Pensieve header: The Alexander polynomial using bridges, tunnels, and Alexander numbering.

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2023-12"];
Once[
 << KnotTheory`;
 << "../Talks/Oaxaca-2210/Rot.m"
]
```

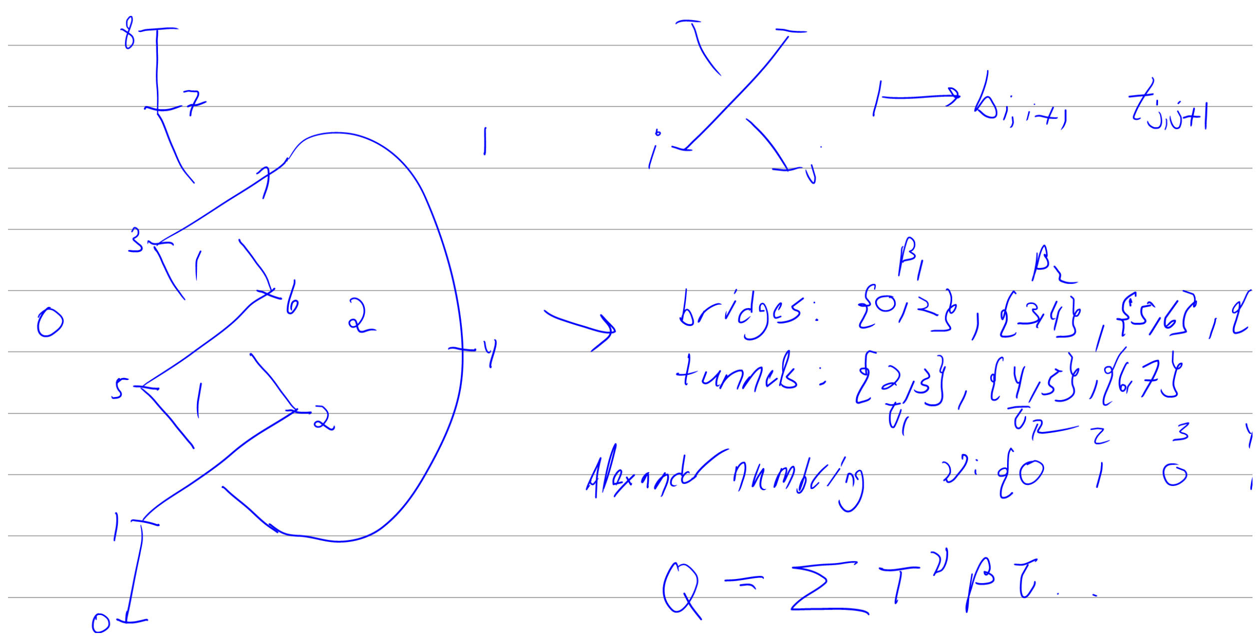
```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
Loading Rot.m from http://drorbn.net/la22/ap to compute rotation numbers.
```

In[ ]:=
```
K1 = EPD[X_{1,2}];
K3 = EPD[X_{1,4}, X_{5,2}, X_{3,6}];
K8 = Knot[8, 17];
K10 = Knot[10, 165];
{Cs, φ} = Rot[K = K10] /. {s_, i_, j_} ↦ {s, j, i}
n = Length[Cs]
```

Out[ ]=
```
{{{-1, 1, 6}, {-1, 7, 18}, {1, 3, 8}, {1, 17, 2},
  {1, 5, 14}, {1, 9, 16}, {1, 15, 10}, {1, 11, 4}, {1, 20, 13}, {1, 12, 19}},
 {0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1, -1, 0, 0, 0}}
```

Out[ ]=
```
10
```



In[∘]:=
```
bridges = tunnels = {};
bn = tn = 0; (* completed features *)
cfb = False; (* current feautre is bridge *);
cfs = 0; (* current feature start *)
ν = {lν = 0};
ϕ = {};
Q = 0;
For[k = 1, k ≤ 2 n, ++k,
  Cs /. {
    {s_, k, j_} ⧴ (
      AppendTo[ϕ, bn + 1];
      If[¬ cfb, AppendTo[tunnels, {cfs, k}]; ++tn;
       cfb = True;
       cfs = k; Q += T^lν (T - 1) β_(bn+1) τ_tn];
      AppendTo[ν, lν += s];
     ),
    {s_, i_, k} ⧴ (
      AppendTo[ϕ, tn + 1];
      If[cfb, AppendTo[bridges, {cfs, k}]; ++bn;
       cfb = False;
       cfs = k; Q += T^lν (1 - T) β_bn τ_(tn+1) ];
      AppendTo[ν, lν -= s];
     )
   };
  Cs /. {
    {s_, k, j_} /; k > j ⧴ (Q += T^ν[[k]] (T - 1) (T^s - 1) β_ϕ[[k]] τ_ϕ[[j]]),
    {s_, i_, k} /; k > i ⧴ (Q += T^ν[[i]] (T - 1) (T^s - 1) β_ϕ[[i]] τ_ϕ[[k]])
   }
 ];
{ν, bridges, tunnels, ϕ, Q} // Column
```

Out[∘]=

{0, -1, -2, -1, -2, -1, 0, -1, -2, -1, -2, -1, 0, -1, -2, -1, -2, -1, 0, -1, 0}
{{1, 2}, {3, 4}, {5, 6}, {7, 8}, {9, 10}, {11, 13}, {15, 16}, {17, 18}}
{{0, 1}, {2, 3}, {4, 5}, {6, 7}, {8, 9}, {10, 11}, {13, 15}, {16, 17}, {18, 20}}
{1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 6, 7, 7, 7, 8, 8, 9, 9, 9}

$$(-1+T)\,\beta_1\,\tau_1+\frac{(1-T)\,\beta_1\,\tau_2}{T}+\frac{(-1+T)\,\beta_2\,\tau_2}{T^2}+\frac{(-1+T)^2\,\beta_8\,\tau_2}{T^2}+\frac{(1-T)\,\beta_2\,\tau_3}{T}+\frac{(-1+T)\,\beta_3\,\tau_3}{T^2}+\frac{(-1+T)^2\,\beta_6\,\tau_3}{T^2}+$$
$$\left(-1+\frac{1}{T}\right)(-1+T)\,\beta_1\,\tau_4+\frac{(1-T)\,\beta_3\,\tau_4}{T}+(-1+T)\,\beta_4\,\tau_4+\frac{(-1+T)^2\,\beta_2\,\tau_5}{T^2}+\frac{(1-T)\,\beta_4\,\tau_5}{T}+\frac{(-1+T)\,\beta_5\,\tau_5}{T^2}+$$
$$\frac{(1-T)\,\beta_5\,\tau_6}{T}+\frac{(-1+T)\,\beta_6\,\tau_6}{T^2}+\frac{(-1+T)^2\,\beta_7\,\tau_6}{T^2}+\frac{(-1+T)^2\,\beta_3\,\tau_7}{T^2}+(1-T)\,\beta_6\,\tau_7+\frac{(-1+T)\,\beta_7\,\tau_7}{T^2}+\frac{(-1+T)^2\,\beta_9\,\tau_7}{T}+$$
$$\frac{(-1+T)^2\,\beta_5\,\tau_8}{T^2}+\frac{(1-T)\,\beta_7\,\tau_8}{T}+\frac{(-1+T)\,\beta_8\,\tau_8}{T^2}+\left(-1+\frac{1}{T}\right)(-1+T)\,\beta_4\,\tau_9+\frac{(-1+T)^2\,\beta_6\,\tau_9}{T}+\frac{(1-T)\,\beta_8\,\tau_9}{T}+\frac{(-1+T)\,\beta_9\,\tau_9}{T}$$

In[∘]:=
```
{bn, tn}
```

Out[∘]=

{8, 9}

In[∘]:= MatrixForm$\left[\text{mat} = \text{Factor@Table}\left[\frac{\partial_{\beta_i,\tau_j} Q}{T-1}, \{i, 1, bn+1\}, \{j, 1, tn\}\right]\right]$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & -\frac{1}{T} & 0 & -\frac{-1+T}{T} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{T^2} & -\frac{1}{T} & 0 & \frac{-1+T}{T^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{T^2} & -\frac{1}{T} & 0 & 0 & \frac{-1+T}{T^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & -\frac{-1+T}{T} \\ 0 & 0 & 0 & 0 & \frac{1}{T^2} & -\frac{1}{T} & 0 & \frac{-1+T}{T^2} & 0 \\ 0 & 0 & \frac{-1+T}{T^2} & 0 & 0 & \frac{1}{T^2} & -1 & 0 & \frac{-1+T}{T} \\ 0 & 0 & 0 & 0 & 0 & \frac{-1+T}{T^2} & \frac{1}{T^2} & -\frac{1}{T} & 0 \\ 0 & \frac{-1+T}{T^2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{T^2} & -\frac{1}{T} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+T}{T} & 0 & \frac{1}{T} \end{pmatrix}$$

In[∘]:= Factor$\left[-\text{mat}0^{\mathsf{T}} \mathbin{/.} T \to T^{-1}\right]$ // MatrixForm

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & -\frac{1}{T} & 0 & -\frac{-1+T}{T} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{T^2} & -\frac{1}{T} & 0 & \frac{-1+T}{T^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{T^2} & -\frac{1}{T} & 0 & 0 & \frac{-1+T}{T^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & -\frac{-1+T}{T} \\ 0 & 0 & 0 & 0 & \frac{1}{T^2} & -\frac{1}{T} & 0 & \frac{-1+T}{T^2} & 0 \\ 0 & 0 & \frac{-1+T}{T^2} & 0 & 0 & \frac{1}{T^2} & -1 & 0 & \frac{-1+T}{T} \\ 0 & 0 & 0 & 0 & 0 & \frac{-1+T}{T^2} & \frac{1}{T^2} & -\frac{1}{T} & 0 \\ 0 & \frac{-1+T}{T^2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{T^2} & -\frac{1}{T} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1+T}{T} & 0 & \frac{1}{T} \end{pmatrix}$$

In[∘]:= Factor$\left[\text{mat} + \left(\text{mat}0^{\mathsf{T}} \mathbin{/.} T \to T^{-1}\right)\right]$

Out[∘]=

```
{{0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0},
 {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0}}
```

In[∘]:= Det[mat]

Out[∘]=

$$\frac{-2\,T^3 + 10\,T^4 - 15\,T^5 + 10\,T^6 - 2\,T^7}{T^{14}}$$

In[∘]:= Alexander[K][T]

Out[∘]=

$$-15 - \frac{2}{T^2} + \frac{10}{T} + 10\,T - 2\,T^2$$