Pensieve header: Searching for perturbations of the Heisenberg R-matrix.

$$R_{ij} = \mathbb{O}_{px}\left(e^{(e^t-1)(p_i-p_j)x_j}\right)$$
$$\mathcal{G}(hm_k^{ij}) = e^{-\xi_i\pi_j+(\pi_i+\pi_j)p_k+(\xi_i+\xi_j)x_k}$$

hp[i,j] is the positive Heisenberg R-matrix.

```
In[]:= hp /: P_𝒫 ** hp[i_, j_] := Simplify[P /. {x_i → x_i - (T - 1) x_j, x_j → T x_j}];
       hp /: hp[i_, j_] ** P_𝒫 := Simplify[P /. {p_j → p_j - (T - 1) (p_i - p_j)}]
```

```
In[]:= 𝒫[P[x1, x2, x3]] ** hp[1, 2]
Out[]= 𝒫[P[x1 - (-1 + T) x2, T x2, x3]]
```

```
In[]:= 𝒫[P[x1, x2, x3]] ** hp[1, 2] ** hp[1, 3]
Out[]= 𝒫[P[x1 - (-1 + T) (x2 + x3), T x2, T x3]]
```

```
In[]:= 𝒫[P[x1, x2, x3]] ** hp[1, 2] ** hp[1, 3] ** hp[2, 3]
Out[]= 𝒫[P[x1 - (-1 + T) (x2 + x3), T (x2 - (-1 + T) x3), T^2 x3]]
```

```
In[]:= 𝒫[P[x1, x2, x3]] ** hp[2, 3] ** hp[1, 3] ** hp[1, 2]
Out[]= 𝒫[P[x1 - (-1 + T) (x2 + x3), T (x2 - (-1 + T) x3), T^2 x3]]
```

```
In[]:= hp[1, 2] ** hp[1, 3] ** hp[2, 3] ** 𝒫[P[p1, p2, p3]]
Out[]= 𝒫[P[p1, -(-1 + T) p1 + T p2, -(-1 + T) p1 + T (-(-1 + T) p2 + T p3)]]
```

```
In[]:= hp[2, 3] ** hp[1, 3] ** hp[1, 2] ** 𝒫[P[p1, p2, p3]]
Out[]= 𝒫[P[p1, -(-1 + T) p1 + T p2, -(-1 + T) p1 + T (-(-1 + T) p2 + T p3)]]
```

```
In[]:= F_{i_,j_} := 𝒫[F[p_i, x_i, p_j, x_j]];
       eq = F_{1,2} ** hp[1, 3] ** hp[2, 3] + hp[1, 2] ** F_{1,3} ** hp[2, 3] + hp[1, 2] ** hp[1, 3] ** F_{2,3} -
         F_{2,3} ** hp[1, 3] ** hp[1, 2] - hp[2, 3] ** F_{1,3} ** hp[1, 2] - hp[2, 3] ** hp[1, 3] ** F_{1,2}
Out[]= -𝒫[F[p1, x1, p2, x2]] + 𝒫[F[p1, x1, p3, T x3]] -
        𝒫[F[p1, x1 + x2 - T x2, p2 - T p2 + T p3, x3]] + 𝒫[F[p1, x1 - (-1 + T) T x3, p2, x2 - (-1 + T) x3]] -
        𝒫[F[p2, T x2, p3, T x3]] + 𝒫[F[p1 - T p1 + T p2, x2, p1 - T p1 + T p3, x3]]
```

```
In[]:= Expand[eq /. 𝒫[F[p1_, x1_, p2_, x2_]] :→ (p1 - p2) x2]
Out[]= 0
```

```
In[]:= Expand[eq /. 𝒫[F[p1_, x1_, p2_, x2_]] :→ p1 x1 - T p1 x2]
Out[]= 0
```

```
In[]:= AllMonomials[{}, 0] = {1};
       AllMonomials[{}, d_Integer] /; d > 0 := {};
       AllMonomials[{v_, vs___}, d_Integer] :=
         Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
       AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
```

In[∘]:=
```
AllMonomials[{p1, p2, p3}, {2}]
```

Out[∘]= $\{1, p1, p2, p3, p1^2, p1\,p2, p1\,p3, p2^2, p2\,p3, p3^2\}$

In[∘]:=
```
Basis[n_, m_] := Flatten@
   Outer[Times, AllMonomials[Table[p_j, {j, n}], m], AllMonomials[Table[x_j, {j, n}], m]];
Basis[n_, {m_}] := Flatten@Table[Basis[n, k], {k, 1, m}]
```

In[∘]:=
```
{Basis[2, 2], Basis[2, {2}]}
```

Out[∘]= $\{\{p_1^2 x_1^2, p_1^2 x_1 x_2, p_1^2 x_2^2, p_1 p_2 x_1^2, p_1 p_2 x_1 x_2, p_1 p_2 x_2^2, p_2^2 x_1^2, p_2^2 x_1 x_2, p_2^2 x_2^2\},$
$\{p_1 x_1, p_1 x_2, p_2 x_1, p_2 x_2, p_1^2 x_1^2, p_1^2 x_1 x_2, p_1^2 x_2^2, p_1 p_2 x_1^2, p_1 p_2 x_1 x_2, p_1 p_2 x_2^2, p_2^2 x_1^2, p_2^2 x_1 x_2, p_2^2 x_2^2\}\}$

In[∘]:=
```
MatrixForm[mat = Table[
   Coefficient[
     eq /. 𝒫[F[p1_, x1_, p2_, x2_]] :→ Expand[in /. {p₁ → p1, p₂ → p2, x₁ → x1, x₂ → x2}],
     out
    ] /. (p | x)_ → 0,
   {in, Basis[2, {2}]}, {out, Basis[3, {2}]}
  ]]
```

Out[∘]//MatrixForm=

0	0	$T - T^2$	0	0	0	0	0	0	0	0	0	0	0
0	0	$1 - T$	0	0	0	0	0	0	0	0	0	0	0
0	$1 - T$	0	$-1 + T$	$-1 + 2T - T^2$	$T - T^2$	$1 - T$	$-T + T^2$	0	0	0	0	0	0
0	0	$1 - T$	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	$-2 + 2T$	$2T - 2T^2$	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	$2 - 2T$
0	0	0	0	0	0	0	0	0	0	0	0	$1 - 2T + T^2$	0
0	0	0	0	0	0	0	0	0	0	0	0	0	$1 - 2T +$
0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	$1 - 2T + T^2$	0
0	0	0	0	0	0	0	0	0	0	0	0	0	$1 - 2T +$
0	0	0	0	0	0	0	0	0	0	0	0	0	0

In[∘]:=
```
Simplify[T NullSpace[Transpose@mat].Basis[2, {2}]]
```

Out[∘]= $\{p_1 x_2 (-p_1 x_1 + T p_2 x_2),$
$\frac{1}{2} p_1 x_2 \left(2 T p_2 x_1 + p_1 \left((1 - 3T) x_1 + (-1 + T) T x_2\right)\right), -p_1 x_1 + T p_2 x_2, p_1 (-x_1 + T x_2)\}$

In[∘]:=
```
MatrixForm[mat = Table[
    Coefficient[
      eq /. 𝒫[F[p1_, x1_, p2_, x2_]] :→ Expand[in /. {p₁ → p1, p₂ → p2, x₁ → x1, x₂ → x2}],
      out
     ] /. (p | x)_ → 0,
    {in, Basis[2, {4}]}, {out, Basis[3, {4}]}
   ]]
```

Out[∘]//MatrixForm=

(⋯ 1 ⋯)

large output | show less | show more | show all | set size limit...

In[∘]:=
```
Factor[T³ NullSpace[Transpose@mat].Basis[2, {4}]]
```

Out[∘]=
$$\left\{\frac{1}{2} p_1 x_2 (-p_1 x_1 + T p_2 x_2) \left(2 p_1^2 x_1^2 + 3 p_1^2 x_1 x_2 - 3 T p_1^2 x_1 x_2 + 2 T p_1 p_2 x_1 x_2 + 2 p_1^2 x_2^2 - 4 T p_1^2 x_2^2 + 2 T^2 p_1^2 x_2^2 + 3 T p_1 p_2 x_2^2 - 3 T^2 p_1 p_2 x_2^2 + 2 T^2 p_2^2 x_2^2\right),\right.$$
$$-T p_1 x_2 (-p_1 x_1 - p_1 x_2 + T p_1 x_2 - T p_2 x_2) (-p_1 x_1 + T p_2 x_2), T^2 p_1 x_2 (-p_1 x_1 + T p_2 x_2),$$
$$\frac{1}{2} T^2 p_1 x_2 \left(p_1 x_1 - 3 T p_1 x_1 + 2 T p_2 x_1 - T p_1 x_2 + T^2 p_1 x_2\right),$$
$$\left. T^2 (-p_1 x_1 + T p_2 x_2), T^2 p_1 (-x_1 + T x_2)\right\}$$

In[∘]:=
```
Unprotect[SeriesData];
SeriesData /: Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
Protect[SeriesData];
```

In[∘]:=
```
𝔼n_[Q_, P_]^m_[L___] ^:= 𝔼m[Q, P] /. Flatten@Table[{pᵢ → p{L}〚i〛, xᵢ → x{L}〚i〛}, {i, n}]
```

In[∘]:=
```
Unprotect[NonCommutativeMultiply];
𝔼n_[Q1_, P1_] ** 𝔼n_[Q2_, P2_] := Module[{i, k, pp, xx},
   Expand /@ 𝔼n[
     Q1 + Q2 + Sum[(∂xᵢ Q1) (∂pᵢ Q2), {i, n}],
     Expand[(P1 /. Table[xᵢ → xᵢ + xxᵢ + ∂pᵢ Q2, {i, n}])
        (P2 /. Table[pᵢ → pᵢ + ppᵢ + ∂xᵢ Q1, {i, n}])] //. ppᵢ_^k_. xxᵢ_^k_. → k! /. (pp | xx)_ → 0
    ]]
```

In[◦]:=
```
RandomPolynomial[specs___] :=
  Basis[specs].Table[RandomInteger[{-9, 9}], Length@Basis[specs]];
Clear[RE];
RE[k_] := RE[k] = 𝔼2[RandomPolynomial[2, 1],
    1 + ϵ RandomPolynomial[2, {2}] + ϵ^2 RandomPolynomial[2, {4}] + O[ϵ]^3];
RE[
 1]
```

Out[◦]=
$$\mathbb{E}_2\Big[6 p_1 x_1 - 8 p_2 x_1 - 7 p_1 x_2 + 8 p_2 x_2,\ 1 + \big(p_1 x_1 - 6 p_2 x_1 - 5 p_1^2 x_1^2 + 2 p_1 p_2 x_1^2 - 5 p_2^2 x_1^2 + 6 p_1 x_2 - 4 p_2 x_2 - 7 p_1^2 x_1 x_2 - 7 p_1 p_2 x_1 x_2 + 6 p_2^2 x_1 x_2 - 8 p_1^2 x_2^2 - 6 p_1 p_2 x_2^2 + 9 p_2^2 x_2^2\big) \epsilon + \big(5 p_1 x_1 + p_2 x_1 + p_1^2 x_1^2 - p_1 p_2 x_1^2 + 7 p_2^2 x_1^2 + 7 p_1^3 x_1^3 - 2 p_1^2 p_2 x_1^3 + 5 p_1 p_2^2 x_1^3 - 2 p_2^3 x_1^3 - 3 p_1^4 x_1^4 + 2 p_1^3 p_2 x_1^4 - 9 p_1^2 p_2^2 x_1^4 + 7 p_1 p_2^3 x_1^4 + 3 p_2^4 x_1^4 + 9 p_1 x_2 - p_2 x_2 + 3 p_1^2 x_1 x_2 + 6 p_1 p_2 x_1 x_2 + 4 p_2^2 x_1 x_2 + 6 p_1^3 x_1^2 x_2 - 3 p_1^2 p_2 x_1^2 x_2 + 9 p_1 p_2^2 x_1^2 x_2 + 6 p_2^3 x_1^2 x_2 - 9 p_1^4 x_1^3 x_2 + 8 p_1^3 p_2 x_1^3 x_2 - 7 p_1^2 p_2^2 x_1^3 x_2 + 5 p_1 p_2^3 x_1^3 x_2 - 4 p_2^4 x_1^3 x_2 + 6 p_1^2 x_2^2 - 7 p_1 p_2 x_2^2 + 3 p_2^2 x_2^2 + 4 p_1^3 x_1 x_2^2 - 6 p_1^2 p_2 x_1 x_2^2 + 5 p_1 p_2^2 x_1 x_2^2 - 9 p_2^3 x_1 x_2^2 - p_1^4 x_1^2 x_2^2 + 4 p_1^3 p_2 x_1^2 x_2^2 + 6 p_1^2 p_2^2 x_1^2 x_2^2 + 5 p_1 p_2^3 x_1^2 x_2^2 - 3 p_2^4 x_1^2 x_2^2 + 3 p_1^3 x_2^3 - 9 p_1^2 p_2 x_2^3 - 5 p_1 p_2^2 x_2^3 - 8 p_2^3 x_2^3 - 4 p_1^3 p_2 x_1 x_2^3 - 3 p_1^2 p_2^2 x_1 x_2^3 - 6 p_1 p_2^3 x_1 x_2^3 + 5 p_2^4 x_1 x_2^3 - 3 p_1^3 p_2 x_2^4 - p_1^2 p_2^2 x_2^4 + 3 p_1 p_2^3 x_2^4 + 6 p_2^4 x_2^4\big) \epsilon^2 + O[\epsilon]^3\Big]$$

In[◦]:=
```
l = RE[1] ** RE[2];
r = RE[2] ** RE[3];
l ** RE[3] == RE[1] ** r
```

Out[◦]= True

In[◦]:=
```
R = 𝔼2[(T - 1) (p1 - p2) x2,
    1 + ϵ 1/2 p1 x2 (2 T p2 x1 + p1 ((1 - 3 T) x1 + (-1 + T) T x2)) + O[ϵ]^3] /. p_i_ :→ -p_i
```

Out[◦]=
$$\mathbb{E}_2\Big[(-1+T)(-p_1+p_2)\,x_2,\ 1 - \frac{1}{2}\big(p_1 x_2 (-2 T p_2 x_1 - p_1 ((1-3T) x_1 + (-1+T) T x_2))\big)\epsilon + O[\epsilon]^3\Big]$$

In[◦]:=
```
R^3[1,3]
```

Out[◦]=
$$\mathbb{E}_3\Big[(-1+T)(-p_1+p_3)\,x_3,\ 1 - \frac{1}{2}\big(p_1 x_3 (-2 T p_3 x_1 - p_1 ((1-3T) x_1 + (-1+T) T x_3))\big)\epsilon + O[\epsilon]^2\Big]$$

In[◦]:=
```
R^3[1,2] ** R^3[1,3] ** R^3[2,3]
```

Out[◦]=
$$\mathbb{E}_3\Big[p_1 x_2 - T p_1 x_2 - p_2 x_2 + T p_2 x_2 + p_1 x_3 - T p_1 x_3 + T p_2 x_3 - T^2 p_2 x_3 - p_3 x_3 + T^2 p_3 x_3,\ 1 + \Big(\frac{1}{2} p_1^2 x_1 x_2 - \frac{3}{2} T p_1^2 x_1 x_2 + T p_1 p_2 x_1 x_2 - \frac{1}{2} T p_1^2 x_2^2 + \frac{1}{2} T^2 p_1^2 x_2^2 + \frac{1}{2} p_1^2 x_1 x_3 - \frac{3}{2} T p_1^2 x_1 x_3 + T p_1 p_2 x_1 x_3 - T^2 p_1 p_2 x_1 x_3 + T^2 p_1 p_3 x_1 x_3 + \frac{1}{2} p_1^2 x_2 x_3 - 2 T p_1^2 x_2 x_3 + \frac{3}{2} T^2 p_1^2 x_2 x_3 + T p_1 p_2 x_2 x_3 - 2 T^2 p_1 p_2 x_2 x_3 + T^3 p_1 p_2 x_2 x_3 + \frac{1}{2} T^2 p_2^2 x_2 x_3 - \frac{3}{2} T^3 p_2^2 x_2 x_3 + T^2 p_1 p_3 x_2 x_3 - T^3 p_1 p_3 x_2 x_3 + T^3 p_2 p_3 x_2 x_3 - \frac{1}{2} T p_1^2 x_3^2 + \frac{1}{2} T^2 p_1^2 x_3^2 - \frac{1}{2} T^3 p_2^2 x_3^2 + \frac{1}{2} T^4 p_2^2 x_3^2\Big)\epsilon + O[\epsilon]^2\Big]$$

In[◦]:=
```
R^3[1,2] ** R^3[1,3] ** R^3[2,3] == R^3[2,3] ** R^3[1,3] ** R^3[1,2]
```

Out[◦]= True

In[∘]:= $R = \mathbb{E}_2\left[(T-1)\ (p_1-p_2)\ x_2,\ 1+\epsilon\, p_1\, x_2\ (-p_1\, x_1+T\, p_2\, x_2)+O[\epsilon]^3\right]\ /.\ p_{i\_} \mapsto -p_i$

Out[∘]= $\mathbb{E}_2\left[(-1+T)\ (-p_1+p_2)\ x_2,\ 1-p_1\, x_2\ (p_1\, x_1-T\, p_2\, x_2)\ \epsilon+O[\epsilon]^3\right]$

In[∘]:= $R = \mathbb{E}_2\Big[(T-1)\ (p_1-p_2)\ x_2,$
$\quad 1+\epsilon\,\frac{1}{2}\, p_1\, x_2\ \left(2\, T\, p_2\, x_1+p_1\ \left((1-3\,T)\ x_1+(-1+T)\ T\, x_2\right)\right)+O[\epsilon]^3\Big]\ /.\ p_{i\_} \mapsto -p_i;$
$\text{err} = \epsilon^{-2}\ \text{Normal}\left[\text{Last}\left[R^{3[1,2]} ** R^{3[1,3]} ** R^{3[2,3]}\right]-\text{Last}\left[R^{3[2,3]} ** R^{3[1,3]} ** R^{3[1,2]}\right]\right]$

Out[∘]= $T^2 p_1 p_2 x_2 x_3 - T^3 p_1 p_2 x_2 x_3 + T p_1^3 x_1 x_2 x_3 - 4 T^2 p_1^3 x_1 x_2 x_3 + 3 T^3 p_1^3 x_1 x_2 x_3 + T^2 p_1^2 p_2 x_1 x_2 x_3 -$
$T^3 p_1^2 p_2 x_1 x_2 x_3 - T^2 p_1 p_2^2 x_1 x_2 x_3 + T^3 p_1 p_2^2 x_1 x_2 x_3 + T^2 p_1^2 p_3 x_1 x_2 x_3 - T^3 p_1^2 p_3 x_1 x_2 x_3 - \frac{1}{4} p_1^4 x_1^2 x_2 x_3 +$
$\frac{7}{4} T p_1^4 x_1^2 x_2 x_3 - \frac{15}{4} T^2 p_1^4 x_1^2 x_2 x_3 + \frac{9}{4} T^3 p_1^4 x_1^2 x_2 x_3 - T p_1^3 p_2 x_1^2 x_2 x_3 + 4 T^2 p_1^3 p_2 x_1^2 x_2 x_3 -$
$3 T^3 p_1^3 p_2 x_1^2 x_2 x_3 - T^2 p_1^2 p_2^2 x_1^2 x_2 x_3 + T^3 p_1^2 p_2^2 x_1^2 x_2 x_3 - \frac{1}{2} T^2 p_1^3 x_2^2 x_3 + \frac{1}{2} T^4 p_1^3 x_2^2 x_3 + \frac{3}{2} T^2 p_1^2 p_2 x_2^2 x_3 -$
$2 T^3 p_1^2 p_2 x_2^2 x_3 + \frac{1}{2} T^4 p_1^2 p_2 x_2^2 x_3 - \frac{1}{2} T^2 p_1 p_2^2 x_2^2 x_3 + 3 T^3 p_1 p_2^2 x_2^2 x_3 - \frac{5}{2} T^4 p_1 p_2^2 x_2^2 x_3 - T^3 p_1 p_2 p_3 x_2^2 x_3 +$
$T^4 p_1 p_2 p_3 x_2^2 x_3 + \frac{1}{2} T p_1^4 x_1 x_2^2 x_3 - 2 T^2 p_1^4 x_1 x_2^2 x_3 + \frac{3}{2} T^3 p_1^4 x_1 x_2^2 x_3 + \frac{1}{2} T^2 p_1^3 p_2 x_1 x_2^2 x_3 +$
$T^3 p_1^3 p_2 x_1 x_2^2 x_3 - \frac{3}{2} T^4 p_1^3 p_2 x_1 x_2^2 x_3 - T^3 p_1^2 p_2^2 x_1 x_2^2 x_3 + T^4 p_1^2 p_2^2 x_1 x_2^2 x_3 - \frac{1}{4} T^2 p_1^4 x_2^3 x_3 + \frac{1}{4} T^3 p_1^4 x_2^3 x_3 +$
$\frac{1}{4} T^4 p_1^4 x_2^3 x_3 - \frac{1}{4} T^5 p_1^4 x_2^3 x_3 + \frac{1}{2} T^3 p_1^3 p_2 x_2^3 x_3 - T^4 p_1^3 p_2 x_2^3 x_3 + \frac{1}{2} T^5 p_1^3 p_2 x_2^3 x_3 + \frac{1}{2} T^2 p_1^2 x_3^2 -$
$T^3 p_1^2 x_3^2 + \frac{1}{2} T^4 p_1^2 x_3^2 - T^2 p_1^3 x_1 x_3^2 + 4 T^3 p_1^3 x_1 x_3^2 - 3 T^4 p_1^3 x_1 x_3^2 - \frac{1}{2} T^2 p_1^2 p_2 x_1 x_3^2 + \frac{1}{2} T^4 p_1^2 p_2 x_1 x_3^2 +$
$\frac{1}{2} T^2 p_1^2 p_3 x_1 x_3^2 - 2 T^3 p_1^2 p_3 x_1 x_3^2 + \frac{3}{2} T^4 p_1^2 p_3 x_1 x_3^2 + T^3 p_1 p_2 p_3 x_1 x_3^2 - T^4 p_1 p_2 p_3 x_1 x_3^2 + \frac{1}{4} T p_1^4 x_1^2 x_3^2 -$
$\frac{7}{4} T^2 p_1^4 x_1^2 x_3^2 + \frac{15}{4} T^3 p_1^4 x_1^2 x_3^2 - \frac{9}{4} T^4 p_1^4 x_1^2 x_3^2 + \frac{1}{2} T^2 p_1^3 p_2 x_1^2 x_3^2 - 2 T^3 p_1^3 p_2 x_1^2 x_3^2 + \frac{3}{2} T^4 p_1^3 p_2 x_1^2 x_3^2 +$
$\frac{1}{2} T^2 p_1^3 p_3 x_1^2 x_3^2 - 2 T^3 p_1^3 p_3 x_1^2 x_3^2 + \frac{3}{2} T^4 p_1^3 p_3 x_1^2 x_3^2 + T^3 p_1^2 p_2 p_3 x_1^2 x_3^2 - T^4 p_1^2 p_2 p_3 x_1^2 x_3^2 - \frac{1}{2} T^2 p_1^3 x_2 x_3^2 +$
$T^3 p_1^3 x_2 x_3^2 + \frac{1}{2} T^4 p_1^3 x_2 x_3^2 - T^5 p_1^3 x_2 x_3^2 + 2 T^2 p_1^2 p_2 x_2 x_3^2 - 7 T^3 p_1^2 p_2 x_2 x_3^2 + 8 T^4 p_1^2 p_2 x_2 x_3^2 -$
$3 T^5 p_1^2 p_2 x_2 x_3^2 + 3 T^3 p_1 p_2^2 x_2 x_3^2 - 8 T^4 p_1 p_2^2 x_2 x_3^2 + 5 T^5 p_1 p_2^2 x_2 x_3^2 - \frac{1}{2} T^3 p_1^2 p_3 x_2 x_3^2 + \frac{1}{2} T^5 p_1^2 p_3 x_2 x_3^2 +$
$T^4 p_1 p_2 p_3 x_2 x_3^2 - T^5 p_1 p_2 p_3 x_2 x_3^2 + \frac{1}{4} p_1^4 x_1 x_2 x_3^2 - \frac{3}{2} T p_1^4 x_1 x_2 x_3^2 + \frac{13}{4} T^2 p_1^4 x_1 x_2 x_3^2 -$
$\frac{17}{4} T^3 p_1^4 x_1 x_2 x_3^2 + \frac{9}{2} T^4 p_1^4 x_1 x_2 x_3^2 - \frac{9}{4} T^5 p_1^4 x_1 x_2 x_3^2 + T p_1^3 p_2 x_1 x_2 x_3^2 - \frac{11}{2} T^2 p_1^3 p_2 x_1 x_2 x_3^2 +$
$\frac{25}{2} T^3 p_1^3 p_2 x_1 x_2 x_3^2 - \frac{25}{2} T^4 p_1^3 p_2 x_1 x_2 x_3^2 + \frac{9}{2} T^5 p_1^3 p_2 x_1 x_2 x_3^2 + T^2 p_1^2 p_2^2 x_1 x_2 x_3^2 - 5 T^3 p_1^2 p_2^2 x_1 x_2 x_3^2 +$
$7 T^4 p_1^2 p_2^2 x_1 x_2 x_3^2 - 3 T^5 p_1^2 p_2^2 x_1 x_2 x_3^2 + T^2 p_1^3 p_3 x_1 x_2 x_3^2 - 4 T^3 p_1^3 p_3 x_1 x_2 x_3^2 + 3 T^4 p_1^3 p_3 x_1 x_2 x_3^2 +$
$2 T^3 p_1^2 p_2 p_3 x_1 x_2 x_3^2 - 4 T^4 p_1^2 p_2 p_3 x_1 x_2 x_3^2 + 2 T^5 p_1^2 p_2 p_3 x_1 x_2 x_3^2 + T^4 p_1^2 p_3^2 x_1 x_2 x_3^2 - T^5 p_1^2 p_3^2 x_1 x_2 x_3^2 -$
$\frac{1}{4} T p_1^4 x_2^2 x_3^2 + \frac{3}{4} T^2 p_1^4 x_2^2 x_3^2 - T^3 p_1^4 x_2^2 x_3^2 + T^4 p_1^4 x_2^2 x_3^2 - \frac{3}{4} T^5 p_1^4 x_2^2 x_3^2 + \frac{1}{4} T^6 p_1^4 x_2^2 x_3^2 + T^4 p_1^3 p_2 x_2^2 x_3^2 -$
$2 T^5 p_1^3 p_2 x_2^2 x_3^2 + T^6 p_1^3 p_2 x_2^2 x_3^2 - \frac{1}{4} T^2 p_1^2 p_2^2 x_2^2 x_3^2 + \frac{5}{2} T^3 p_1^2 p_2^2 x_2^2 x_3^2 - 7 T^4 p_1^2 p_2^2 x_2^2 x_3^2 + \frac{15}{2} T^5 p_1^2 p_2^2 x_2^2 x_3^2 -$

$$\frac{11}{4} T^6 p_1^2 p_2^2 x_2^2 x_3^2 - \frac{1}{2} T^3 p_1 p_2^3 x_2^2 x_3^2 + 3 T^4 p_1 p_2^3 x_2^2 x_3^2 - \frac{11}{2} T^5 p_1 p_2^3 x_2^2 x_3^2 + 3 T^6 p_1 p_2^3 x_2^2 x_3^2 - \frac{1}{2} T^3 p_1^3 p_3 x_2^2 x_3^2 +$$
$$\frac{1}{2} T^4 p_1^3 p_3 x_2^2 x_3^2 + \frac{1}{2} T^5 p_1^3 p_3 x_2^2 x_3^2 - \frac{1}{2} T^6 p_1^3 p_3 x_2^2 x_3^2 - \frac{1}{2} T^3 p_1^2 p_2 p_3 x_2^2 x_3^2 + \frac{5}{2} T^4 p_1^2 p_2 p_3 x_2^2 x_3^2 -$$
$$\frac{7}{2} T^5 p_1^2 p_2 p_3 x_2^2 x_3^2 + \frac{3}{2} T^6 p_1^2 p_2 p_3 x_2^2 x_3^2 - \frac{3}{2} T^4 p_1 p_2^2 p_3 x_2^2 x_3^2 + 5 T^5 p_1 p_2^2 p_3 x_2^2 x_3^2 - \frac{7}{2} T^6 p_1 p_2^2 p_3 x_2^2 x_3^2 -$$
$$T^5 p_1 p_2 p_3^2 x_2^2 x_3^2 + T^6 p_1 p_2 p_3^2 x_2^2 x_3^2 + \frac{1}{2} T^2 p_1^3 x_3^3 - \frac{3}{2} T^3 p_1^3 x_3^3 + \frac{5}{2} T^4 p_1^3 x_3^3 - \frac{5}{2} T^5 p_1^3 x_3^3 + T^6 p_1^3 x_3^3 -$$
$$\frac{1}{2} T^3 p_1^2 p_2 x_3^3 + \frac{3}{2} T^4 p_1^2 p_2 x_3^3 - \frac{3}{2} T^5 p_1^2 p_2 x_3^3 + \frac{1}{2} T^6 p_1^2 p_2 x_3^3 - T^4 p_1 p_2^2 x_3^3 + 2 T^5 p_1 p_2^2 x_3^3 - T^6 p_1 p_2^2 x_3^3 -$$
$$\frac{1}{4} T p_1^4 x_1 x_3^3 + \frac{3}{2} T^2 p_1^4 x_1 x_3^3 - \frac{15}{4} T^3 p_1^4 x_1 x_3^3 + \frac{25}{4} T^4 p_1^4 x_1 x_3^3 - 6 T^5 p_1^4 x_1 x_3^3 + \frac{9}{4} T^6 p_1^4 x_1 x_3^3 -$$
$$T^2 p_1^3 p_2 x_1 x_3^3 + 5 T^3 p_1^3 p_2 x_1 x_3^3 - \frac{19}{2} T^4 p_1^3 p_2 x_1 x_3^3 + 8 T^5 p_1^3 p_2 x_1 x_3^3 - \frac{5}{2} T^6 p_1^3 p_2 x_1 x_3^3 - T^3 p_1^2 p_2^2 x_1 x_3^3 +$$
$$3 T^4 p_1^2 p_2^2 x_1 x_3^3 - 3 T^5 p_1^2 p_2^2 x_1 x_3^3 + T^6 p_1^2 p_2^2 x_1 x_3^3 - \frac{1}{2} T^3 p_1^3 p_3 x_1 x_3^3 + \frac{1}{2} T^4 p_1^3 p_3 x_1 x_3^3 + \frac{1}{2} T^5 p_1^3 p_3 x_1 x_3^3 -$$
$$\frac{1}{2} T^6 p_1^3 p_3 x_1 x_3^3 - \frac{1}{4} T^2 p_1^4 x_2 x_3^3 + \frac{3}{2} T^3 p_1^4 x_2 x_3^3 - \frac{13}{4} T^4 p_1^4 x_2 x_3^3 + \frac{15}{4} T^5 p_1^4 x_2 x_3^3 - \frac{5}{2} T^6 p_1^4 x_2 x_3^3 +$$
$$\frac{3}{4} T^7 p_1^4 x_2 x_3^3 + \frac{1}{2} T^2 p_1^3 p_2 x_2 x_3^3 - \frac{7}{2} T^3 p_1^3 p_2 x_2 x_3^3 + \frac{21}{2} T^4 p_1^3 p_2 x_2 x_3^3 - \frac{31}{2} T^5 p_1^3 p_2 x_2 x_3^3 + 11 T^6 p_1^3 p_2 x_2 x_3^3 -$$
$$3 T^7 p_1^3 p_2 x_2 x_3^3 + \frac{3}{2} T^3 p_1^2 p_2^2 x_2 x_3^3 - \frac{17}{2} T^4 p_1^2 p_2^2 x_2 x_3^3 + \frac{33}{2} T^5 p_1^2 p_2^2 x_2 x_3^3 - \frac{27}{2} T^6 p_1^2 p_2^2 x_2 x_3^3 +$$
$$4 T^7 p_1^2 p_2^2 x_2 x_3^3 + T^4 p_1 p_2^3 x_2 x_3^3 - \frac{9}{2} T^5 p_1 p_2^3 x_2 x_3^3 + 6 T^6 p_1 p_2^3 x_2 x_3^3 - \frac{5}{2} T^7 p_1 p_2^3 x_2 x_3^3 - T^4 p_1^3 p_3 x_2 x_3^3 +$$
$$\frac{5}{2} T^5 p_1^3 p_3 x_2 x_3^3 - 2 T^6 p_1^3 p_3 x_2 x_3^3 + \frac{1}{2} T^7 p_1^3 p_3 x_2 x_3^3 + \frac{3}{2} T^4 p_1^2 p_2 p_3 x_2 x_3^3 - \frac{9}{2} T^5 p_1^2 p_2 p_3 x_2 x_3^3 +$$
$$\frac{9}{2} T^6 p_1^2 p_2 p_3 x_2 x_3^3 - \frac{3}{2} T^7 p_1^2 p_2 p_3 x_2 x_3^3 + \frac{3}{2} T^5 p_1 p_2^2 p_3 x_2 x_3^3 - 3 T^6 p_1 p_2^2 p_3 x_2 x_3^3 + \frac{3}{2} T^7 p_1 p_2^2 p_3 x_2 x_3^3 +$$
$$\frac{1}{2} T^3 p_1^4 x_3^4 - \frac{9}{4} T^4 p_1^4 x_3^4 + \frac{9}{2} T^5 p_1^4 x_3^4 - 5 T^6 p_1^4 x_3^4 + 3 T^7 p_1^4 x_3^4 - \frac{3}{4} T^8 p_1^4 x_3^4 - \frac{1}{2} T^3 p_1^3 p_2 x_3^4 + \frac{7}{2} T^4 p_1^3 p_2 x_3^4 -$$
$$9 T^5 p_1^3 p_2 x_3^4 + 11 T^6 p_1^3 p_2 x_3^4 - \frac{13}{2} T^7 p_1^3 p_2 x_3^4 + \frac{3}{2} T^8 p_1^3 p_2 x_3^4 - \frac{5}{4} T^4 p_1^2 p_2^2 x_3^4 + 5 T^5 p_1^2 p_2^2 x_3^4 -$$
$$\frac{15}{2} T^6 p_1^2 p_2^2 x_3^4 + 5 T^7 p_1^2 p_2^2 x_3^4 - \frac{5}{4} T^8 p_1^2 p_2^2 x_3^4 - \frac{1}{2} T^5 p_1 p_2^3 x_3^4 + \frac{3}{2} T^6 p_1 p_2^3 x_3^4 - \frac{3}{2} T^7 p_1 p_2^3 x_3^4 + \frac{1}{2} T^8 p_1 p_2^3 x_3^4$$

```
In[∘]:= v = Table[Coefficient[err, c] /. (p | x)_ → 0, {c, Basis[3, {4}]}]
```

Out[•]=

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{T^2}{2} - T^3 + \frac{T^4}{2}, 0, 0, 0, 0, T^2 - T^3, 0, 0, 0, 0, 0, 0, 0, 0,$$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, T - 4 T^2 + 3 T^3, -T^2 + 4 T^3 - 3 T^4,$$

$$0, -\frac{T^2}{2} + \frac{T^4}{2}, -\frac{T^2}{2} + T^3 + \frac{T^4}{2} - T^5, \frac{T^2}{2} - \frac{3 T^3}{2} + \frac{5 T^4}{2} - \frac{5 T^5}{2} + T^6, 0, 0, 0, 0, T^2 - T^3, -\frac{T^2}{2} + \frac{T^4}{2},$$

$$0, \frac{3 T^2}{2} - 2 T^3 + \frac{T^4}{2}, 2 T^2 - 7 T^3 + 8 T^4 - 3 T^5, -\frac{T^3}{2} + \frac{3 T^4}{2} - \frac{3 T^5}{2} + \frac{T^6}{2}, 0, 0, 0, 0, T^2 - T^3,$$

$$\frac{T^2}{2} - 2 T^3 + \frac{3 T^4}{2}, 0, 0, -\frac{T^3}{2} + \frac{T^5}{2}, 0, 0, 0, 0, 0, -T^2 + T^3, 0, 0, -\frac{T^2}{2} + 3 T^3 - \frac{5 T^4}{2}, 3 T^3 - 8 T^4 + 5 T^5,$$

$$-T^4 + 2 T^5 - T^6, 0, 0, 0, 0, 0, T^3 - T^4, 0, -T^3 + T^4, T^4 - T^5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{1}{4} + \frac{7 T}{4} - \frac{15 T^2}{4} + \frac{9 T^3}{4}, \frac{T}{4} - \frac{7 T^2}{4} + \frac{15 T^3}{4} - \frac{9 T^4}{4}, 0,$$

$$\frac{T}{2} - 2 T^2 + \frac{3 T^3}{2}, \frac{1}{4} - \frac{3 T}{2} + \frac{13 T^2}{4} - \frac{17 T^3}{4} + \frac{9 T^4}{2} - \frac{9 T^5}{4}, -\frac{T}{4} + \frac{3 T^2}{2} - \frac{15 T^3}{4} + \frac{25 T^4}{4} - 6 T^5 + \frac{9 T^6}{4}, 0,$$

$$-\frac{T^2}{4} + \frac{T^3}{4} + \frac{T^4}{4} - \frac{T^5}{4}, -\frac{T}{4} + \frac{3 T^2}{4} - T^3 + T^4 - \frac{3 T^5}{4} + \frac{T^6}{4}, -\frac{T^2}{4} + \frac{3 T^3}{2} - \frac{13 T^4}{4} + \frac{15 T^5}{4} - \frac{5 T^6}{2} + \frac{3 T^7}{4},$$

$$\frac{T^3}{2} - \frac{9 T^4}{4} + \frac{9 T^5}{2} - 5 T^6 + 3 T^7 - \frac{3 T^8}{4}, 0, 0, 0, 0, -T + 4 T^2 - 3 T^3, \frac{T^2}{2} - 2 T^3 + \frac{3 T^4}{2}, 0, \frac{T^2}{2} + T^3 - \frac{3 T^4}{2},$$

$$T - \frac{11 T^2}{2} + \frac{25 T^3}{2} - \frac{25 T^4}{2} + \frac{9 T^5}{2}, -T^2 + 5 T^3 - \frac{19 T^4}{2} + 8 T^5 - \frac{5 T^6}{2}, 0, \frac{T^3}{2} - T^4 + \frac{T^5}{2}, T^4 - 2 T^5 + T^6,$$

$$\frac{T^2}{2} - \frac{7 T^3}{2} + \frac{21 T^4}{2} - \frac{31 T^5}{2} + 11 T^6 - 3 T^7, -\frac{T^3}{2} + \frac{7 T^4}{2} - 9 T^5 + 11 T^6 - \frac{13 T^7}{2} + \frac{3 T^8}{2}, 0, 0, 0, 0,$$

$$0, \frac{T^2}{2} - 2 T^3 + \frac{3 T^4}{2}, 0, 0, T^2 - 4 T^3 + 3 T^4, -\frac{T^3}{2} + \frac{T^4}{2} + \frac{T^5}{2} - \frac{T^6}{2}, 0, 0, -\frac{T^3}{2} + \frac{T^4}{2} + \frac{T^5}{2} - \frac{T^6}{2},$$

$$-T^4 + \frac{5 T^5}{2} - 2 T^6 + \frac{T^7}{2}, 0, 0, 0, 0, 0, -T^2 + T^3, 0, 0, -T^3 + T^4, T^2 - 5 T^3 + 7 T^4 - 3 T^5,$$

$$-T^3 + 3 T^4 - 3 T^5 + T^6, 0, 0, -\frac{T^2}{4} + \frac{5 T^3}{2} - 7 T^4 + \frac{15 T^5}{2} - \frac{11 T^6}{4}, \frac{3 T^3}{2} - \frac{17 T^4}{2} + \frac{33 T^5}{2} - \frac{27 T^6}{2} + 4 T^7,$$

$$-\frac{5 T^4}{4} + 5 T^5 - \frac{15 T^6}{2} + 5 T^7 - \frac{5 T^8}{4}, 0, 0, 0, 0, 0, T^3 - T^4, 0, 0, 2 T^3 - 4 T^4 + 2 T^5, 0,$$

$$0, 0, -\frac{T^3}{2} + \frac{5 T^4}{2} - \frac{7 T^5}{2} + \frac{3 T^6}{2}, \frac{3 T^4}{2} - \frac{9 T^5}{2} + \frac{9 T^6}{2} - \frac{3 T^7}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$

$$T^4 - T^5, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{T^3}{2} + 3 T^4 - \frac{11 T^5}{2} + 3 T^6,$$

$$T^4 - \frac{9 T^5}{2} + 6 T^6 - \frac{5 T^7}{2}, -\frac{T^5}{2} + \frac{3 T^6}{2} - \frac{3 T^7}{2} + \frac{T^8}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$

$$-\frac{3 T^4}{2} + 5 T^5 - \frac{7 T^6}{2}, \frac{3 T^5}{2} - 3 T^6 + \frac{3 T^7}{2}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -T^5 + T^6, 0,$$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,$$

$$0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

In[ ]:=
```
mat = Table[
  R = 𝔼₂[(T - 1) (p₂ - p₁) x₂, 1 + ϵ (1/2) p₁ x₂ (2 T p₂ x₁ + p₁ ((1 - 3 T) x₁ + (-1 + T) T x₂)) + ϵ² b + O[ϵ]³];
  db = ϵ^-2 Normal[Last[R^3[1,2] ** R^3[1,3] ** R^3[2,3]] - Last[R^3[2,3] ** R^3[1,3] ** R^3[1,2]]];
  Table[Coefficient[db, c] /. (p | x)_ → 0, {c, Basis[3, {4}]}],
  {b, Basis[2, {4}]}]
```

Out[ ]=
```
{ ⋯ 1 ⋯ }
large output | show less | show more | show all | set size limit...
```

In[ ]:=
```
LinearSolve[mat^T, v].Basis[2, {4}]
```

Out[ ]=
$$\frac{p_1 x_1}{1 - T} + \frac{T p_1 x_2}{-1 + T}$$

In[ ]:=
```
Module[{A, B1, B2},
  A = {{1, -1, 0}, {-1, 1, 0}, {0, 0, 0}}; B1 = {{0, 0, 0}, {0, 1, -1}, {0, -1, 1}}; B2 = {{1, 0, -1}, {0, 0, 0}, {-1, 0, 1}};
  MatrixForm /@ {A.B1 - B1.A, A.B2 - B2.A}
]
```

Out[ ]=
$$\left\{\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\right\}$$