Pensieve header: A talk and a program about Archibald- and Γ-calculus and the Halacheva map between them. Continued pensieve://2021-03/.

Title. I Still don’t Understand the Alexander Polynomial

Abstract. As an algebraic knot theorist, I still don’t understand the Alexander polynomial. There are two conventions as for how to present tangle theory in algebra: one may name the strands of a tangle, or one may name their ends. The distinction might seem too minor to matter, yet it leads to a completely different view of the set of tangles as an algebraic structure. There are lovely formulas for the Alexander polynomial as viewed from either perspective, and they even agree where they meet. But the “strands” formulas know about strand doubling while the “ends” ones don’t, and the “ends” formulas know about skein relations while the “strands” ones don’t. There ought to be a common generalization, but I don’t know what it is.

General

In[∘]:=
```
Xp_{a_,b_} := Xp[a, b]; Xm_{a_,b_} := Xm[a, b];
```

In[∘]:=
```
SXForm[L_] := SXForm[
   Skeleton[L],
   Times @@ PD[L] /.
    X[i_, j_, k_, l_] :> If[PositiveQ[X[i, j, k, l]], Xp[l, i], Xm[j, i]]
  ];
Z[L_] := Z[Identity, L];
Z[χ_, L_] := Module[{s, z},
   {s, z} = List @@ SXForm[L];
   z = χ[z];
   Do[z = z // dm[s⟦c, 1⟧, s⟦c, k⟧ , s⟦c, 1⟧], {c, Length[s]}, {k, 2, Length[s⟦c⟧]}];
   z
  ];
```

In[∘]:=
```
dA[a_, rest__][α_] := α // dA[a] // dA[rest];
dA[l_List] := dA @@ l;
dA[All][α_] := α // dA[dL[α]];
dS[a_, rest__][α_] := α // dS[a] // dS[rest];
dS[l_List] := dS @@ l;
dS[All][α_] := α // dS[dL[α]];
```

Γ-Calculus

In[∘]:=
```
ΓSimp = Factor; SetAttributes[ΓCollect, Listable];
ΓCollect[Γ[ω_, σ_, λ_]] := ΓCollect[ΓSimp][Γ[ω, σ, λ]];
ΓCollect[simp_][Γ[ω_, σ_, λ_]] := Γ[simp[ω], simp[σ],
   Collect[λ, h_, Collect[#, t_, simp] &]];
dL[Γ[_, _, λ_]] := Union[Cases[λ, (h | t)a_ ⧴ a, Infinity]];
Γ[ω1_, _, _][ω] := ω1;
Γ[ω_, σ_, λ_][Σ] := (∂h# σ) & /@ dL[Γ[ω, σ, λ]];
Γ[ω_, σ_, λ_][A] := Module[{S = dL[Γ[ω, σ, λ]]}, Outer[ΓSimp[(∂t#1h#2 λ)] &, S, S]];
ΓForm[Γ[ω_, σ_, λ_]] := Module[{S, M},
   S = dL[Γ[ω, σ, λ]];
   M = Γ[ω, σ, λ][A] // Transpose;
   PrependTo[M, s# & /@ S];
   M = Join[
     {Prepend[s# & /@ S, ω]},
     Transpose[M],
     {Prepend[Γ[ω, σ, λ][Σ], "Γ"]}
    ];
   MatrixForm[M]
  ];
ΓForm[else_] := else /. Γ[ω_, σ_, λ_] ⧴ ΓForm[Γ[ω, σ, λ]];
Format[Γ[ω_, σ_, λ_], StandardForm] := ΓForm[Γ[ω, σ, λ]];
```

In[∘]:=
```
Γ /: Γ[ω1_, σ1_, μ1_] == Γ[ω2_, σ2_, μ2_] := Module[
  {S},
  S = dL[Γ[ω1, σ1, μ1]] ⋃ dL[Γ[ω2, σ2, μ2]];
  (ω1 == ω2) && (And @@ ((∂h# σ1 == ∂h# σ2) & /@ S)) && (
    And @@ Flatten[Outer[
       (∂t#1,h#2 μ1 == ∂t#1,h#2 μ2) &,
       S, S
      ]]
   )
 ]
```

In[∘]:=

```
Γ /: Γ[ω1_, σ1_, λ1_] Γ[ω2_, σ2_, λ2_] := Γ[ω1 * ω2, σ1 + σ2, λ1 + λ2];
dm_{i_ j_ →k_}[Γ[ω_, σ_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
   ( α β θ ; γ δ ϵ ; ϕ ψ Ξ ) = ( ∂_{t_i,h_i} λ  ∂_{t_i,h_j} λ  ∂_{t_i} λ ; ∂_{t_j,h_i} λ  ∂_{t_j,h_j} λ  ∂_{t_j} λ ; ∂_{h_i} λ  ∂_{h_j} λ  λ ) /. (t | h)_{i|j} → 0;
   ΓCollect[Γ[(μ = 1 - β) ω,
       h_k (∂_{h_i} σ) (∂_{h_j} σ) + (σ /. h_{i|j} → 0),
       {t_k, 1}.( γ + α δ / μ   ϵ + δ θ / μ ; ϕ + α ψ / μ   Ξ + θ ψ / μ ).{h_k, 1}
      ]] /. {T_i → T_k, T_j → T_k, b_i → b_k, b_j → b_k} // ΓCollect
  ];
dm[a_, b_, c_][Γ[ω_, σ_, λ_]] := dm_{ab→c}[Γ[ω, σ, λ]];
dη[a_][γ_Γ] := γ /. {(h | t)_a → 0, T_a → 1};
```

In[∘]:=

```
tr[a_][Γ[ω_, σ_, λ_]] := Module[{α, θ, ψ, Ξ},
   ( α θ ; ψ Ξ ) = ( ∂_{t_a,h_a} λ  ∂_{t_a} λ ; ∂_{h_a} λ  λ ) /. (t | h)_a → 0;
   Γ[ω (1 - α), σ /. h_a → 0, Ξ + ψ * θ / (1 - α)] // ΓCollect];
```

In[∘]:=

```
FullStitch[γ1_Γ, γ2_Γ] := Module[{S1, S2, S, γ, τ},
   S = (S1 = dL[γ1]) ⋃ (S2 = dL[γ2]);
   γ = γ1 (Times @@ (Γ /@ ϵ /@ Complement[S, S1]));
   γ *= (γ2 /. {h_a_ :→ h_τ[a], t_a_ :→ t_τ[a], T_a_ :→ T_τ[a]})
     (Times @@ (Γ /@ ϵ /@ τ /@ Complement[S, S2]));
   Do[
    γ = γ // dm[s, τ[s], s],
    {s, S}
   ];
   γ
  ];
Γ /: γ1_Γ ** γ2_Γ := Module[{S1, S2, S, γ1p, γ2p},
   S = (S1 = dL[γ1]) ⋃ (S2 = dL[γ2]);
   γ1p = γ1 (Times @@ (Γ /@ ϵ /@ Complement[S, S1]));
   γ2p = γ2 (Times @@ (Γ /@ ϵ /@ Complement[S, S2]));
   Γ[
    γ1p[ω] * γ2p[ω],
    (γ1p[Σ] γ2p[Σ]).(h_# & /@ S),
    (t_# & /@ S).(γ2p[A].γ1p[A]).(h_# & /@ S)
   ]
  ];
```

In[•]:=
```
Γ /: Γ[ω_, σ_, λ_]^-1 := Module[{S = dL[Γ[ω, σ, λ]]},
   Γ[
    ω^-1, Collect[σ, h_, (1 / #) &],
    (t_# & /@ S).Inverse[Outer[ΓSimp[(∂_{t_#1 h_#2} λ)] &, S, S]].(h_# & /@ S)
   ]
  ];
```

In[•]:=
```
dA[a_][Γ[ω_, σ_, λ_]] := Module[
   {α, θ, ϕ, Ξ, σa},
   ( α  θ )   ( ∂_{t_a,h_a} λ   ∂_{t_a} λ )
   ( ϕ  Ξ ) = ( ∂_{h_a} λ        λ        ) /. (h | t)_a → 0;
   σa = ∂_{h_a} σ;
   ΓCollect[Γ[
     α ω / σa,
     ((σ /. h_a → 0) + h_a / σa),
     {t_a, 1}.( 1     θ         ).{h_a, 1} / α
              ( -ϕ   α Ξ - ϕ θ )
    ]]
  ];
dS[a_][γ_Γ] := ΓCollect[dA[a][γ] /. {T_a → 1 / T_a, b_a → -b_a}];
```

In[•]:=
```
Mirror[γ_Γ] := Module[{γ1},
   γ1 = γ // (dS @@ dL[γ]);
   γ1[[3]] = γ1[[3]] /. {t_a_ :→ h_a, h_a_ :→ t_a};
   γ1];
```

In[•]:=
```
tσ[rules___Rule][γ_Γ] := ΓCollect[
   γ /. {t_u_ :→ t_{u /. {rules}}, T_u_ :→ T_{u /. {rules}}, b_u_ :→ b_{u /. {rules}}}
  ];
hσ[rules___Rule][γ_Γ] := ΓCollect[γ /. h_x_ :→ h_{x /. {rules}}];
```

In[•]:=
```
SetAttributes[Γ, Listable];
Γ[p_Times | p_NonCommutativeMultiply] := Γ /@ p;
Γ[ϵ[a_]] := Γ[1, h_a, h_a t_a];
Γ[Xp[a_, b_]] := Γ[1, h_a + h_b T_a, {t_a, t_b}.( 1  1 - T_a ).{h_a, h_b}];
                                                ( 0  T_a     )
Γ[Xm[a_, b_]] := Γ[Xp[a, b]] /. T_a → 1 / T_a;
```

In[∘]:=
```
MVA[Γ, L_Link] := Module[{Hs, ω, σ, μ, A},
  {ω, σ, μ} = List @@ Z[Γ, L];
  Hs = Rest[h_# & /@ (First /@ Skeleton[L])];
  A = Outer[Coefficient[μ, #1 * #2] &, Hs, Hs /. h_a_ :> t_a];
  Factor[ω Det[A - IdentityMatrix@Length@Hs] / (1 - T_Skeleton[L][[1,1]])]
 ]
```

𝒜-Calculus

In[∘]:=
```
WP[Wedge[u___], Wedge[v___]] := Signature[{u, v}] * Wedge @@ Sort[{u, v}];
WP[0, _] = WP[_, 0] = 0;
WP[A_, B_] :=
  Expand[Distribute[A ** B] /. (a_. * u_Wedge) ** (b_. * v_Wedge) :> a b WP[u, v]];
```

In[∘]:= `WP[Wedge[x], Wedge[y]]`

Out[∘]= $x \wedge y$

In[∘]:= `WP[Wedge[y], Wedge[x]]`

Out[∘]= $-(x \wedge y)$

In[∘]:=
```
WExp[A_] := Module[{s = Wedge[], t = Wedge[], k = 0},
  While[t =!= 0, s += (t = Expand[WP[t, A] / (++k)])]; s]
```

In[∘]:= `A = Sum[a_i (x_i ∧ y_i), {i, 2}]`

Out[∘]= $a_1\, x_1 \wedge y_1 + a_2\, x_2 \wedge y_2$

In[∘]:= `t = Wedge[]`

Out[∘]= Wedge[]

In[∘]:= `WP[t, A]`

Out[∘]= $a_1\, x_1 \wedge y_1 + a_2\, x_2 \wedge y_2$

In[∘]:= `WExp[Sum[a_i (x_i ∧ y_i), {i, 4}]]`

Out[∘]= $\text{Wedge}[\,] + a_1\, x_1 \wedge y_1 + a_2\, x_2 \wedge y_2 + a_3\, x_3 \wedge y_3 + a_4\, x_4 \wedge y_4 - a_1 a_2\, x_1 \wedge x_2 \wedge y_1 \wedge y_2 - a_1 a_3\, x_1 \wedge x_3 \wedge y_1 \wedge y_3 - a_1 a_4\, x_1 \wedge x_4 \wedge y_1 \wedge y_4 - a_2 a_3\, x_2 \wedge x_3 \wedge y_2 \wedge y_3 - a_2 a_4\, x_2 \wedge x_4 \wedge y_2 \wedge y_4 - a_3 a_4\, x_3 \wedge x_4 \wedge y_3 \wedge y_4 - a_1 a_2 a_3\, x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3 - a_1 a_2 a_4\, x_1 \wedge x_2 \wedge x_4 \wedge y_1 \wedge y_2 \wedge y_4 - a_1 a_3 a_4\, x_1 \wedge x_3 \wedge x_4 \wedge y_1 \wedge y_3 \wedge y_4 - a_2 a_3 a_4\, x_2 \wedge x_3 \wedge x_4 \wedge y_2 \wedge y_3 \wedge y_4 + a_1 a_2 a_3 a_4\, x_1 \wedge x_2 \wedge x_3 \wedge x_4 \wedge y_1 \wedge y_2 \wedge y_3 \wedge y_4$

In[∘]:=
```
c_{x_,y_}[w_Wedge] := Module[{i, j},
  {i} = FirstPosition[w, x, {0}]; {j} = FirstPosition[w, y, {0}];
  Which[
   (i == 0) ∧ (j == 0), w,
   (i == 0) ∨ (j == 0), 0,
   True, (-1)^(i+j+If[i>j,1,0]) Delete[w, {{i}, {j}}]
  ]];
c_{x_,y_}[ℰ_] := ℰ /. w_Wedge :> c_{x,y}[w]
```

In[∘]:= $c_{x_6,x_5}[x_1 \wedge x_2 \wedge x_3 \wedge x_4]$

Out[∘]= $x_1 \wedge x_2 \wedge x_3 \wedge x_4$

In[∘]:= lhs = c_{x_4,y_4}[WExp[Sum[a_i ($x_i \wedge y_i$), {i, 4}]]]

Out[∘]= $\text{Wedge}[] - a_4\,\text{Wedge}[] + a_1\,x_1 \wedge y_1 - a_1 a_4\,x_1 \wedge y_1 + a_2\,x_2 \wedge y_2 - a_2 a_4\,x_2 \wedge y_2 + a_3\,x_3 \wedge y_3 - a_3 a_4\,x_3 \wedge y_3 - a_1 a_2\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 + a_1 a_2 a_4\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 - a_1 a_3\,x_1 \wedge x_3 \wedge y_1 \wedge y_3 + a_1 a_3 a_4\,x_1 \wedge x_3 \wedge y_1 \wedge y_3 - a_2 a_3\,x_2 \wedge x_3 \wedge y_2 \wedge y_3 + a_2 a_3 a_4\,x_2 \wedge x_3 \wedge y_2 \wedge y_3 - a_1 a_2 a_3\,x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3 + a_1 a_2 a_3 a_4\,x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3$

In[∘]:= rhs = Expand[(1 - a_4) WExp[Sum[a_i ($x_i \wedge y_i$), {i, 3}]]]

Out[∘]= $\text{Wedge}[] - a_4\,\text{Wedge}[] + a_1\,x_1 \wedge y_1 - a_1 a_4\,x_1 \wedge y_1 + a_2\,x_2 \wedge y_2 - a_2 a_4\,x_2 \wedge y_2 + a_3\,x_3 \wedge y_3 - a_3 a_4\,x_3 \wedge y_3 - a_1 a_2\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 + a_1 a_2 a_4\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 - a_1 a_3\,x_1 \wedge x_3 \wedge y_1 \wedge y_3 + a_1 a_3 a_4\,x_1 \wedge x_3 \wedge y_1 \wedge y_3 - a_2 a_3\,x_2 \wedge x_3 \wedge y_2 \wedge y_3 + a_2 a_3 a_4\,x_2 \wedge x_3 \wedge y_2 \wedge y_3 - a_1 a_2 a_3\,x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3 + a_1 a_2 a_3 a_4\,x_1 \wedge x_2 \wedge x_3 \wedge y_1 \wedge y_2 \wedge y_3$

In[∘]:= lhs == rhs

Out[∘]= True

In[∘]:= lhs = c_{x_3,y_3}[c_{x_4,y_4}[WExp[Sum[a_i ($x_i \wedge y_i$), {i, 4}]]]]

Out[∘]= $\text{Wedge}[] - a_3\,\text{Wedge}[] - a_4\,\text{Wedge}[] + a_3 a_4\,\text{Wedge}[] + a_1\,x_1 \wedge y_1 - a_1 a_3\,x_1 \wedge y_1 - a_1 a_4\,x_1 \wedge y_1 + a_1 a_3 a_4\,x_1 \wedge y_1 + a_2\,x_2 \wedge y_2 - a_2 a_3\,x_2 \wedge y_2 - a_2 a_4\,x_2 \wedge y_2 + a_2 a_3 a_4\,x_2 \wedge y_2 - a_1 a_2\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 + a_1 a_2 a_3\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 + a_1 a_2 a_4\,x_1 \wedge x_2 \wedge y_1 \wedge y_2 - a_1 a_2 a_3 a_4\,x_1 \wedge x_2 \wedge y_1 \wedge y_2$

In[∘]:=
```
n = 4; γ0 = Γ[ω, Sum[h_a σ_a, {a, 1, n}], Sum[Sum[t_a h_b α_{10 a+b}, {a, 1, n}], {b, 1, n}]]
γ0 // tr[2]
```

Out[∘]=

$$\begin{pmatrix} \omega & s_1 & s_2 & s_3 & s_4 \\ s_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ s_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ s_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ s_4 & \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \\ \Gamma & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \end{pmatrix}$$

Out[∘]=

$$\begin{pmatrix} -\omega\,(-1+\alpha_{22}) & s_1 & s_3 & s_4 \\ s_1 & \frac{-\alpha_{11}-\alpha_{12}\,\alpha_{21}+\alpha_{11}\,\alpha_{22}}{-1+\alpha_{22}} & \frac{-\alpha_{13}+\alpha_{13}\,\alpha_{22}-\alpha_{12}\,\alpha_{23}}{-1+\alpha_{22}} & \frac{-\alpha_{14}+\alpha_{14}\,\alpha_{22}-\alpha_{12}\,\alpha_{24}}{-1+\alpha_{22}} \\ s_3 & \frac{-\alpha_{31}+\alpha_{22}\,\alpha_{31}-\alpha_{21}\,\alpha_{32}}{-1+\alpha_{22}} & \frac{-\alpha_{23}\,\alpha_{32}-\alpha_{33}+\alpha_{22}\,\alpha_{33}}{-1+\alpha_{22}} & \frac{-\alpha_{24}\,\alpha_{32}-\alpha_{34}+\alpha_{22}\,\alpha_{34}}{-1+\alpha_{22}} \\ s_4 & \frac{-\alpha_{41}+\alpha_{22}\,\alpha_{41}-\alpha_{21}\,\alpha_{42}}{-1+\alpha_{22}} & \frac{-\alpha_{23}\,\alpha_{42}-\alpha_{43}+\alpha_{22}\,\alpha_{43}}{-1+\alpha_{22}} & \frac{-\alpha_{24}\,\alpha_{42}-\alpha_{44}+\alpha_{22}\,\alpha_{44}}{-1+\alpha_{22}} \\ \Gamma & \sigma_1 & \sigma_3 & \sigma_4 \end{pmatrix}$$

In[∘]:=
```
𝒜[Γ[ω_, _, λ_]] := Expand[ω WExp[Expand[λ] /. t_a_ h_b_ :> ξ_a ∧ x_b]]
```

In[ ]:= **lhs = 𝒜[γ0 // tr[1]] // Simplify**

Out[ ]=
$$-\omega\,(-\text{Wedge}[\,] + \alpha_{22}\, x_2 \wedge \xi_2 + \alpha_{32}\, x_2 \wedge \xi_3 + \alpha_{42}\, x_2 \wedge \xi_4 +$$
$$\alpha_{13}\,\alpha_{21}\, x_3 \wedge \xi_2 + \alpha_{23}\, x_3 \wedge \xi_2 + \alpha_{13}\,\alpha_{31}\, x_3 \wedge \xi_3 + \alpha_{33}\, x_3 \wedge \xi_3 + \alpha_{13}\,\alpha_{41}\, x_3 \wedge \xi_4 + \alpha_{43}\, x_3 \wedge \xi_4 +$$
$$\alpha_{14}\,\alpha_{21}\, x_4 \wedge \xi_2 + \alpha_{24}\, x_4 \wedge \xi_2 + \alpha_{14}\,\alpha_{31}\, x_4 \wedge \xi_3 + \alpha_{34}\, x_4 \wedge \xi_3 + \alpha_{14}\,\alpha_{41}\, x_4 \wedge \xi_4 + \alpha_{44}\, x_4 \wedge \xi_4 +$$
$$\alpha_{13}\,\alpha_{22}\,\alpha_{31}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 - \alpha_{13}\,\alpha_{21}\,\alpha_{32}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 - \alpha_{23}\,\alpha_{32}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 +$$
$$\alpha_{22}\,\alpha_{33}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 + \alpha_{13}\,\alpha_{22}\,\alpha_{41}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 - \alpha_{13}\,\alpha_{21}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 -$$
$$\alpha_{23}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{22}\,\alpha_{43}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{13}\,\alpha_{32}\,\alpha_{41}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{13}\,\alpha_{31}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 - \alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 + \alpha_{32}\,\alpha_{43}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 +$$
$$\alpha_{14}\,\alpha_{22}\,\alpha_{31}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{24}\,\alpha_{32}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 +$$
$$\alpha_{22}\,\alpha_{34}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{14}\,\alpha_{22}\,\alpha_{41}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 -$$
$$\alpha_{24}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{22}\,\alpha_{44}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{14}\,\alpha_{32}\,\alpha_{41}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{14}\,\alpha_{31}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 - \alpha_{34}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 + \alpha_{32}\,\alpha_{44}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 +$$
$$\alpha_{14}\,\alpha_{23}\,\alpha_{31}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{13}\,\alpha_{24}\,\alpha_{31}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{14}\,\alpha_{21}\,\alpha_{33}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 -$$
$$\alpha_{24}\,\alpha_{33}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{13}\,\alpha_{21}\,\alpha_{34}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{23}\,\alpha_{34}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 +$$
$$\alpha_{14}\,\alpha_{23}\,\alpha_{41}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 - \alpha_{13}\,\alpha_{24}\,\alpha_{41}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 -$$
$$\alpha_{24}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{13}\,\alpha_{21}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{23}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 +$$
$$\alpha_{14}\,\alpha_{33}\,\alpha_{41}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 - \alpha_{13}\,\alpha_{34}\,\alpha_{41}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 - \alpha_{14}\,\alpha_{31}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{34}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 + \alpha_{13}\,\alpha_{31}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 + \alpha_{33}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 +$$
$$\alpha_{14}\,\alpha_{23}\,\alpha_{32}\,\alpha_{41}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{13}\,\alpha_{24}\,\alpha_{32}\,\alpha_{41}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{14}\,\alpha_{22}\,\alpha_{33}\,\alpha_{41}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{13}\,\alpha_{22}\,\alpha_{34}\,\alpha_{41}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{14}\,\alpha_{23}\,\alpha_{31}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{13}\,\alpha_{24}\,\alpha_{31}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 +$$
$$\alpha_{14}\,\alpha_{21}\,\alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{24}\,\alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{13}\,\alpha_{21}\,\alpha_{34}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{23}\,\alpha_{34}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 +$$
$$\alpha_{14}\,\alpha_{22}\,\alpha_{31}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{14}\,\alpha_{21}\,\alpha_{32}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{24}\,\alpha_{32}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{22}\,\alpha_{34}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{13}\,\alpha_{22}\,\alpha_{31}\,\alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{13}\,\alpha_{21}\,\alpha_{32}\,\alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 +$$
$$\alpha_{23}\,\alpha_{32}\,\alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{22}\,\alpha_{33}\,\alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 +$$
$$\alpha_{12}\,(\alpha_{41}\,(x_2 \wedge \xi_4 - \alpha_{33}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 - \alpha_{24}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 - \alpha_{34}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 +$$
$$\alpha_{24}\,\alpha_{33}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 - \alpha_{23}\,(x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{34}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4)) +$$
$$\alpha_{21}\,(x_2 \wedge \xi_2 + \alpha_{43}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{34}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{44}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 +$$
$$\alpha_{34}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{33}\,(x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 - \alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4)) +$$
$$\alpha_{31}\,(x_2 \wedge \xi_3 + \alpha_{43}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 - \alpha_{24}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{44}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{24}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{23}\,(-(x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3) + \alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4))) +$$
$$\alpha_{11}\,(\text{Wedge}[\,] - \alpha_{42}\, x_2 \wedge \xi_4 - \alpha_{23}\, x_3 \wedge \xi_2 - \alpha_{33}\, x_3 \wedge \xi_3 - \alpha_{43}\, x_3 \wedge \xi_4 - \alpha_{24}\, x_4 \wedge \xi_2 - \alpha_{34}\, x_4 \wedge \xi_3 -$$
$$\alpha_{44}\, x_4 \wedge \xi_4 + \alpha_{23}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 + \alpha_{24}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 +$$
$$\alpha_{34}\,\alpha_{42}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 + \alpha_{24}\,\alpha_{33}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 - \alpha_{23}\,\alpha_{34}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{24}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 -$$
$$\alpha_{23}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_4 + \alpha_{34}\,\alpha_{43}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 - \alpha_{33}\,\alpha_{44}\, x_3 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{24}\,\alpha_{33}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{23}\,\alpha_{34}\,\alpha_{42}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{22}\,(x_2 \wedge \xi_2 + \alpha_{43}\, x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_4 + \alpha_{34}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{44}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_4 +$$
$$\alpha_{34}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{33}\,(x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3 - \alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4)) -$$
$$\alpha_{32}\,(x_2 \wedge \xi_3 + \alpha_{43}\, x_2 \wedge x_3 \wedge \xi_3 \wedge \xi_4 - \alpha_{24}\, x_2 \wedge x_4 \wedge \xi_2 \wedge \xi_3 + \alpha_{44}\, x_2 \wedge x_4 \wedge \xi_3 \wedge \xi_4 -$$
$$\alpha_{24}\,\alpha_{43}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 + \alpha_{23}\,(-(x_2 \wedge x_3 \wedge \xi_2 \wedge \xi_3) + \alpha_{44}\, x_2 \wedge x_3 \wedge x_4 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4))))$$

In[•]:= **rhs = c_{ξ_1,x_1}[𝒜[γ0]] // Simplify**

Out[•]=

$$-\omega\,(-\mathrm{Wedge}[\,]+\alpha_{22}\,x_2\wedge\xi_2+\alpha_{32}\,x_2\wedge\xi_3+\alpha_{42}\,x_2\wedge\xi_4+$$
$$\alpha_{13}\alpha_{21}\,x_3\wedge\xi_2+\alpha_{23}\,x_3\wedge\xi_2+\alpha_{13}\alpha_{31}\,x_3\wedge\xi_3+\alpha_{33}\,x_3\wedge\xi_3+\alpha_{13}\alpha_{41}\,x_3\wedge\xi_4+\alpha_{43}\,x_3\wedge\xi_4+$$
$$\alpha_{14}\alpha_{21}\,x_4\wedge\xi_2+\alpha_{24}\,x_4\wedge\xi_2+\alpha_{14}\alpha_{31}\,x_4\wedge\xi_3+\alpha_{34}\,x_4\wedge\xi_3+\alpha_{14}\alpha_{41}\,x_4\wedge\xi_4+\alpha_{44}\,x_4\wedge\xi_4+$$
$$\alpha_{13}\alpha_{22}\alpha_{31}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_3-\alpha_{13}\alpha_{21}\alpha_{32}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_3-\alpha_{23}\alpha_{32}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_3+$$
$$\alpha_{22}\alpha_{33}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_3+\alpha_{13}\alpha_{22}\alpha_{41}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4-\alpha_{13}\alpha_{21}\alpha_{42}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4-$$
$$\alpha_{23}\alpha_{42}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4+\alpha_{22}\alpha_{43}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4+\alpha_{13}\alpha_{32}\alpha_{41}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{13}\alpha_{31}\alpha_{42}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4-\alpha_{33}\alpha_{42}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4+\alpha_{32}\alpha_{43}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4+$$
$$\alpha_{14}\alpha_{22}\alpha_{31}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3-\alpha_{14}\alpha_{21}\alpha_{32}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3-\alpha_{24}\alpha_{32}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3+$$
$$\alpha_{22}\alpha_{34}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3+\alpha_{14}\alpha_{22}\alpha_{41}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4-\alpha_{14}\alpha_{21}\alpha_{42}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4-$$
$$\alpha_{24}\alpha_{42}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4+\alpha_{22}\alpha_{44}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4+\alpha_{14}\alpha_{32}\alpha_{41}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{14}\alpha_{31}\alpha_{42}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4-\alpha_{34}\alpha_{42}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4+\alpha_{32}\alpha_{44}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4+$$
$$\alpha_{14}\alpha_{23}\alpha_{31}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3-\alpha_{13}\alpha_{24}\alpha_{31}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3-\alpha_{14}\alpha_{21}\alpha_{33}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3-$$
$$\alpha_{24}\alpha_{33}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3+\alpha_{13}\alpha_{21}\alpha_{34}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3+\alpha_{23}\alpha_{34}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3+$$
$$\alpha_{14}\alpha_{23}\alpha_{41}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4-\alpha_{13}\alpha_{24}\alpha_{41}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4-\alpha_{14}\alpha_{21}\alpha_{43}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4-$$
$$\alpha_{24}\alpha_{43}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4+\alpha_{13}\alpha_{21}\alpha_{44}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4+\alpha_{23}\alpha_{44}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4+$$
$$\alpha_{14}\alpha_{33}\alpha_{41}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4-\alpha_{13}\alpha_{34}\alpha_{41}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4-\alpha_{14}\alpha_{31}\alpha_{43}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{34}\alpha_{43}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4+\alpha_{13}\alpha_{31}\alpha_{44}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4+\alpha_{33}\alpha_{44}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4+$$
$$\alpha_{14}\alpha_{23}\alpha_{32}\alpha_{41}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-\alpha_{13}\alpha_{24}\alpha_{32}\alpha_{41}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{14}\alpha_{22}\alpha_{33}\alpha_{41}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{13}\alpha_{22}\alpha_{34}\alpha_{41}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{14}\alpha_{23}\alpha_{31}\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{13}\alpha_{24}\alpha_{31}\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+$$
$$\alpha_{14}\alpha_{21}\alpha_{33}\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{24}\alpha_{33}\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{13}\alpha_{21}\alpha_{34}\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-\alpha_{23}\alpha_{34}\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+$$
$$\alpha_{14}\alpha_{22}\alpha_{31}\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-\alpha_{14}\alpha_{21}\alpha_{32}\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{24}\alpha_{32}\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{22}\alpha_{34}\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{13}\alpha_{22}\alpha_{31}\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{13}\alpha_{21}\alpha_{32}\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+$$
$$\alpha_{23}\alpha_{32}\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-\alpha_{22}\alpha_{33}\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+$$
$$\alpha_{12}\,(\alpha_{41}\,(x_2\wedge\xi_4-\alpha_{33}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4-\alpha_{24}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4-\alpha_{34}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4+$$
$$\alpha_{24}\alpha_{33}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-\alpha_{23}\,(x_2\wedge x_3\wedge\xi_2\wedge\xi_4+\alpha_{34}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4))+$$
$$\alpha_{21}\,(x_2\wedge\xi_2+\alpha_{43}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4+\alpha_{34}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3+\alpha_{44}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4+$$
$$\alpha_{34}\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{33}\,(x_2\wedge x_3\wedge\xi_2\wedge\xi_3-\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4))+$$
$$\alpha_{31}\,(x_2\wedge\xi_3+\alpha_{43}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4-\alpha_{24}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3+\alpha_{44}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{24}\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{23}\,(-(x_2\wedge x_3\wedge\xi_2\wedge\xi_3)+\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4)))+$$
$$\alpha_{11}\,(\mathrm{Wedge}[\,]-\alpha_{42}\,x_2\wedge\xi_4-\alpha_{23}\,x_3\wedge\xi_2-\alpha_{33}\,x_3\wedge\xi_3-\alpha_{43}\,x_3\wedge\xi_4-\alpha_{24}\,x_4\wedge\xi_2-\alpha_{34}\,x_4\wedge\xi_3-$$
$$\alpha_{44}\,x_4\wedge\xi_4+\alpha_{23}\alpha_{42}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4+\alpha_{33}\alpha_{42}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4+\alpha_{24}\alpha_{42}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4+$$
$$\alpha_{34}\alpha_{42}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4+\alpha_{24}\alpha_{33}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3-\alpha_{23}\alpha_{34}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_3+\alpha_{24}\alpha_{43}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4-$$
$$\alpha_{23}\alpha_{44}\,x_3\wedge x_4\wedge\xi_2\wedge\xi_4+\alpha_{34}\alpha_{43}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4-\alpha_{33}\alpha_{44}\,x_3\wedge x_4\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{24}\alpha_{33}\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{23}\alpha_{34}\alpha_{42}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{22}\,(x_2\wedge\xi_2+\alpha_{43}\,x_2\wedge x_3\wedge\xi_2\wedge\xi_4+\alpha_{34}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3+\alpha_{44}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_4+$$
$$\alpha_{34}\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{33}\,(x_2\wedge x_3\wedge\xi_2\wedge\xi_3-\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4))-$$
$$\alpha_{32}\,(x_2\wedge\xi_3+\alpha_{43}\,x_2\wedge x_3\wedge\xi_3\wedge\xi_4-\alpha_{24}\,x_2\wedge x_4\wedge\xi_2\wedge\xi_3+\alpha_{44}\,x_2\wedge x_4\wedge\xi_3\wedge\xi_4-$$
$$\alpha_{24}\alpha_{43}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4+\alpha_{23}\,(-(x_2\wedge x_3\wedge\xi_2\wedge\xi_3)+\alpha_{44}\,x_2\wedge x_3\wedge x_4\wedge\xi_2\wedge\xi_3\wedge\xi_4))))$$

In[•]:= **Simplify[lhs == rhs]**

Out[•]= True

In[•]:=
```
Import[
 "C:\\drorbn\\AcademicPensieve\\Talks\\Sandbjerg-0810/pA.txt",
 "Text"
]
```

Out[•]=
```
        (* WP: Wedge Product *)
WSort[expr_] := Expand[expr /. w_W :> Signature[w]*Sort[w]];
WP[0, _] = WP[_, 0] = 0;
WP[a_, b_] := WSort[Distribute[a ** b] /.
    (c1_. * w1_W) ** (c2_. * w2_W) :> c1 c2 Join[w1, w2]];

        (* IM: Interior Multiplication *)
IM[{}, expr_] := expr;
IM[i_, w_W] := If[FreeQ[w, i], 0,
    -(-1)^Position[w, i][[1,1]]*DeleteCases[w, i] ];
IM[{is___, i_}, w_W] := IM[{is}, IM[i, w]];
IM[is_List, expr_] := expr /. w_W :> IM[is, w]

        (* pA on Crossings *)
pA[Xp[i_,j_,k_,l_]] := AHD[(t[i]==t[k])(t[j]==t[l]), {i,l}, W[j,k],
    W[l,i] + (t[i]-1)W[l,j] - t[l]W[l,k] + W[i,j] + t[l]W[j,k] ];
pA[Xm[i_,j_,k_,l_]] := AHD[(t[i]==t[k])(t[j]==t[l]), {i,j}, W[k,l],
    t[j]W[i,j] - t[j]W[i,l] + W[j,k] + (t[i]-1)W[j,l] + W[k,l] ]

        (* Variable Equivalences *)
ReductionRules[Times[]] = {};
ReductionRules[Equal[a_, b__]] := (# -> a)& /@ {b};
ReductionRules[eqs_Times] := Join @@ (ReductionRules /@ List@@eqs)

        (* AHD: Alexander Half Densities *)
AHD[eqs_, is_, -os_, p_] := AHD[eqs, is, os, Expand[-p]];
AHD /: Reduce[AHD[eqs_, is_, os_, p_]] :=
  AHD[eqs, Sort[is], WSort[os], WSort[p /. ReductionRules[eqs]]];
AHD /: AHD[eqs1_,is1_,os1_,p1_] AHD[eqs2_,is2_,os2_,p2_] := Module[
  {glued = Intersection[Union[is1, is2], List@@Union[os1, os2]]},
  Reduce[AHD[
    eqs1*eqs2 //. eq1_Equal*eq2_Equal /;
      Intersection[List@@eq1, List@@eq2] =!= {} :> Union[eq1, eq2],
    Complement[Union[is1, is2], glued],
    IM[glued, WP[os1, os2]],
    IM[glued, WP[p1, p2]]
]] ]

        (* pA on Circuit Diagrams *)
pA[cd_CircuitDiagram, eqs___] := pA[cd, {}, AHD[Times[eqs], {}, W[], W[]]];
pA[cd_CircuitDiagram, done_, ahd_AHD] := Module[
  {pos = First[Ordering[Length[Complement[List @@ #, done]] & /@ cd]]},
  pA[Delete[cd, pos], Union[done, List @@ cd[[pos]]], ahd*pA[cd[[pos]]]]
];
pA[CircuitDiagram[], _, ahd_AHD] := ahd
```

In[∘]:=
```
Expand[(Xp_{a,b} // Γ)[[3]]] /. t_a_ h_b_ :> ξ_a ∧ x_b /. {ξ_a → ξ_l, ξ_b → ξ_i, x_a → x_j, x_b → x_k, T_a → T_l}
```

Out[∘]= $T_l\, \xi_i \wedge x_k + \xi_l \wedge x_j + \xi_l \wedge x_k - T_l\, \xi_l \wedge x_k$

In[∘]:=
```
Expand[(Xm_{a,b} // Γ)[[3]]] /. t_a_ h_b_ :> ξ_a ∧ x_b /. {ξ_a → ξ_j, ξ_b → ξ_i, x_a → x_l, x_b → x_k, T_a → T_j}
```

Out[∘]= $\frac{\xi_i \wedge x_k}{T_j} + \xi_j \wedge x_k - \frac{\xi_j \wedge x_k}{T_j} + \xi_j \wedge x_l$

The $\mathcal{A}$-invariants of the crossings.

In[∘]:=
```
𝒜[X_{i_,j_,k_,l_}] :=
  𝒜[{i, l}, {j, k}, (T_i == T_k) (T_l == T_j), WExp[T_l ξ_i ∧ x_k + ξ_l ∧ x_j + ξ_l ∧ x_k - T_l ξ_l ∧ x_k]];
𝒜[X̄_{i_,j_,k_,l_}] := 𝒜[{i, j}, {k, l}, (T_i == T_k) (T_j == T_l),
  WExp[T_j^{-1} ξ_i ∧ x_k + ξ_j ∧ x_k - T_j^{-1} ξ_j ∧ x_k + ξ_j ∧ x_l]]
```

In[∘]:= `𝒜[X_{i,j,k,l}]`

Out[∘]= $\mathcal{A}[\{i, l\}, \{j, k\}, (T_i == T_k)\,(T_l == T_j),$
$\text{Wedge}[\,] - x_j \wedge \xi_l - T_l\, x_k \wedge \xi_i - x_k \wedge \xi_l + T_l\, x_k \wedge \xi_l + T_l\, x_j \wedge x_k \wedge \xi_i \wedge \xi_l]$

In[∘]:= `𝒜[X̄_{i,j,k,l}]`

Out[∘]= $\mathcal{A}[\{i, j\}, \{k, l\}, (T_i == T_k)\,(T_j == T_l),$
$\text{Wedge}[\,] - \frac{x_k \wedge \xi_i}{T_j} - x_k \wedge \xi_j + \frac{x_k \wedge \xi_j}{T_j} - x_l \wedge \xi_j - \frac{x_k \wedge x_l \wedge \xi_i \wedge \xi_j}{T_j}]$

RR - Reduction Rules for Variable Equivalences.

In[∘]:=
```
RR[Times[]] = {};
RR[Equal[a_, b__]] := (# → a) & /@ {b};
RR[eqs_Times] := Join @@ (RR /@ List @@ eqs)
```

In[∘]:= `RR[(T_i == T_k) (T_l == T_j)]`

Out[∘]= $\{T_k \to T_i,\ T_j \to T_l\}$

In[∘]:=
```
𝒜 /: 𝒜[is1_, os1_, e1_, w1_] 𝒜[is2_, os2_, e2_, w2_] :=
  𝒜[is1 ⋃ is2, os1 ⋃ os2, e1 e2, WP[w1, w2]]
```

In[∘]:= `𝒜[X_{2,5,4,1}] 𝒜[X_{3,7,6,5}] 𝒜[X_{6,9,8,4}] // Short`

Out[∘]//Short= $\mathcal{A}[\{1, 2, 3, 4, 5, 6\}, \{4, 5, \ll 2 \gg, 8, 9\}, \ll 1 \gg,$
$\ll 331 \gg + T_1 T_4 T_5\, x_4 \wedge x_5 \wedge x_6 \wedge x_7 \wedge x_8 \wedge x_9 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \wedge \xi_4 \wedge \xi_5 \wedge \xi_6]$

```
c@𝒜[is_, os_, eqs_, w_] :=
```

In[∘]:= `skel1 = <|x_1 → ξ_1, ξ_1 → x_1, x_3 → T_1|>`

Out[∘]= $\langle| x_1 \to \xi_1,\ \xi_1 \to x_1,\ x_3 \to T_1 |\rangle$

In[∘]:= **skel1[x_3]**

Out[∘]= T_1

In[∘]:= **skel2 = <|$x_2 \to \xi_2$, $\xi_2 \to x_2$, $x_4 \to T_1$|>**

Out[∘]= $\langle| x_2 \to \xi_2,\ \xi_2 \to x_2,\ x_4 \to T_1 |\rangle$

In[∘]:= **Merge[{skel1, skel2}, First]**

Out[∘]= $\langle| x_1 \to \xi_1,\ \xi_1 \to x_1,\ x_3 \to T_1,\ x_2 \to \xi_2,\ \xi_2 \to x_2,\ x_4 \to T_1 |\rangle$

In[∘]:= **KeyUnion[{skel1, skel2}]**

Out[∘]= $\{\langle| x_1 \to \xi_1,\ \xi_1 \to x_1,\ x_3 \to T_1,\ x_2 \to$ Missing[KeyAbsent, x_2], $\xi_2 \to$ Missing[KeyAbsent, ξ_2],
$x_4 \to$ Missing[KeyAbsent, x_4] $|\rangle$, $\langle| x_1 \to$ Missing[KeyAbsent, x_1],
$\xi_1 \to$ Missing[KeyAbsent, ξ_1], $x_3 \to$ Missing[KeyAbsent, x_3], $x_2 \to \xi_2,\ \xi_2 \to x_2,\ x_4 \to T_1 |\rangle\}$

In[∘]:= **Join[skel1, skel2]**

Out[∘]= $\langle| x_1 \to \xi_1,\ \xi_1 \to x_1,\ x_3 \to T_1,\ x_2 \to \xi_2,\ \xi_2 \to x_2,\ x_4 \to T_1 |\rangle$

In[∘]:= **Normal[skel1]**

Out[∘]= $\{x_1 \to \xi_1,\ \xi_1 \to x_1,\ x_3 \to T_1\}$

In[∘]:= **x_1 /. skel1**

Out[∘]= ξ_1