the relations coming from the axioms of a quaitriangular Hopf algebra. A quasitriangular Hopf algebra in $\mathcal{N}$ is an object with a linear algebraic structure of type QTHA.

5. Classical Yang-Baxter algebras. In this case the set S consists of three elements of bidegrees (2,1),(0,1),(0,2) ("the universal product, unit, and r-matrix"), and the category $\mathcal{C} =$ CYBA is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by the associativity relation and the classical Yang-Baxter equation. A classical Yang-Baxter algebra in $\mathcal{N}$ is an object with a linear algebraic structure of type CYBA.

6. Quantum Yang-Baxter algebras. In this case the set S consists of three elements of bidegrees (2,1),(0,1),(0,2) ("the universal product, unit, and R-matrix"), and the category $\mathcal{C} =$ QYBA is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by the associativity relation and the quantum Yang-Baxter equation. A quantum Yang-Baxter algebra in $\mathcal{N}$ is an object with a linear algebraic structure of type QYBA.

7. Co-Poisson Hopf algebras [Dr1]. In this case the set S consists of six elements of bidegrees (2,1),(1,2),(0,1),(1,0),(1,1),(1,2) ("the universal product, coproduct, unit, counit, antipode, Poisson cobracket"), and the category $\mathcal{C} =$ CPHA is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by the relations coming from the axioms of a co-Poisson Hopf algebra. A co-Poisson Hopf algebra in $\mathcal{N}$ is an object with a linear algebraic structure of type CPHA.

8. Quasitriangular co-Poisson Hopf algebras. In this case the set S consists of six elements of bidegrees (2,1),(1,2),(0,1),(1,0),(1,1),(0,2) ("the universal product, coproduct, unit, counit, antipode, r-matrix"), and the category $\mathcal{C} =$ QTCPHA is $\mathcal{F}_S/\mathcal{I}$, where $\mathcal{I}$ is generated by the relations coming from the axioms of a quaitriangular co-Poisson Hopf algebra. A quasitriangular co-Poisson Hopf algebra in $\mathcal{N}$ is an object with a linear algebraic structure of type QTCPHA.

1.3. *The functor of universal quantization.*

Let $\mathcal{C}_1$, $\mathcal{C}_2$ be cyclic categories.

Definition. *A universal construction is a symmetric tensor functor* $Q : \mathcal{C}_2 \to \bar{\mathcal{C}}_1$.

The functor Q defines a functor $\widehat{Q} : \mathcal{G}(\mathcal{C}_1, \mathcal{N}) \to \mathcal{G}(\mathcal{C}_2, \bar{\mathcal{N}})$.

Examples. 1. Let $\mathcal{C}_1 = AA$, $\mathcal{C}_2 = LA$, and $Q : LA \to AA$, such that $Q([1]) = [1]$, $Q([,]) = * - *^{op}$, where $*$ is the product and $*^{op}$ is the opposite product. This functor gives rise to a functor $\widehat{Q}$ from the category of associative algebras in $\mathcal{N}$ to the category of Lie algebras in $\mathcal{N}$, which assigns to every associative algebra itself regarded as a Lie algebra.

2. Let $\mathcal{C}_1 = LA$, $\mathcal{C}_2 = AA$, and $Q : AA \to LA$ be the functor such that $\widehat{Q}$ assigns to every Lie algebra $\mathfrak{g}$ in $\mathcal{N}$ its universal enveloping algebra $U(\mathfrak{g})$ in $\bar{\mathcal{N}}$. The functor Q satisfies $Q([1]) = S[1]$, and $Q(*)$ is described in terms of the commutator using the standard method of computing the product of monomials. For example, $Q(*)|_{[1]\otimes[1]} = \mathrm{Sym}_2 + [,]/2$, where $[1] \otimes [1]$ is the subobject in $S[1] \otimes S[1]$ obtained by tensoring two subobjects $[1] \subset S[1]$.

S, presumably is the sum of all symmetric powers.

This is the S(g) view of U(g). I wonder if the setup can be amended to directly include the U(g) view of U(g).

3. The functor Q of taking the universal enveloping algebra (example 2) extends to a functor $Q : HA \to LA$, since the universal enveloping algebra of a Lie algebra carries a natural structure of a Hopf algebra. Moreover, it extends to a functor $Q : CPHA \to LBA$, so that the corresponding functor $\widehat{Q}$ assigns to any Lie bialgebra its universal enveloping algebra regarded as a co-Poisson Hopf algebra.

May 23, 2011, 11:30PM

The main result of Chapter 1 is the following theorem, which establishes the universality of the quantization of Lie bialgebras, r-matrices, quasitriangular and triangular Lie bialgebras, obtained in the previous chapters.

Let $(\mathfrak{a}, [,], \delta)$ be a Lie bialgebra over k. Define a Lie bialgebra $\mathfrak{a}_h$ over $k[[h]]$ to be $(\mathfrak{a}[[h]], [,], h\delta)$. Similarly, if $(\mathfrak{a}, [,], r)$ is a quasitriangular Lie bialgebra over k, we define $\mathfrak{a}_h = (\mathfrak{a}[[h]], [,], hr)$, and if $(A, *, r)$ is a classical Yang-Baxter algebra, we define $A_h = (A[[h]], *, hr)$.

Read as a further grading by cob-vertices?

Theorem 1.2.

See Subpage!

There exist "universal quantization functors"

(i) $Q : HA_{\langle \Delta - \Delta^{op}, S - S^{-1}\rangle} \to \overline{LBA_{\langle\delta\rangle}}$ *such that for any Lie bialgebra* $\mathfrak{a}$ *over* k $\widehat{Q}(\mathfrak{a}_h) = U_h(\mathfrak{a})$*;*

(ii) $Q^{qt} : QTHA_{\langle \Delta - \Delta^{op}, R-1, S - S^{-1}\rangle} \to \overline{QTLBA_{\langle r\rangle}}$ *such that for any quasitriangular Lie bialgebra* $\mathfrak{a}$ *over* k $\widehat{Q^{qt}}(\mathfrak{a}_h) = U_h^{qt}(\mathfrak{a})$*, where* $U_h^{qt}(\mathfrak{a})$ *is the quasitriangular quantization defined in Section 6.1;*

(iii) $Q^{YB} : QYBA_{\langle R-1\rangle} \to CYBA_{\langle r\rangle}$ *such that for any classical Yang-Baxter algebra* (A, r) *over* k *one has* $\widehat{Q^{YB}}(A_h) = (A, R)$*, where* R *is constructed from* r *as explained in Chapter 5.*

If you have three theorems it means you didn't bother to unify them.

What do the subscripts mean?

Forgotten overline?

In the language of Drinfeld [Dr3], Theorem 1.2 implies that the multiplication and comultiplication in $U_h(\mathfrak{a})$ are expressed via the commutator and cocommutator in $\mathfrak{a}$ in terms of acyclic tensor calculus, and the Hopf algebra relations in $U_h(\mathfrak{a})$ can be formally deduced from the axioms of a Lie bialgebra. Thus it answers positively Question 1.2 in [Dr3] (the existence of universal quantization of Lie bialgebras), and hence question 2.1 (the existence of a quantum Campbell-Hausdorff series). Together with the material of Chapters 5,6, it also answers positively the "universality" questions in Sections 3,4 of [Dr3].

The functoriality of quantization implies the existence of quantization functors over local Artinian or pro-Artinian algebras. Namely, let K be a commutative local Artinian or pro-Artinian $\mathbb{Q}$-algebra, I be the maximal ideal in K, $k = K/I$ be the residue field (main example: $K = k[[h]]$). Let $\mathcal{A}_K$ be the category of topologically free K-modules, i.e. modules of the form $\varprojlim V \otimes K/I^n$, where V is a free abelian group.

Let:

$LBA_0(K)$ be the category of Lie bialgebras in $\mathcal{A}_K$ with $\delta = 0$ mod I;

$HA_0(K)$ be the category of QUE algebras, i.e. Hopf algebras in $\mathcal{A}_K$ which are cocommutative mod I;

$QTLBA_0(K)$ the category of quasitriangular Lie bialgebras in $\mathcal{A}_K$ with $r = 0$ mod I;

$QTHA_0(K)$ the category of quasitriangular QUE algebras, i.e. quasitriangular Hopf algebras in $\mathcal{A}_K$ which are cocommutative and have $R = 1$ mod I;

$CYBA_0(K)$ be the category of classical Yang-Baxter algebras in $\mathcal{A}_K$ with $r = 0$ mod I;

$QYBA_0(K)$ be the category of quantum Yang-Baxter algebras in $\mathcal{A}_K$ with $R = 1$ mod I.

Then:

(i) The functor Q defines a functor $\widehat{Q}$ from $LBA_0(K)$ to $HA_0(K)$;

(ii) The functor Q^{qt} defines a functor $\widehat{Q^{qt}}$ from $QTLBA_0(K)$ to $QTHA_0(K)$;

(ii) The functor Q^{YB} defines a functor $\widehat{Q^{YB}}$ from $CYBA_0(K)$ to $QYBA_0(K)$.

Remark. Quantization of triangular Lie bialgebras is also universal and functorial. Formulations and proofs of the results are the same as in the quasitriangular case.

It is important to remember that the functors $Q, Q^{qt}, Q^{YB}, \widehat{Q}, \widehat{Q^{qt}}, \widehat{Q^{YB}}$ depend on the choice of the Lie associator Φ, which was fixed in Section 1.3. We will later show that different choices of Φ give isomorphic functors, but this is a non-trivial fact. Therefore, when we want to emphasize the dependence of these functors on Φ, we will write them as $Q_\Phi, Q_\Phi^{qt}, Q_\Phi^{YB}, \widehat{Q_\Phi}, \widehat{Q_\Phi^{qt}}, \widehat{Q_\Phi^{YB}}$.

The proof of Theorem 1.2 is given in the next section. The idea of the proof is to show that that the construction of quantization described in [EK] actually defines some universal formulas which can be used to construct the quantization in a general tensor category.

In section 1.5, we give an application of functoriality.

1.4. *Proof of Theorem 1.2*

Part (i).

Let $\mathfrak{a}_+$ be the canonical Lie bialgebra [1] in the tensor category $\mathcal{C} = \overline{LBA_{\langle\delta\rangle}}$, with commutator μ and cocommutator δ. Let $U(\mathfrak{a}_+) := S\mathfrak{a}_+ \in \mathcal{C}$ be the universal enveloping algebra of $\mathfrak{a}_+$. Our goal is to introduce a Hopf algebra structure on $U(\mathfrak{a}_+)$, which coincides with the standard one modulo $\langle\delta\rangle$, and yields the Lie bialgebra structure on $\mathfrak{a}_+$ when considered modulo $\langle\delta\rangle^2$. This Hopf algebra will be exactly $Q([1])$, where [1] is the generating object of the category $HA_{\langle\Delta-\Delta^{op}, S-S^{-1}\rangle}$.

We will perform virtually the same constructions as in Part II of [EK], remembering, though, that now $\mathfrak{a}_+$ is not a vector space and even not a set but an object in the category.

First, we define the notion of a module, comodule, and dimodule over a Lie algebra, coalgebra, and bialgebra, respectively, in a general tensor category. The notion of a dimodule over a Lie bialgebra is equivalent, for a finite dimensional Lie bialgebra over a field, to the notion of a module over its double.

I'm not sure about that.

Definition.

(i) Let $\mathfrak{a}_+$ be a Lie algebra in a tensor category $\mathcal{N}$ with commutator μ. An object $X \in \mathcal{N}$ is said to be equipped with the structure of a left $\mathfrak{a}_+$-module if it is endowed with a morphism $\pi : \mathfrak{a}_+ \otimes X \to X$ (the action of $\mathfrak{a}_+$ on X), such that $\pi \circ (1 \otimes \pi) = \pi \circ (\mu \otimes 1)$ on $\Lambda^2\mathfrak{a}_+ \otimes X$.

(ii) Let $\mathfrak{a}_+$ be a Lie coalgebra in a tensor category $\mathcal{N}$ with cocommutator δ. An object $X \in \mathcal{N}$ is said to be equipped with the structure of a right $\mathfrak{a}_+$-comodule if it is endowed with a morphism $\pi^* : X \to \mathfrak{a}_+ \otimes X$ (the coaction of $\mathfrak{a}_+$ on X), such that $Alt_{21} \circ (1 \otimes \pi^*) \circ \pi^* = (\delta \otimes 1) \circ \pi^*$, where Alt_{21} is the alternator of the first and second components ($Alt(a \otimes b) := b \otimes a - a \otimes b$).

might they mean $\pi^*: X^* \to \mathfrak{a}_+ \otimes X^*$

(iii) Let $\mathfrak{a}_+$ be a Lie bialgebra in a tensor category $\mathcal{N}$. An object $X \in \mathcal{N}$ is said to be equipped with the structure of a $\mathfrak{a}_+$-dimodule if it is endowed with two morphisms $\pi : \mathfrak{a}_+ \otimes X \to X$, $\pi^* : X \to \mathfrak{a}_+ \otimes X$, such that π is a left action of $\mathfrak{a}_+$ on X as a Lie algebra, π^* is a right coaction of $\mathfrak{a}_+$ on X as a Lie coalgebra, and they agree according to the formula

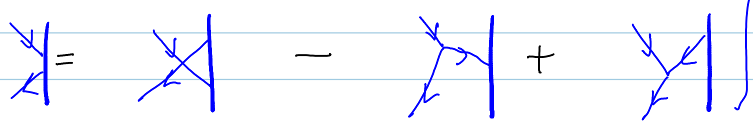
Later examples: $M_- \cong U(\mathfrak{a}_+)$, M_+^*

why this left-right distinction?

$$\pi^* \circ \pi = (\pi \otimes 1) \circ \sigma_{12} \circ (1 \otimes \pi^*) - (1 \otimes \pi) \circ (\delta \otimes 1) + (\mu \otimes 1) \circ (1 \otimes \pi^*), \tag{1.1}$$

both: $\mathfrak{a}_+ \otimes X \to \mathfrak{a}_+ \otimes X$
A variant of STU!

where σ_{ij} denotes the permutation of the components i, j.



Let X be any $\mathfrak{a}_+$-module (comodule, dimodule). Define X_0 to be the object X with the zero structure of a module (comodule, dimodule).

There is an obvious notion of tensor product of modules and comodules. The tensor product of dimodules is just the tensor product of the underlying modules and comodules. Thus, modules, comodules, and dimodules over $\mathfrak{a}_+$ in $\mathcal{N}$ form a tensor category. We denote the first category by $\mathcal{M}_{\mathfrak{a}_+}$, the second by $\mathcal{M}^{\mathfrak{a}_+}$, and the third by $\mathcal{M}_{\mathfrak{a}_+}^{\mathfrak{a}_+}$.

Now we define the Verma dimodules M_-, M_+^* over $\mathfrak{a}_+$. As objects of $\mathcal{C}$, $M_- = S\mathfrak{a}_+$, $M_+^* = \hat{S}\mathfrak{a}_+$. Let $1_- : \mathbf{1} \to M_-$, $1_+^* : \mathbf{1} \to M_+^*$ be embeddings of the identity subobjects into M_-, M_+^*, and $1_-^* : M_- \to \mathbf{1}$, $1_+ : M_+^* \to \mathbf{1}$ be the corresponding projections.

Verma modules are named after what they kill.

Funny naming for the $1_\pm, 1_\pm^*$.

The action of $\mathfrak{a}_+$ in M_- is the same as the standard left action of $\mathfrak{a}_+$ in $U(\mathfrak{a}_+)$. The coaction of $\mathfrak{a}_+$ in M_- is then completely determined (via formula (1.1)) by the coaction on the identity component $\mathbf{1}$ of M_-, which we define to be zero.

The formulas for the action (respectively, coaction) of $\mathfrak{a}_+$ in M_+^* are obtained from the formulas for the coaction (respectively, action) in M_- by interchanging μ and δ, reversing the order of compositions, and changing the signs.

Let M_1, M_2 be $\mathfrak{a}_+$-dimodules. Define the classical r-matrix and the Casimir operator in $\mathrm{End}_{\mathcal{C}}(M_1 \otimes M_2)$ by

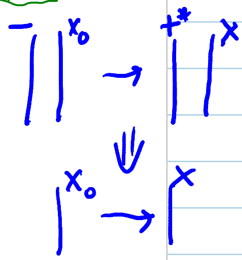
$$r = (\pi \otimes 1) \circ \sigma_{12} \circ (1 \otimes \pi^*), \Omega = r + r^{op}. \tag{1.2}$$

This implies that if Φ is any Lie associator, then for any three $\mathfrak{a}_+$-dimodules M_1, M_2, M_3 we can define an invertible element of $\mathrm{End}_{\mathcal{C}}(M_1 \otimes M_2 \otimes M_3)[[h]]$, which represents the action of Φ in $M_1 \otimes M_2 \otimes M_3$.

Now we describe an intertwiner ψ, which is used in the construction of quantization.

June 4, 2011

For any object $X \in \mathcal{M}_{\mathfrak{a}_+}^{\mathfrak{a}_+}$, define the map $\theta : \mathrm{Hom}_{\mathcal{M}_{\mathfrak{a}_+}^{\mathfrak{a}_+}}(M_- \otimes X_0, M_+^* \otimes X) \to \mathrm{Hom}_{\mathcal{C}}(X_0, X)$ by $\theta(f) = (1_+ \otimes 1_X) \circ f \circ (1_- \otimes 1_{X_0})$.



Lemma 1.3. *The map θ is an isomorphism.*

Proof. By Frobenius reciprocity, for any two dimodules X, Y over $\mathfrak{a}_+$, $\mathrm{Hom}_{\mathcal{M}_{\mathfrak{a}_+}^{\mathfrak{a}_+}}(M_- \otimes X, M_+^* \otimes Y) = \mathrm{Hom}_{\mathcal{M}^{\mathfrak{a}_+}}(X, M_+^* \otimes Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$. This implies the Lemma.

Now set $X = M_-$. Denote by ψ the morphism $\theta^{-1}(1_X)$. Let $\eta = \psi \circ (1_- \otimes 1_{X_0}) : X_0 \to M_+^* \otimes X$.

Consider the category $\mathcal{C}_h$, with the same objects as in $\mathcal{C}$, and morphisms defined by $\mathrm{Hom}_{\mathcal{C}_h}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)[[h]]$. We define the Hopf algebra $U_h(\mathfrak{a}_+)$ in $\mathcal{C}_h$ as follows. By the definition, this is the object M_-, with the product $m : M_- \otimes M_- \to M_-$ and coproduct $\Delta : M_- \to M_- \otimes M_-$ defined as follows.

The product is

$$m^{op} = (1_+ \otimes 1_+ \otimes 1) \circ (1 \otimes \psi) \circ \Phi \circ (\eta \otimes 1). \tag{1.3}$$

Is this the first Φ in this paper? Is it evaluated at multiples of h?

The coproduct is

$$\Delta = J^{-1} \circ \Delta_0, \tag{1.4}$$

where Δ_0 is the standard coproduct in M_-, and $J : M_- \otimes M_- \to M_- \otimes M_-$ is given by

$$J = (1_+ \otimes 1 \otimes 1_+ \otimes 1) \circ \Phi_{1,3,24}^{-1} \circ \Phi_{3,2,4} \circ e^{-h\Omega_{23}/2} \circ \Phi_{2,3,4}^{-1} \circ \Phi_{1,2,34} \circ (\eta \otimes \eta). \tag{1.5}$$

(cf. formula (3.1)).

Proposition 1.4. *Formulas (1.3)-(1.5) define a Hopf algebra structure on M_-.*

Proof. The axioms of a Hopf algebra are demonstrated similarly to the case of Lie bialgebras in the category of vector spaces, which was done in [EK]. For example, associativity of multiplication is shown by a computation in Chapter 9; associativity of comultiplication is shown by the computation given in the proof of Proposition 8.2.

Define the functor $P_h : \mathcal{C} \to \mathcal{C}_h$ between topological categories, which maps objects to themselves, and $P_h(\mu) = \mu, P_h(\delta) = h\delta$. The image of this functor is the closed subcategory $\mathcal{C}'_h \subset \mathcal{C}_h$, generated over $\mathbb{Q}$ by the morphisms $\mu, h\delta$, and the functor $P_h : \mathcal{C} \to \mathcal{C}'_h$ is an equivalence.

Proposition 1.5. *The morphisms m, Δ defined by (1.3)-(1.5) belong to $\mathcal{C}'_h$.*

Proof. It is clear that a morphism f of $\mathcal{C}_h$ belongs to $\mathcal{C}'_h$ if and only if it is invariant under the transformation a^t of $\mathcal{C}_h$ defined by $h \to th$, $\mu \to \mu$, $\delta \to \delta/t$, for all $t \in \mathbb{Q}^*$. Therefore, it is enough to show that the Hopf algebra structure on M_- defined by (1.3)-(1.5) is invariant under the automorphism a^t.

Denote by $\mathfrak{a}^t_+$ the Lie bialgebra $a^t(\mathfrak{a}_+)$. Let $\mathcal{M}^t$ be the category of $\mathfrak{a}^t_+$-dimodules in $\mathcal{C}_h$.

We have an equivalence of categories $\theta^*_t : \mathcal{M}^1 \to \mathcal{M}^t$, which maps any $\mathfrak{a}_+$-dimodule X to X, with an action and coaction of $\mathfrak{a}^t_+$ defined by the formulas $\pi_t = \pi, \pi^*_t = t\pi^*$. In the case of Lie bialgebras over a field, this equivalence comes from the natural Lie algebra isomorphism between the doubles $D(\mathfrak{g}^t_+)$, $D(\mathfrak{g}_+)$, $\theta_t : D(\mathfrak{g}^t_+) \to D(\mathfrak{g}_+)$, which is the identity on $\mathfrak{g}_+$ and multiplication by t on $\mathfrak{g}_-$. By the definition, this functor maps the Verma modules to the Verma modules, sends the classical r-matrix r to $t^{-1}r$, and preserves the morphism ψ.

Since the associator Φ is a function of $h\Omega_{12}, h\Omega_{23}$, it is invariant under the transformation a^t. By Proposition 1.4, this implies that $U_h(\mathfrak{a}_+)$ is fixed by a^t, and Proposition 1.5 follows.

Let m', Δ' be the morphisms in the category $\mathcal{C}$ such that $P_h(m') = m$, $P_h(\Delta') = \Delta$. They exist by Proposition 1.5 and are obviously unique. The morphisms m', Δ' define a Hopf algebra structure on M_- in $\mathcal{C}$.

Now define $Q([1])$ to be the object $S[1]$ with the Hopf algebra structure (m', Δ'). This determines the functor Q whose existence is claimed in Part (i) of Theorem 1.2. Part (i) is proved.

Now let us prove parts (ii) and (iii) of Theorem 1.2.

Let $\mathcal{N}$ be a symmetric tensor category, $X \subset \mathcal{N}$ an object. By an element of X we mean a morphism $x : \mathbf{1} \to X$ (we will write $x \in X$).

Part (ii). Let $\mathfrak{g} = [1]$ be the canonical quasitriangular Lie bialgebra in the tensor category $\mathcal{C} = \overline{QTLBA_{\langle r \rangle}}$. Let $U(\mathfrak{g})$ be its universal enveloping algebra. We have an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ – the classical r-matrix of $\mathfrak{g}$. Our purpose is to define a Hopf algebra H in $\mathcal{N}$, which coincides with $U(\mathfrak{g})$ as an object of $\mathcal{N}$.

The definition of H is as follows. The product and unit in H are the same as in $U(\mathfrak{g})$. The coproduct is defined by the formula

$$\Delta = J^{-1}\Delta_0 J, \qquad (1.6)$$

where $J = J(r)$ is an invertible element of $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ (cf. (3.1),(3.2)).

The element $J(r)$ is defined by a universal formula which is obtained from (3.1): $J(r) = 1 + r/2 + O(\langle r \rangle^2)$.

Define the functor Q^{qt} by $Q^{qt}([1]) = H$. This functor satisfies the conditions of Part (ii) of Theorem 1.2. Part (ii) is proved.

Part (iii). Let A be the canonical classical Yang-Baxter algebra in the tensor category $CYBA_{<r>}$, and $r \in A \otimes A$ be the classical r-matrix. Let $J(r)$ be the element of $A \otimes A$ given by the same formula as in Part (ii). Define $R = (J^{op})^{-1} e^{(r+r^{op})/2} J \in A \otimes A$. (cf. (3.10)). As shown in Part I of [EK], this element satisfies the quantum Yang-Baxter equation.

Define the functor Q^{YB} by $Q^{YB}([1]) = (A, R)$. This functor satisfies the conditions of Part (iii) of Theorem 1.2. Part (iii) is proved.

This completes the proof of Theorem 1.2.

1.5. *Identification of two quantizations of a quasitriangular Lie bialgebra.*

Let $\mathfrak{a}$ be a finite dimensional quasitriangular Lie bialgebra over k. Let $U_h(\mathfrak{a})$ be the quantization of $\mathfrak{a}$ constructed in Chapter 4, and $U_h^{qt}(\mathfrak{a})$ be the quasitriangular quantization of $\mathfrak{a}$ constructed in Section 6.1.

Theorem 1.6. *The quantized universal enveloping algebras $U_h(\mathfrak{a})$, $U_h^{qt}(\mathfrak{a})$ are isomorphic.*

To prove Theorem 1.6, we first need the following result, which appears (in somewhat different form) in [RS].

Lemma 1.7. *Let $\mathfrak{a}$ be a quasitriangular Lie bialgebra, and $\mathfrak{g}$ be the double of $\mathfrak{a}$. Then the linear map $\tau : \mathfrak{g} \to \mathfrak{a}$ defined by*

$$\tau(x + f) = x + (f \otimes 1)(r), x \in \mathfrak{a}, f \in \mathfrak{a}^*, \tag{1.6}$$

is a homomorphism of quasitriangular Lie bialgebras.

Proof. First we show that τ is a homomorphism of Lie algebras, i.e. $\tau([g_1 g_2]) = [\tau(g_1)\tau(g_2)]$. This is obvious when $g_1, g_2 \in \mathfrak{a}$. Assume that $f, g \in \mathfrak{a}^*$. Then, using the classical Yang-Baxter equation, we get

$$\begin{gathered} \tau([fg]) = ([fg] \otimes 1)(r) = (f \otimes g \otimes 1)((\delta \otimes 1)(r)) = \\ (f \otimes g \otimes 1)([r_{13} + r_{23}, r_{12}]) = (f \otimes g \otimes 1)([r_{13}, r_{23}]) = [\tau(f)\tau(g)]. \end{gathered} \tag{1.7}$$

Now assume that $x \in \mathfrak{a}, f \in \mathfrak{a}^*$. Then

$$\begin{gathered} \tau([xf]) = \tau(\mathrm{ad}^* x(f)) - \tau(\mathrm{ad}^* f(x)) = \\ \tau((f \otimes 1)([r, x \otimes 1])) + \tau((f \otimes 1)([x \otimes 1 + 1 \otimes x, r])) = \\ \tau((f \otimes 1)([1 \otimes x, r])) = [\tau(x)\tau(f)]. \end{gathered} \tag{1.8}$$

Now we check that τ is a homomorphism of quasitriangular Lie bialgebras. Let $\tilde{r}$ be the quasitriangular structure on $\mathfrak{g}$. If x_i is a basis of $\mathfrak{a}$, and f_i is the dual basis of $\mathfrak{a}^*$, then $\tilde{r}$ is given by the formula $\tilde{r} = \sum_i x_i \otimes f_i$. Thus we have

$$(\tau \otimes \tau)(\tilde{r}) = \sum_i \tau(x_i) \otimes \tau(f_i) = \sum_i x_i \otimes (f_i \otimes 1)(r) = r. \tag{1.9}$$

The Lemma is proved. □

Proof of Theorem 1.6. Lemma 1.8 claims that there exists a morphism of quasitriangular Lie bialgebras $\tau : \mathfrak{g} \to \mathfrak{a}$ which is the identity on $\mathfrak{a}$. Theorem 1.2 states that quasitriangular quantization of Section 6.1 is a functor from the category of quasitriangular Lie bialgebras to the category of quasitriangular topological Hopf algebras over $k[[h]]$. Thus, τ defines a morphism $\hat{\tau} : U_h^{qt}(\mathfrak{g}) \to U_h^{qt}(\mathfrak{a})$. On the other hand, $U_h(\mathfrak{a})$ was constructed as a subalgebra in $U_h^{qt}(\mathfrak{g})$, so we have an embedding $\eta : U_h(\mathfrak{a}) \to U_h^{qt}(\mathfrak{g})$. Consider the morphism $\tau \circ \eta : U_h(\mathfrak{a}) \to U_h^{qt}(\mathfrak{a})$. This morphism is an isomorphism since it equals to 1 modulo h. The theorem is proved. □

Corollary 1.8. *The quantization of the double $\mathfrak{g}$ of a finite dimensional Lie bialgebra $\mathfrak{a}$ constructed in Chapter 3 is isomorphic to the quantization of $\mathfrak{g}$ as a Lie bialgebra, constructed in Chapter 4.*

Remark. The analog of Theorem 1.6 holds for infinite dimensional Lie bialgebras. Namely, the "usual" quantization of $\mathfrak{a}$ defined in Section 9 is isomorphic to its quasitriangular quantization. The proof is analogous to the finite dimensional case.

2. Dequantization of QUE algebras

2.1. *The main result.*

The main result of this paper is the following theorem.

Theorem 2.1. *The functor $\widehat{Q}$ is an equivalence of categories.*

Remark. We plan to prove in a forthcoming paper that the functors $\widehat{Q^{qt}}$, $\widehat{Q^{YB}}$ are also equivalences of categories.

We will prove the theorem for $K = k[[h]]$. It is not difficult to generalize the proof to the case when K is a general pro-Artinian algebra over k.

In order to prove the theorem, we construct the functors of dequantization (quasiclassical limit) which are inverse to the functors of quantization $\widehat{Q}$, $\widehat{Q^{qt}}$, $\widehat{Q^{YB}}$. The usual notion of quasiclassical limit described in Chapter 3 is not sufficient for us since it assigns to a QUE algebra over $k[[h]]$ a Lie bialgebra over k, not over $k[[h]]$, and thus erases nearly all the information about the QUE algebra. Our construction of quasiclassical limit is different and uses the action of the Grothendieck-Teichmuller semigroup on braided structures on a tensor category, defined by Drinfeld. This construction is described in the next sections.

2.2. *The Grothendieck-Teichmuller semigroup.*

In this section we follow [Dr4].

Let $\mathcal{B}$ be a $\mathbb{Q}$-linear topological braided tensor category, with a tensor ideal $\mathcal{I}$ such that $\mathcal{B}$ is complete with respect to the topology defined by $\mathcal{I}$. This means, any series of morphisms of the form $\sum_{n=0}^{\infty} a_n$, where $a_n \in \mathcal{I}^n$, is convergent (for example, $\mathcal{B}$ is K-linear, and $\mathcal{I}_{X,Y} = I\mathrm{Hom}_{\mathcal{B}}(X,Y)$, where I is the maximal ideal in K). Let Φ be the associativity morphism, and β the commutativity morphism (braiding) of $\mathcal{B}$.

Definition. *We say that $\mathcal{B}$ is quasisymmetric if $\beta^2 = 1$ mod $\mathcal{I}$.*

Denote by $\overline{GT}$ the set of all pairs $a = (\lambda, f)$ such that

1) $\lambda \in \mathbb{Q}$, $f(X,Y)$ is a formal series of the form $\exp(P(\ln X, \ln Y))$, where P is a Lie formal series over $\mathbb{Q}$, and

2) for any quasisymmetric category with associativity morphism Φ and braiding β the morphisms

$$a(\Phi) = \Phi f(\beta_{21}\beta_{12}, \Phi^{-1}\beta_{32}\beta_{23}\Phi), \ a(\beta) = \beta \circ (\beta^2)^m, m = \frac{1}{2}(\lambda - 1) \tag{2.1}$$

define a new structure of a quasisymmetric category on $\mathcal{B}$.

The set $\overline{GT}$ is a monoid under the composition law

$$(\lambda_1, f_1)(\lambda_2, f_2) = (\lambda_1\lambda_2, f_1 \circ_{\lambda_2} f_2),$$
$$f_1 \circ_\lambda f_2(X,Y) := f_1(f_2(X,Y)X^\lambda f_2(X,Y)^{-1}, Y^\lambda)f_2(X,Y). \tag{2.2}$$

The subset $GT = \{(\lambda, f) \in \overline{GT} : \lambda \neq 0\}$ is a group under the same composition law.

For $\mu \in \mathbb{Q}$, denote by GT_μ the set of elements of $\overline{GT}$ of the form (μ, f). It is clear that $\overline{GT}$, GT, GT_μ have a natural structure of proalgebraic $\mathbb{Q}$-varieties.

The following result is due to Drinfeld ([Dr4], p. 855).

Theorem 2.2. *For any $a_0 \in GT_0$ there exists a unique algebraic homomorphism of semigroups $a : \mathbb{Q} \to \overline{GT}$, such that $a(0) = a_0$, and $a(\mu) \in GT_\mu$, $\mu \in \mathbb{Q}$.*

Remark. An algebraic homomorphism means a homomorphism, which is polynomial modulo any finite degree.

Given an element $a = (\lambda, f) \in \overline{GT}$ and a quasisymmetric tensor category $\mathcal{B}$ with associativity isomorphism Φ and braiding β, we can define a new quasisymmetric category $\mathcal{B}^a$, which is the same as $\mathcal{B}$ as an additive category with tensor product, and the associativity isomorphism and braiding given by formula (2.1). This category is symmetric if $a \in GT_0$. Also, if $a, b \in \overline{GT}$, then $\mathcal{B}^{ab} = (\mathcal{B}^b)^a$. Thus, the semigroup $\overline{GT}$ acts on the set of structures of a quasisymmetric category on $\mathcal{B}$.

2.3. *Dimodules over a Hopf algebra.*

In this section we will introduce the notion of a dimodule over a Hopf algebra H in an arbitrary symmetric tensor category. It is analogous to the notion of a dimodule over a Lie bialgebra, introduced in Chapter 1, and, for the case of finite-dimensional Hopf algebras, is equivalent to the notion of a module over the quantum double $D(H)$.

Definition.

(i) Let A_+ be an associative algebra in a symmetric tensor category $\mathcal{N}$, with product m and unit ι. An object $X \in \mathcal{N}$ is said to be equipped with the structure of a left A_+-module if it is endowed with a morphism $\pi : A_+ \otimes X \to X$ (the action of A_+ on X), such that $\pi \circ (1 \otimes \pi) = \pi \circ (m \otimes 1)$ on $A_+ \otimes A_+ \otimes X$, and $\pi \circ (\iota \otimes 1) = id$ on X.

(ii) Let A_+ be a coassociative coalgebra in a symmetric tensor category $\mathcal{N}$, with coproduct Δ and counit ε. An object $X \in \mathcal{N}$ is said to be equipped with the structure of a right A_+-comodule if it is endowed with a morphism $\pi^* : X \to A_+ \otimes X$ (the coaction of A_+ on X), such that $(1 \otimes \pi^*) \circ \pi^* = (\Delta^{op} \otimes 1) \circ \pi^*$ on X, and $(\varepsilon \otimes 1) \circ \pi^* = id$ on X.

(iii) Let A_+ be a Hopf algebra in a symmetric tensor category $\mathcal{N}$. An object $X \in \mathcal{N}$ is said to be equipped with the structure of an A_+-dimodule if it is endowed

with two morphisms $\pi : A_+ \otimes X \to X$, $\pi^* : X \to A_+ \otimes X$, such that π is a left action of A_+ on X as an algebra, π^* is a right coaction of A_+ on X as a coalgebra, and they agree according to the formula (cf [Dr1], *p. 816)*

$$\pi^* \circ \pi = (m_3 \otimes \pi) \circ \sigma_{13}\sigma_{24} \circ (S^{-1} \otimes 1^{\otimes 4}) \circ (\Delta_3 \otimes \pi^*), \tag{2.3}$$

where $m_3 := m \circ (m \otimes 1)$, and $\Delta_3 := (\Delta \otimes 1) \circ \Delta$.

Let A_+ be a Hopf algebra in $\mathcal{N}$. We say that an A_+-module (comodule) X is trivial if $\pi = \varepsilon \otimes 1_X$ (respectively, $\pi^* = \iota \otimes 1_X$). An A_+-dimodule is called trivial if it is trivial both as a module and a comodule. For any A_+-module, comodule, or dimodule X, let X_0 be the object X equipped with the trivial structure of a module (comodule, dimodule).

There is an obvious notion of tensor product of modules and comodules. Namely, for any two modules (comodules) V, W

$$\pi_{V\otimes W} = (\pi_V \otimes \pi_W) \circ \sigma_{23} \circ (\Delta \otimes 1 \otimes 1); \pi^*_{V\otimes W} = (m^{op} \otimes 1 \otimes 1) \circ \sigma_{23} \circ (\pi^*_V \otimes \pi^*_W). \tag{2.4}$$

The tensor product of dimodules is just the tensor product of the underlying modules and comodules. It follows from [Dr1], p. 816, and can be checked by a direct computation, that in this way one indeed obtains a new dimodule.

Thus, modules, comodules, and dimodules over A_+ in $\mathcal{N}$ form a tensor category. We denote the first category by $\mathcal{M}_{A_+}$, the second by $\mathcal{M}^{A_+}$, and the third by $\mathcal{M}^{A_+}_{A_+}$.

According to the results of Drinfeld [Dr1], the category $\mathcal{M}^{A_+}_{A_+}$ has a natural structure of a braided tensor category. The braiding is defined by the formula

$$\beta = \sigma \circ R, R = (\pi \otimes 1) \circ \sigma_{12} \circ (1 \otimes \pi^*). \tag{2.5}$$

Drinfeld proved that (2.5) satisfies the hexagon relations.

Remark. The existence of this braiding corresponds to the fact that the double of a Hopf algebra is a quasitriangular Hopf algebra.

Now we define the Verma dimodules M_-, $\hat{M}^*_+$ over A_+. As objects of $\mathcal{N}$, $M_- = \hat{M}^*_+ = A_+$. Let $1_- : \mathbf{1} \to M_-$, $1^*_+ : \mathbf{1} \to \hat{M}^*_+$ be the unit of A_+, and $1_+ : \hat{M}^*_+ \to \mathbf{1}$, $1^*_- : M_- \to \mathbf{1}$ be the counit.

Define the action of A_+ in M_- by $\pi_- = m$; this is the same as the standard left action of A_+ on itself. The coaction of A_+ in M_- is then completely determined (via formula (2.3)) by its composition with 1_-, which we define by $\pi^* \circ 1_- = \iota \otimes 1_-$. It is easy to see from (2.3) that this coaction has the form

$$\pi^*_- = (m^{op} \otimes 1) \circ (S^{-1} \otimes \sigma_{23}) \circ \Delta_3. \tag{2.6}$$

The formulas for the action and coaction of A_+ in $\hat{M}^*_+$ have the form

$$\pi_+ = m_3 \circ \sigma_{23} \circ (1 \otimes S^{-1} \otimes 1) \circ (\Delta^{op} \otimes 1), \pi^*_+ = \Delta^{op}. \tag{2.7}$$

For any object $X \in \mathcal{M}^{A_+}_{A_+}$ consider the map $\theta : \mathrm{Hom}_{\mathcal{M}^{A_+}_{A_+}}(M_- \otimes X_0, \hat{M}^*_+ \otimes X) \to \mathrm{Hom}_{\mathcal{N}}(X_0, X)$ given by $\theta(f) = (\varepsilon \otimes 1_X) \circ f \circ (\iota \otimes 1_{X_0})$.

Lemma 2.3. *The map θ is an isomorphism.*

Proof. By Frobenius reciprocity, for any two dimodules X, Y over A_+, $\mathrm{Hom}_{\mathcal{M}_{A_+}^{A_+}}(M_- \otimes X, \hat{M}_+^* \otimes Y) = \mathrm{Hom}_{\mathcal{M}^{A_+}}(X, \hat{M}_+^* \otimes Y) = \mathrm{Hom}_{\mathcal{N}}(X, Y)$. This implies the Lemma.

Denote by ψ the morphism $\theta^{-1}(1_X)$. Let $\eta = \psi \circ (\iota \otimes 1_{X_0}) : X_0 \to \hat{M}_+^* \otimes X$.

2.4. *Dequantization of Hopf algebras.*

In the next three sections we will give the construction of the functor $\widehat{DQ}$ of dequantization, and show that this functor is inverse to the functor $\widehat{Q}$.

Let us explain the idea of the construction of $\widehat{DQ}$. We will start with a Hopf algebra A_+ in some symmetric tensor category $\mathcal{N}$, which satisfies a certain condition called quasisymmetry. We will assign to it a family of Hopf algebras $A_+(t)$ depending algebraically on $t \in \mathbb{Q}$ (all realized on the same object of $\mathcal{N}$), such that $A_+(1) = A_+$, and $A_+(0)$ is cocommutative. Then $A_+(0)$ has a natural co-Poisson Hopf structure, obtained as the quasiclassical limit of coproducts in $A_+(t)$ as $t \to 0$. Since this construction works in any symmetric tensor category, it is defined by universal acyclic formulas, and so it defines a functor G from the category of quasisymmetric Hopf algebras in $\mathcal{N}$ to the category of co-Poisson Hopf algebras in $\mathcal{N}$.

Now we can consider the case when $\mathcal{N}$ is the category of topologically free $k[[h]]$-modules, and A_+ is a QUE algebra. In this case A_+ happens to be quasisymmetric. So we can define the Poisson-Hopf algebra $A_+(0)$ as above, and $\mathfrak{a}_+$ to be the set of primitive elements of $A_+(0)$. The object $\mathfrak{a}_+$ has a natural Lie bialgebra structure. Thus we establish a functor $\widehat{DQ}$ from the category of QUE algebras over $k[[h]]$ to the category of Lie bialgebras over $k[[h]]$, which will later be shown to be an inverse to $\widehat{Q}$.

Let us now explain the main technical point – how to construct $A_+(t)$ from A_+. For this purpose we will consider the category $\mathcal{M}_{A_+}^{A_+}$ of dimodules over A_+. This category is a braided tensor category, which has two remarkable objects – M_- and M_+^*. In Section 1.4 we have explained how to recover the Hopf algebra A_+ using these objects.

To construct $A_+(t)$, we choose a 1-parameter subsemigroup $a(t)$ in $\overline{GT}$, and twist the category $\mathcal{M}_{A_+}^{A_+}$ by $a(t)$, as explained in Section 2.2. The objects M_-, M_+^*, which are crucial in the construction of Section 1.4, are also present in the twisted category $(\mathcal{M}_{A_+}^{A_+})^{a(t)}$, and have the same properties. Therefore, one can apply the recovering procedure of Section 1.4 to this twisted category, which will yield the Hopf algebra $A_+(t)$, as desired.

Now we describe the details of this construction.

Let $A_+ \in \mathcal{N}$ be a Hopf algebra.

Let $\mathcal{M}$ be the full tensor subcategory of the category $\mathcal{M}_{A_+}^{A_+}$ generated by M_-, $\hat{M}_+^*$. That is, objects of $\mathcal{M}$ are arbitrary tensor products of copies of $\hat{M}_+^*$ and M_-, and morphisms are homomorphisms of dimodules.

Let $\mathcal{I}$ be a tensor ideal in $\mathcal{M}$ generated by the morphisms $R_{VW} - 1, V, W \in \mathcal{M}$ where R is the R-matrix defined by (2.5), and assume that $\mathcal{M}$ is separated and complete in the topology defined by $\mathcal{I}$. In this case, we call A_+ a quasisymmetric Hopf algebra.

If A_+ is a quasisymmetric Hopf algebra, then the category $\mathcal{M}$ is quasisymmetric. Pick $a \in \overline{GT}$.

Define a Hopf algebra A_+^a as follows. As an object of $\mathcal{N}$, A_+^a coincides with A_+. The product is defined by formula (1.3), where Φ is the associativity morphism of the category $\mathcal{M}^a$, and $1_+ : \hat{M}_+^* \to \mathbf{1}$ is the counit of A_+.

The coproduct is defined by the formula (1.4), where Δ_0 is the coproduct in A_+, and $J : M_- \otimes M_- \to M_- \otimes M_-$ is given by the formula

(2.8)

$$J = (1_+ \otimes 1 \otimes 1_+ \otimes 1) \circ \Phi_{1,3,24}^{-1} \circ \Phi_{3,2,4} \circ (RR^{op})^{(t-1)/2} (R^{op})^{-1} \circ \Phi_{2,3,4}^{-1} \circ \Phi_{1,2,34} \circ (\eta \otimes \eta)$$

(see (1.5)).

Using similar arguments to those given in Part II of [EK], one shows that these operations define a structure of a Hopf algebra on A_+^a.

Let $QS(\mathcal{N})$ be the category of quasisymmetric Hopf algebras in $\mathcal{N}$. We have constructed a functor $G_a : QS(\mathcal{N}) \to QS(\mathcal{N})$ such that $G_b \circ G_a$ is isomorphic to G_{ab}. Thus, we have an action of the semigroup $\widehat{GT}$ on the category $QS(\mathcal{N})$.

Now fix $a_0 \in GT_0$ be any point. Let $a(t)$ be the 1-parameter semigroup in $\overline{GT}$ which is defined by Theorem 2.2.

Let $\mathcal{M}(t)$, $t \in \mathbb{Q}$, be the tensor category $\mathcal{M}^{a(t)}$. Define a family of Hopf algebras $A_+(t) = A_+^{a(t)}$, $t \in \mathbb{Q}$. It is clear that the operations in $A_+(t)$ depend polynomially on t modulo any finite power of the ideal $\mathcal{I}$.

Proposition 2.4. *The Hopf algebra $A_+(0)$ is cocommutative.*

Proof. A direct computation.

Corollary 2.5. *The Hopf algebra $A_+(0)$ has a natural structure of a co-Poisson Hopf algebra, defined by $\delta = \lim_{t\to 0}(\Delta - \Delta^{op})/t$.*

We call the co-Poisson Hopf algebra $A_+(0)$ in $\mathcal{N}$ the dequantization of A_+.

2.5. *Dequantization of quantized universal enveloping algebras.*

Let $\mathcal{N}$ be the category of topologically free K-modules, and $A_+ \in HA_0(K)$ be a quantized universal enveloping algebra.

Proposition 2.6. *A_+ is quasisymmetric.*

Proof. Easy

Therefore, applying the construction of Section 2.4, we obtain a co-Poisson Hopf algebra $A_+(0)$. It is clear that the assignment $A_+ \to A_+(0)$ is a functor from the category of QUE algebras to the category of co-Poisson Hopf algebras, since it is given by a universal construction in the sense of Chapter 1.

According to Drinfeld's Proposition 3.7 of [Dr2], any cocommutative Hopf QUE algebra B is equal to $U(\mathfrak{b})$, where $\mathfrak{b}$ is the Lie algebra of primitive elements in B. Let $\mathfrak{a}_+$ be the Lie algebra of primitive elements in $A_+(0)$. It has a natural structure of a Lie bialgebra, induced by the co-Poisson Hopf structure on $A_+(0)$.

Thus, we have assigned to any QUE algebra A_+ a Lie bialgebra $\mathfrak{a}_+$, such that A_+ is isomorphic to $U(\mathfrak{a}_+)$ mod h. This assignment is clearly a functor $HA_0(K) \to LBA_0(K)$. We will call it the functor of dequantization, and denote it by $\widehat{DQ}_a$, or, shortly, $\widehat{DQ}$.

2.6. *Invertibility of the quantization functor.*

Let $M_1(k)$ be the set of Lie associators over k. For any $a = (\lambda, f) \in \overline{GT}(k)$, $\Phi \in M_1(k)$ define $a\Phi$ by formula (2.1):

$$a\Phi = \Phi f(e^{h\Omega_{12}}, \Phi^{-1}e^{h\Omega_{23}}\Phi). \tag{2.9}$$

Let Φ be the Lie associator over k fixed in Section 1.3. Denote by $f_\Phi(X,Y)$ the expression of the form

$$f_\Phi(X,Y) = e^{P_\Phi(\ln X, \ln Y)}, \tag{2.10}$$

such that $\Phi = f_\Phi(e^{ht_{12}}, e^{ht_{23}})$. For any $\lambda \in k$, denote by Φ_λ the expression

$$\Phi_\lambda = f_\Phi(X^\lambda, Y^\lambda). \tag{2.11}$$

The following result is due to Drinfeld, [Dr4].

Proposition 2.7. *Let Ψ be a Lie associator over k. There exists a unique 1-parameter semigroup $a_\Psi \subset \overline{GT}(k)$, such that $a_\Psi(\lambda) \in GT_\lambda(k)$ for all $\lambda \in k$, and $a_\Psi(\lambda)\Psi = \Phi_\lambda$.*

Proposition 2.8. *The functors $\widehat{Q_\Phi}$, $\widehat{DQ}_{a_\Phi}$ are quasi-inverse to each other. That is, the functors $\widehat{Q_\Phi} \circ \widehat{DQ}_{a_\Phi}, \widehat{DQ}_{a_\Phi} \circ \widehat{Q_\Phi}$ are isomorphic to the identity.*

Proof. Let $\mathfrak{a}_+ \in LBA_0(K)$ be a Lie bialgebra, $A_+ = \widehat{Q}(\mathfrak{a}_+)$. For any $\lambda \in k$, let $\mathfrak{a}_+(\lambda)$ be the Lie bialgebra obtained from $\mathfrak{a}_+$ by multiplying the cocommutator by λ. Let $A_+(\lambda)$ be the Hopf algebra obtained from A_+ as described in Section 2.5, using the semigroup $a = a_\Phi$. As $a_\Phi(\lambda)\Phi = \Phi^\lambda$, we get that $A_+(\lambda)$ coincides with $\widehat{Q}(\mathfrak{a}_+(\lambda))$. Tending λ to 0, we obtain that the Lie bialgebra $\widehat{DQ}(A_+)$ is naturally isomorphic to $\mathfrak{a}_+$. Thus, $\widehat{DQ}_{a_\Phi} \circ \widehat{Q_\Phi}$ is isomorphic to the identity.

Now let us prove that $\widehat{Q}_\Phi \circ \widehat{DQ}_{a_\Phi}$ is isomorphic to the identity. Let $a_\Phi(0) = (0, g_\Phi)$. Let $A_+ \in HA_0(K)$, $\mathfrak{a}_+ = \widehat{DQ}_{a_\Phi}(A_+)$, $\mathcal{M}$ the category of A_+-dimodules defined in Section 2.4, R the R-matrix defined by (2.5), and $T = R^{op}R$. Set $(\Phi^0, \sigma R^0) = g_\Phi \cdot (1, \sigma R)$, where σ is the permutation. We have $\Phi^0 = g_\Phi(T_{12}, T_{23})$, $R^0 = RT^{-1/2}$.

Define the Casimir element by $\Omega = \ln T$. Introduce commutativity and associativity morphisms in $\mathcal{M}$ by

$$R' = R^0 e^{\Omega/2}, \Phi' = f_\Phi(\Phi^0 e^{\Omega_{12}}(\Phi^0)^{-1}, e^{\Omega_{23}})\Phi^0. \tag{2.12}$$

By the definition of $\widehat{Q}$, this tensor category is obtained in the process of quantization of the Lie bialgebra $\mathfrak{a}'_+ := \widehat{DQ_{a_\Phi}}(A_+))$. Therefore, our claim follows from the equalities $R' = R$ and $\Phi' = 1$.

The first equlity is obvious. It remains to prove the equality $\Phi' = \Phi$. Using the expression for Φ^0, we see that

$$\Phi' = f_\Phi(g_\Phi(T_{12}, T_{23})^{-1} T_{12} g_\Phi(T_{12}, T_{23}), T_{23}) g_\Phi(T_{12}, T_{23}). \tag{2.13}$$

So, it remains to show that $f_\Phi \circ_1 g_\Phi = 1$, where the operation $\circ_1$ is defined by (2.2).

The set $\mathcal{F}$ of exponentials of formal Lie series equipped with the operation $\circ_1$, is a group. By the definition of g_Φ, we have in this group $g_\Phi \circ f_\Phi = 1$. This implies the desired equality $f_\Phi \circ_1 g_\Phi = 1$. Thus, the QUE algebra $\widehat{Q}(\widehat{DQ}(A_+))$ is canonically isomorphic to A_+. The Proposition is proved.

Thus, we have shown that the functor $\widehat{Q}_\Phi$ is an equivalence of categories and proved Theorem 2.1.

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