Bakalov-Kirillov on the Quantum Double of a Finite Group

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Following Bakalov-Kirillov

http://www.math.sunysb.edu/~kirillov/tensor/tensor.html

3.2. Example: Quantum double of a finite group

We will give the simplest example of a modular tensor category — the category of finite dimensional representations of the Hopf algebra $D(G)$, which is the quantum double of the group algebra $k[G]$ of a finite group G. It is interesting that this example appeared in two seemingly unrelated areas — the theory of characters of reductive groups over finite fields [**L5, L6**] and the orbifold constructions in Conformal Field Theory [**DVVV, KT**].

Let us first fix the notation. Let G be a finite group. Recall that its *group algebra* $k[G]$ over a field k is a Hopf algebra with a k-basis $\{x\}_{x\in G}$ and

multiplication	$x \otimes y \mapsto xy, \qquad x, y \in G,$
unit	$e \qquad$ (the unit element of G),
comultiplication	$\Delta(x) = x \otimes x, \qquad x \in G,$
counit	$\varepsilon(x) = 1,$
antipode	$\gamma(x) = x^{-1}.$

This Hopf algebra is cocommutative. A representation of $k[G]$ is the same as a representation of G. By Maschke's theorem, the category $\mathcal{R}ep_f\, k[G]$ of finite dimensional representations is semisimple. ?

The Hopf algebra dual to $k[G]$ is isomorphic to the function algebra $F(G)$ of the group G. It has a k-basis $\{\delta_g\}_{g\in G}$ consisting of delta functions:

$$\delta_g(x) = \delta_{g,x} = \begin{cases} 1 & \text{for } g = x, \\ 0 & \text{for } g \neq x. \end{cases}$$

It has

multiplication	$\delta_g \delta_h = \delta_{g,h}\delta_g, \qquad g, h \in G,$
unit	$1 = \sum_{g\in G} \delta_g,$
comultiplication	$\Delta(\delta_g) = \sum_{g_1 g_2 = g} \delta_{g_1} \otimes \delta_{g_2}, \qquad g \in G,$
counit	$\varepsilon(\delta_g) = \delta_{g,e},$
antipode	$\gamma(\delta_g) = \delta_{g^{-1}}.$

Is this related to $S(g^*)$?

A representation of $F(G)$ is the same as a G-graded vector space (since $\{\delta_g\}_{g\in G}$ are projectors).

Applying Drinfeld's quantum double construction [**Dr3**] it is easy to describe explicitly the quantum double $D(G)$ of $k[G]$. As a vector space, $D(G) = F(G) \otimes_k k[G]$. It is a Hopf algebra with

multiplication	$(\delta_g \otimes x)(\delta_h \otimes y) = \delta_{gx,xh}(\delta_g \otimes xy), \qquad x, y, g, h \in G,$
unit	$1 = \sum_{g\in G} \delta_g \otimes e,$
comultiplication	$\Delta(\delta_g \otimes x) = \sum_{g_1 g_2 = g} (\delta_{g_1} \otimes x) \otimes (\delta_{g_2} \otimes x), \qquad g, x \in G,$
counit	$\varepsilon(\delta_g \otimes x) = \delta_{g,e},$
antipode	$\gamma(\delta_g \otimes x) = \delta_{x^{-1}g^{-1}x} \otimes x^{-1}.$

The Hopf algebra $D(G)$ is quasitriangular with

R-matrix $\qquad R = \sum_{g\in G} (\delta_g \otimes e) \otimes (1 \otimes g).$

HW: How is this related to $I_{\mathfrak{g}} = \mathfrak{g}^* \rtimes \mathfrak{g}$?

Q. Is there a generalization of this, corresponding to other reps R replacing $\mathfrak{g}^*$?

(Of course, once we know the above formulas, they can be easily checked directly.)

Note that $F(G)$ and $k[G]$ embed in $D(G)$ as k-algebras and $D(G)$ is their semidirect product:

$$D(G) = F(G) \rtimes k[G], \tag{3.2.1}$$

with

$$x\delta_g x^{-1} = \delta_{xgx^{-1}} \quad \text{for } g, x \in G. \tag{3.2.2}$$

Q. Is there an action of $D(G)$ on $F(G)$, "tangential" like the action of $U(\mathfrak{g})$ on $\mathrm{Fun}(\mathfrak{g})$?

Let $\mathcal{R}ep_f D(G)$ be the category of finite dimensional representations of $D(G)$ as a k-algebra. By the above remarks, a representation V of $D(G)$ is the same as a G-module with a G-grading $V = \bigoplus_{g\in G} V_g$ satisfying $xV_g \subset V_{xgx^{-1}}$, $x, g \in G$. In other words, objects of $\mathcal{R}ep_f D(G)$ are finite dimensional G-equivariant vector bundles over G. We will show that the category $\mathcal{R}ep_f D(G)$ is semisimple and will

$k[G]$. It is a Hopf algebra with

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unit	$1 = \sum_{g\in G} \delta_g \otimes e,$
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For $V \in \mathrm{Ob}\,\mathcal{R}ep_f D(G)$ and $v \in V$ the submodule generated by v is

$$D(G)v = \sum_{g\in G} k[G]\delta_g v = \sum_{g\in G} \bigoplus_{xgx^{-1}\in\overline{g}} xZ(g)\delta_g v,$$

where $\overline{g}$ denotes the conjugasy class and $Z(g)$ the centralizer of g in G. Note that $k[Z(g)]\delta_g v$ is an irreducible representation π of $Z(g)$. Hence

(3.2.3) $$V_{\overline{g},\pi} := k[G]\delta_g v = \bigoplus_{xgx^{-1}\in\overline{g}} x\pi,$$

is an irreducible $D(G)$-module which depends only on the conjugacy class $\overline{g}$ and the isomorphism class of the irreducible representation π of $Z(g)$. The action of $D(G)$ on $V_{\overline{g},\pi}$ is given explicitly by:

(3.2.4) $$(\delta_f \otimes h)(xv) = \delta_{f,hxgh^{-1}x^{-1}}\, hxv \quad \text{for } f, h, x \in G,\ v \in \pi.$$

This shows that the category $\mathcal{R}ep_f D(G)$ is semisimple with simple objects $V_{\overline{g},\pi}$ labeled by pairs $(\overline{g}, \pi)$, where $\overline{g} \in \overline{G}$ is a conjugacy class in G and $\pi \in \widehat{Z(g)}$ is an isomorphism class of irreducible representation of the centralizer $Z(g)$ of some element $g \in \overline{g}$ (π is independent of the choice of g).

HW: How is this related to $I\mathfrak{g} = \mathfrak{g}^* \rtimes \mathfrak{g}$?

Q. Is there a generalization of this, corresponding to other reps R replacing $\mathfrak{g}^*$?

Q. Is there an action of D(G) on F(G), "tangentially" like the action of U(g) on Fun(g)?

In what follows we will use the orthogonality relations of irreducible characters of a finite group G:

$$\text{(3.2.5)} \qquad \frac{1}{|G|}\sum_{h\in G}\operatorname{tr}_{\pi^*}(h)\operatorname{tr}_{\pi'}(hg)=\frac{\operatorname{tr}_\pi(g)}{\operatorname{tr}_\pi(e)}\delta_{\pi,\pi'}, \qquad \pi,\pi'\in\widehat{G},\ g\in G,$$

$$\text{(3.2.6)} \qquad \frac{1}{|Z(g)|}\sum_{\pi\in\widehat{G}}\operatorname{tr}_{\pi^*}(g)\operatorname{tr}_\pi(h)=\delta_{\overline{g},\overline{h}}, \qquad h,g\in G.$$

Also recall that $|\overline{g}||Z(g)|=|G|$.

THEOREM 3.2.1. *$\mathcal{R}ep_f D(G)$ is a modular tensor category with simple objects $V_{\overline{g},\pi}$ labeled by $(\overline{g},\pi)$, $\overline{g}\in\overline{G}$, $\pi\in\widehat{Z(g)}$ $(g\in\overline{g})$. We have:*

$$\text{(3.2.7)} \qquad V^*_{\overline{g},\pi}\simeq V_{\overline{g^{-1}},\pi^*},$$

$$\text{(3.2.8)} \qquad t_{(\overline{g},\pi),(\overline{g'},\pi')}=\delta_{(\overline{g},\pi),(\overline{g'},\pi')}\frac{\operatorname{tr}_\pi(g)}{\operatorname{tr}_\pi(e)},$$

$$\text{(3.2.9)} \qquad s_{(\overline{g},\pi),(\overline{g'},\pi')}=\frac{1}{|Z(g)||Z(g')|}\sum_{\substack{h\in G\\ hg'h^{-1}\in Z(g)}}\operatorname{tr}_\pi(hg'^{-1}h^{-1})\operatorname{tr}_{\pi'}(h^{-1}g^{-1}h).$$

The numbers $p^\pm$ from (3.1.7) are equal to the order of G.

The s-matrix (3.2.9) was first introduced by Lusztig [**L5**] (see also [**L6, L7**]) under the names "non-abelian Fourier transform" and "exotic Fourier transform". Then it appeared in [**DVVV**] and [**KT**] in connection with "orbifolds". Dijkgraaf, Pasquier and Roche [**DPR**] considered a generalization of the above construction which is also related to orbifolds. They introduced a quasi-Hopf algebra $D^c(G)$, depending on a cohomology class $c\in \mathrm{H}^3(G,\mathrm{U}(1))$, which reduces to $D(G)$ when $c=1$.

PROOF OF THEOREM 3.2.1. Eq. (3.2.7) follows easily from the definitions (note that $Z(g^{-1})=Z(g)$ and $\operatorname{tr}_{\pi^*}(h)=\operatorname{tr}_\pi(h^{-1})$).

To prove (3.2.8), we compute the twists θ using the results of Proposition 2.2.4 and Lemma 2.2.5. Since $\gamma^2=\mathrm{id}$, it follows that $\delta_V=\mathrm{id}$, cf. (2.2.11). Hence,

$$\text{(3.2.10)} \qquad \theta=u^{-1}=\sum_{h\in G}\delta_h\otimes h.$$

As g is central in $Z(g)$, it acts as a constant $=\operatorname{tr}_\pi(g)/\operatorname{tr}_\pi(e)$ on the representation π; hence by (3.2.4), $\theta_{\overline{g},\pi}=\operatorname{tr}_\pi(g)/\operatorname{tr}_\pi(e)$.

To prove (3.2.9), we will use (3.1.2). We compute for $x,x'\in G$, $v\in\pi^*$, $v'\in\pi'$:

$$\begin{aligned}\theta_{V^*_{\overline{g},\pi}\otimes V_{\overline{g'},\pi'}}(xv\otimes x'v')&=\Delta(u^{-1})(xv\otimes x'v')\\ &=\sum_{\substack{h\in G\\ h_1h_2=h}}(\delta_{h_1}\otimes h)(xv)\otimes(\delta_{h_2}\otimes h)(x'v')\\ &=\sum_{\substack{h\in G\\ h_1h_2=h}}\delta_{h_1,hxg^{-1}x^{-1}h^{-1}}hxv\otimes\delta_{h_2,hx'g'x'^{-1}h^{-1}}hx'v'\\ &=(fxv\otimes fx'v'),\qquad\text{where } f=xg^{-1}x^{-1}x'g'x'^{-1}.\end{aligned}$$

Hence,

$$\begin{aligned}\operatorname{tr}\theta_{V^*_{\overline{g},\pi}\otimes V_{\overline{g'},\pi'}}&=\sum_{\substack{xg^{-1}x^{-1}\in\overline{g^{-1}}\\ x'g'x'^{-1}\in\overline{g'}\\ x^{-1}x'g'x'^{-1}x\in Z(g^{-1})}}\operatorname{tr}_{\pi^*}(g^{-1}x^{-1}x'g'x'^{-1}x)\operatorname{tr}_{\pi'}(x'^{-1}xg^{-1}x^{-1}x'g')\\ &=\frac{\operatorname{tr}_{\pi^*}(g^{-1})}{\operatorname{tr}_{\pi^*}(e)}\frac{\operatorname{tr}_{\pi'}(g')}{\operatorname{tr}_{\pi'}(e)}\frac{1}{|Z(g)||Z(g')|}\sum_{\substack{h\in G\\ hg'h^{-1}\in Z(g)}}\operatorname{tr}_{\pi^*}(hg'h^{-1})\operatorname{tr}_{\pi'}(h^{-1}g^{-1}h),\end{aligned}$$

which proves (3.2.9).

The computation of $p^\pm$ is straightforward (using (3.2.5, 3.2.6)), and is left to the reader. $\square$

Preliminary moral. $D(G)$ is the ring of operators on $\operatorname{Fun}(G)$ generated by multiplication operators and by conjugations.

Q: why is $D(G)$ a coalgebra? Define $D(G)$

Q. Why is $D(G)$ a co-algebra? Perhaps $D(G)$ should be interpreted as some sort of "scheme" that functorially takes groups into their doubles?