Pensieve header: Testing and implementing lemmas 1,2,3 of the DoPeGDO handouts. Continues pensieve://2020-02/.

$$[F:\mathcal{E}]_B := e^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad \text{and} \quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}e^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

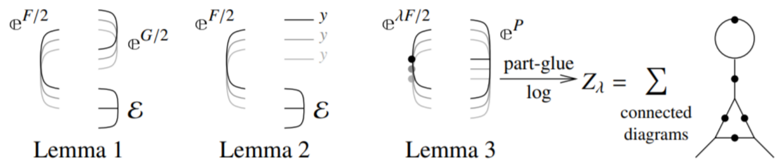
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}e^{\sum_{i\in B} y_iz_i}\right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F: e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Goals:

Implement the container $\mathbb{E}[\omega, Q, P] := \omega e^{Q+P}$.

Implement the containers $|F, \mathcal{E}|_B := [F:\mathcal{E}]_B$ and $\langle F, \mathcal{E}\rangle_B := \langle F:\mathcal{E}\rangle_B$, their evaluator Ev_k as power series in $\hbar$ to degree k, and verify lemmas 1, 2, and 3. Inserting $\hbar$ in the appropriate places is user responsibility.

Implement DaGauss, DeLin, and a Lemma 3 evaluator, PEv.

Utilities

In[∘]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Generic Polynomials:

In[∘]:=
```
GenericPolynomial[d_Integer, vars_List, gc_] := Total[Map[
    gc_Sequence@@# Times @@ (vars^#) &,
    Join @@ (Permutations /@ IntegerPartitions[d + Length@vars, {Length@vars}]) - 1
  ]];
GenericPolynomial[specs_List, vars_List, gc_] :=
 Sum[GenericPolynomial[specs[[1]], vars, gc], specs]
```

Preliminary Definitions

In[∘]:=
```
Unprotect[SeriesData];
Expand[sd_SeriesData] ^:= MapAt[Expand, sd, 3];
Protect[SeriesData];
```

In[∘]:=
```
CF[⟨F_, ℰ_⟩B_] := ⟨Simplify@F, Simplify@ℰ⟩B;
```

Variables and their duals:

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)i; (B_List)* := #* & /@B;
```

Act and Contract:

In[∘]:=
```
Evk_ @ ¦F_, ℰ_¦B_ := Expand[Total[
      CoefficientRules[Normal@Series[e^(B*.F.B*/2), {ℏ, 0, k}], B*] /.
       (ps_ → c_) :→ c D[ℰ, Sequence @@ Thread[{B, ps}]]
    ] + O[ℏ]^(k+1)];
Evk_ @⟨F_, ℰ_⟩B_ := Evk@¦F, ℰ¦B /. Alternatives @@ B → 0
```

In[∘]:= $\left\{\mathrm{Ev}_2@\left|\hbar\begin{pmatrix}0&1\\1&0\end{pmatrix}, e^{x\,y}\right|_{\{x,y\}}, \mathrm{Ev}_3@\left\langle\hbar\begin{pmatrix}0&1\\1&0\end{pmatrix}, e^{3\,x\,y}\right\rangle_{\{x,y\}}\right\}$

Out[∘]= $\left\{e^{x\,y} + \left(e^{x\,y} + e^{x\,y}\,x\,y\right)\hbar + \left(e^{x\,y} + 2\,e^{x\,y}\,x\,y + \frac{1}{2}\,e^{x\,y}\,x^2\,y^2\right)\hbar^2 + O[\hbar]^3,\ 1 + 3\,\hbar + 9\,\hbar^2 + 27\,\hbar^3 + O[\hbar]^4\right\}$

Implementing / Testing Lemma 1

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1} : \mathcal{E}\right\rangle_B$$

In[∘]:=
```
{p = 10, B = {x}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ {{f}}, G = {{g}}, ℰ = GenericPolynomial[{d, 0, 4}, B, c]}
lhs = Ev_p@⟨F, ℰ 𝕖^(B.G.B/2)⟩_B
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) ℰ⟩_B;
HL[lhs == rhs]
```

Out[∘]= $\{10, \{x\}, \{\{1\}\}, \{\{f\,\hbar\}\}, \{\{g\}\}, c_0 + x\,c_1 + x^2\,c_2 + x^3\,c_3 + x^4\,c_4\}$

Out[∘]= $c_0 + \left(\frac{1}{2} f\,g\,c_0 + f\,c_2\right)\hbar + \left(\frac{3}{8} f^2 g^2 c_0 + \frac{3}{2} f^2 g\,c_2 + 3 f^2 c_4\right)\hbar^2 + \left(\frac{5}{16} f^3 g^3 c_0 + \frac{15}{8} f^3 g^2 c_2 + \frac{15}{2} f^3 g\,c_4\right)\hbar^3 + \left(\frac{35}{128} f^4 g^4 c_0 + \frac{35}{16} f^4 g^3 c_2 + \frac{105}{8} f^4 g^2 c_4\right)\hbar^4 + \left(\frac{63}{256} f^5 g^5 c_0 + \frac{315}{128} f^5 g^4 c_2 + \frac{315}{16} f^5 g^3 c_4\right)\hbar^5 + \left(\frac{231 f^6 g^6 c_0}{1024} + \frac{693}{256} f^6 g^5 c_2 + \frac{3465}{128} f^6 g^4 c_4\right)\hbar^6 + \left(\frac{429 f^7 g^7 c_0}{2048} + \frac{3003 f^7 g^6 c_2}{1024} + \frac{9009}{256} f^7 g^5 c_4\right)\hbar^7 + \left(\frac{6435 f^8 g^8 c_0}{32768} + \frac{6435 f^8 g^7 c_2}{2048} + \frac{45045 f^8 g^6 c_4}{1024}\right)\hbar^8 + \left(\frac{12155 f^9 g^9 c_0}{65536} + \frac{109395 f^9 g^8 c_2}{32768} + \frac{109395 f^9 g^7 c_4}{2048}\right)\hbar^9 + \left(\frac{46189 f^{10} g^{10} c_0}{262144} + \frac{230945 f^{10} g^9 c_2}{65536} + \frac{2078505 f^{10} g^8 c_4}{32768}\right)\hbar^{10} + O[\hbar]^{11}$

Out[∘]= True

In[∘]:=
```
{p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
 F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), ℰ = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ e^(B.G.B/2)⟩_B]
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) ℰ⟩_B;
HL[lhs == rhs]
```

Out[∘]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[∘]= $$\Big\{0.0625,\ c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right)\hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right)\hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]= True

```
In[•]:= {p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
         F = ℏ (f11 f12
                f12 f22), G = (g11 g12
                               g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
        Timing[lhs = Ev_p@⟨F, 𝓔 e^(B.G.B/2)⟩_B]
        rhs = Expand[Series[Det[𝓘 - G.F]^(-1/2), {ℏ, 0, p}] Ev_p[⟨F.Inverse[𝓘 - G.F], 𝓔⟩_B]]
        HL[lhs == rhs]
```

Out[•]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[•]= $$\Big\{0.0625,\ c_{0,0} + \Big(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\Big)\hbar + \Big(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\Big)\hbar^2 + O[\hbar]^3\Big\}$$

Out[•]= $$c_{0,0} + \Big(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\Big)\hbar + \Big(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\Big)\hbar^2 + O[\hbar]^3$$

Out[•]= True

$$\left[F\colon \mathcal{E}\, e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right]_B = \det(1-GF)^{-1/2} e^{\frac{1}{2}\sum_{i,j\in B}(G(I-FG)^{-1})_{ij} z_i z_j} \cdot \left(\left[F(1-GF)^{-1}\colon \mathcal{E}\right]_B\right)_{z_B\to(I-}$$

```
In[•]:= {p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
         F = ℏ (f11 f12
                f12 f22), G = (g11 g12
                               g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
        Timing[lhs = Ev_p@|F, 𝓔 e^(B.G.B/2)|_B];
        rhs = Series[Det[𝓘 - G.F]^(-1/2) e^(B.G.Inverse[𝓘-F.G].B/2), {ℏ, 0, p}]
            (Ev_p@|F.Inverse[𝓘 - G.F], 𝓔|_B /. Thread[B → Inverse[𝓘 - F.G].B]);
        HL@FullSimplify[Normal[lhs - rhs] == 0]
```

Out[•]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[•]= True

In[]:=
```
DeGauss@⟨F_, ℰ_⟩B_ := Module[{ℐ, Q, G, M, Δ},
  ℐ = IdentityMatrix@Length@B;
  Q = Log[Normal[ℰ] /. ϵ → 0];
  G = Table[∂_{i,j}Q, {i, B}, {j, B}];
  M = Inverse[ℐ - G.F];
  Δ = Simplify@Det@M;
  CF@⟨F.M, Δ^(1/2) ℰ 𝕖^(-B.G.B/2)⟩_B
 ]
```

In[]:=
```
{p = 2, B = {x, y}, F = ℏ (f11 f12; f12 f22),
  G = (g11 g12; g12 g22), ℰ = 1 + ϵ GenericPolynomial[{d, 0, 2}, B, c]};
lhs = ⟨F, ℰ 𝕖^(B.G.B/2)⟩_B
rhs = DeGauss@⟨F, ℰ 𝕖^(B.G.B/2)⟩_B
HL[Ev_p@lhs == Ev_p@rhs]
```

Out[]=
$$\left\langle \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},\ e^{\frac{1}{2}(x(x g_{11}+y g_{12})+y(x g_{12}+y g_{22}))}\left(1+\epsilon\left(c_{0,0}+y c_{0,1}+y^2 c_{0,2}+x c_{1,0}+x y c_{1,1}+x^2 c_{2,0}\right)\right)\right\rangle_{\{x,y\}}$$

Out[]=
$$\Bigg\langle \Bigg\{\Bigg\{\frac{\hbar\left(\hbar f_{12}^2 g_{22}+f_{11}\left(1-\hbar f_{22} g_{22}\right)\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}\left(-1+\hbar f_{22} g_{22}\right)\right)},$$
$$\frac{\hbar\left(f_{12}-\hbar f_{12}^2 g_{12}+\hbar f_{11} f_{22} g_{12}\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}\left(-1+\hbar f_{22} g_{22}\right)\right)}\Bigg\},$$
$$\Bigg\{\frac{\hbar\left(f_{12}-\hbar f_{12}^2 g_{12}+\hbar f_{11} f_{22} g_{12}\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}\left(-1+\hbar f_{22} g_{22}\right)\right)},$$
$$\frac{\hbar\left(\hbar f_{12}^2 g_{11}+f_{22}\left(1-\hbar f_{11} g_{11}\right)\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}\left(-1+\hbar f_{22} g_{22}\right)\right)}\Bigg\}\Bigg\},$$
$$\sqrt{\frac{1}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}\left(-1+\hbar f_{22} g_{22}\right)\right)}}$$
$$\left(1+\epsilon c_{0,0}+y\epsilon c_{0,1}+y^2\epsilon c_{0,2}+x\epsilon c_{1,0}+x y\epsilon c_{1,1}+x^2\epsilon c_{2,0}\right)\Bigg\rangle_{\{x,y\}}$$

Out[]= True

Implementing / Testing Lemma 2

Lemma 2. $\left\langle F\colon \mathcal{E}\,\mathbb{e}^{\sum_{i\in B} y_i z_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j}\left\langle F\colon\ \mathcal{E}|_{z_B\to z_B+F y_B}\right\rangle_B.$

In[ ]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], 𝓘 = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
  F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ∈ GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ 𝕖^(Y.B)⟩_B]
Timing[rhs = Ev_p@⟨F, 𝕖^(Y.F.Y/2) ℰ /. Thread[B → B + F.Y]⟩_B]
HL@Expand[lhs == rhs]
```

Out[ ]= $\{2, 2, \{z_1, z_2\}, \{\{1, 0\}, \{0, 1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$1 + \in (c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0})\}$

Out[ ]= $\{0.015625,$

$$(1 + \in c_{0,0}) + \Big(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \in f_{22} y_2^2 c_{0,0} +$$
$$\in f_{12} y_1 c_{0,1} + \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0}\Big)\hbar +$$
$$\Big(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 +$$
$$\frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} +$$
$$\frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1} +$$
$$\frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 y_2 c_{0,2} +$$
$$\frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 c_{0,3} + 3 \in f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0} +$$
$$\in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1} + 2 \in f_{12}^2 y_1 y_2 c_{1,1} +$$
$$\in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1} + 2 \in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3 \in f_{12} f_{22} y_2 c_{1,2} +$$
$$\frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 c_{2,1} +$$
$$2 \in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3 \in f_{11}^2 y_1 c_{3,0} + 3 \in f_{11} f_{12} y_2 c_{3,0}\Big)\hbar^2 + O[\hbar]^3\}$$

Out[∘]=
$$\Big\{0.015625,$$
$$\left(1+\epsilon\, c_{0,0}\right)+\Big(\frac{1}{2} f_{11} y_1^2+f_{12} y_1 y_2+\frac{1}{2} f_{22} y_2^2+\frac{1}{2}\epsilon\, f_{11} y_1^2 c_{0,0}+\epsilon\, f_{12} y_1 y_2 c_{0,0}+\frac{1}{2}\epsilon\, f_{22} y_2^2 c_{0,0}+$$
$$\epsilon\, f_{12} y_1 c_{0,1}+\epsilon\, f_{22} y_2 c_{0,1}+\epsilon\, f_{22} c_{0,2}+\epsilon\, f_{11} y_1 c_{1,0}+\epsilon\, f_{12} y_2 c_{1,0}+\epsilon\, f_{12} c_{1,1}+\epsilon\, f_{11} c_{2,0}\Big)\hbar+$$
$$\Big(\frac{1}{8} f_{11}^2 y_1^4+\frac{1}{2} f_{11} f_{12} y_1^3 y_2+\frac{1}{2} f_{12}^2 y_1^2 y_2^2+\frac{1}{4} f_{11} f_{22} y_1^2 y_2^2+\frac{1}{2} f_{12} f_{22} y_1 y_2^3+\frac{1}{8} f_{22}^2 y_2^4+$$
$$\frac{1}{8}\epsilon\, f_{11}^2 y_1^4 c_{0,0}+\frac{1}{2}\epsilon\, f_{11} f_{12} y_1^3 y_2 c_{0,0}+\frac{1}{2}\epsilon\, f_{12}^2 y_1^2 y_2^2 c_{0,0}+\frac{1}{4}\epsilon\, f_{11} f_{22} y_1^2 y_2^2 c_{0,0}+$$
$$\frac{1}{2}\epsilon\, f_{12} f_{22} y_1 y_2^3 c_{0,0}+\frac{1}{8}\epsilon\, f_{22}^2 y_2^4 c_{0,0}+\frac{1}{2}\epsilon\, f_{11} f_{12} y_1^3 c_{0,1}+\epsilon\, f_{12}^2 y_1^2 y_2 c_{0,1}+\frac{1}{2}\epsilon\, f_{11} f_{22} y_1^2 y_2 c_{0,1}+$$
$$\frac{3}{2}\epsilon\, f_{12} f_{22} y_1 y_2^2 c_{0,1}+\frac{1}{2}\epsilon\, f_{22}^2 y_2^3 c_{0,1}+\epsilon\, f_{12}^2 y_1^2 c_{0,2}+\frac{1}{2}\epsilon\, f_{11} f_{22} y_1^2 c_{0,2}+3\epsilon\, f_{12} f_{22} y_1 y_2 c_{0,2}+$$
$$\frac{3}{2}\epsilon\, f_{22}^2 y_2^2 c_{0,2}+3\epsilon\, f_{12} f_{22} y_1 c_{0,3}+3\epsilon\, f_{22}^2 y_2 c_{0,3}+\frac{1}{2}\epsilon\, f_{11}^2 y_1^3 c_{1,0}+\frac{3}{2}\epsilon\, f_{11} f_{12} y_1^2 y_2 c_{1,0}+$$
$$\epsilon\, f_{12}^2 y_1 y_2^2 c_{1,0}+\frac{1}{2}\epsilon\, f_{11} f_{22} y_1 y_2^2 c_{1,0}+\frac{1}{2}\epsilon\, f_{12} f_{22} y_2^3 c_{1,0}+\frac{3}{2}\epsilon\, f_{11} f_{12} y_1^2 c_{1,1}+2\epsilon\, f_{12}^2 y_1 y_2 c_{1,1}+$$
$$\epsilon\, f_{11} f_{22} y_1 y_2 c_{1,1}+\frac{3}{2}\epsilon\, f_{12} f_{22} y_2^2 c_{1,1}+2\epsilon\, f_{12}^2 y_1 c_{1,2}+\epsilon\, f_{11} f_{22} y_1 c_{1,2}+3\epsilon\, f_{12} f_{22} y_2 c_{1,2}+$$
$$\frac{3}{2}\epsilon\, f_{11}^2 y_1^2 c_{2,0}+3\epsilon\, f_{11} f_{12} y_1 y_2 c_{2,0}+\epsilon\, f_{12}^2 y_2^2 c_{2,0}+\frac{1}{2}\epsilon\, f_{11} f_{22} y_2^2 c_{2,0}+3\epsilon\, f_{11} f_{12} y_1 c_{2,1}+$$
$$2\epsilon\, f_{12}^2 y_2 c_{2,1}+\epsilon\, f_{11} f_{22} y_2 c_{2,1}+3\epsilon\, f_{11}^2 y_1 c_{3,0}+3\epsilon\, f_{11} f_{12} y_2 c_{3,0}\Big)\hbar^2+O[\hbar]^3\Big\}$$

Out[∘]= True

$$\left[F\colon \mathcal{E}\,\mathbb{e}^{\sum_{i\in B} y_i z_i}\right]_B=\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j+\sum_{i\in B} y_i z_i}\left[F\colon\ \mathcal{E}\big|_{z_B\to z_B+F y_B}\right]_B$$
$$=\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j+\sum_{i\in B} y_i z_i}\left(\left[F\colon \mathcal{E}\right]_B\big|_{z_B\to z_B}\right.$$

In[∘]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], I = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
 F = ℏ Table[f_{{10,1}.Sort[{i,j}]}, {i, n}, {j, n}], ℰ = 1 + ∈ GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Ev_p@ |F, ℰ e^(Y.B)|_B] // Short
Timing[rhs = e^(Y.B) Ev_p@ |F, e^(Y.F.Y/2) ℰ /. Thread[B → B + F.Y]|_B];
HL@Expand[lhs == rhs]
```

Out[∘]=
$$\Big\{2,\ 2,\ \{z_1, z_2\},\ \{\{1, 0\}, \{0, 1\}\},\ \{y_1, y_2\},\ \{\{\hbar\, f_{11}, \hbar\, f_{12}\}, \{\hbar\, f_{12}, \hbar\, f_{22}\}\},$$
$$1+\epsilon\left(c_{0,0}+z_2 c_{0,1}+z_2^2 c_{0,2}+z_2^3 c_{0,3}+z_1 c_{1,0}+z_1 z_2 c_{1,1}+z_1 z_2^2 c_{1,2}+z_1^2 c_{2,0}+z_1^2 z_2 c_{2,1}+z_1^3 c_{3,0}\right)\Big\}$$

Out[∘]//Short= {0.40625, <<1>> }

Out[∘]= True

In[◦]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], 𝓘 = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
 F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ϵ GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Ev_p@|F, ℰ e^(Y.B)|_B] // Short
Timing[rhs = e^(Y.B) (Ev_p@|F, e^(Y.F.Y/2) ℰ|_B /. Thread[B → B + F.Y])];
HL@Expand[lhs == rhs]
```

Out[◦]= $\{2, 2, \{z_1, z_2\}, \{\{1, 0\}, \{0, 1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$1 + \epsilon\, (c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0})\}$

Out[◦]//Short= $\{0.015625, (e^{y_1 z_1 + y_2 z_2} + e^{y_1 z_1 + y_2 z_2}\, \epsilon\, c_{0,0} + \ll 8 \gg + e^{y_1 z_1 + y_2 z_2}\, \epsilon\, z_1^3 c_{3,0}) +$
$(\ll 1 \gg)\, \hbar + (\ll 1 \gg + \ll 171 \gg)\, \ll 1 \gg + O[\hbar]^3\}$

Out[◦]= True

In[◦]:=
```
DeLin@⟨F_, ℰ_⟩_B_ := Module[{L, Y},
   L = PowerExpand@Log[Normal[ℰ] /. ϵ → 0];
   Y = Table[∂_i L, {i, B}];
   CF@⟨F, e^(Y.F.Y/2) (e^(-B.Y) ℰ /. Thread[B → B + F.Y])⟩_B
  ]
```

In[◦]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], Y = Table[y_i, {i, n}],
 F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ϵ GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ e^(Y.B)⟩_B]
Timing[rhs = Ev_p@DeLin@⟨F, ℰ e^(Y.B)⟩_B]
HL@Simplify[lhs == rhs]
```

Out[◦]= $\{2, 2, \{z_1, z_2\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$1 + \epsilon\, (c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0})\}$

Out[•]= $\{0.03125,\ (1+\in c_{0,0}) + \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} + \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0}\right) \hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 c_{0,3} + 3 \in f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0} + \in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1} + 2 \in f_{12}^2 y_1 y_2 c_{1,1} + \in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1} + 2 \in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3 \in f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 c_{2,1} + 2 \in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3 \in f_{11}^2 y_1 c_{3,0} + 3 \in f_{11} f_{12} y_2 c_{3,0}\right) \hbar^2 + O[\hbar]^3\}$

Out[•]= $\{0.,\ (1+\in c_{0,0}) + \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} + \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0}\right) \hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 c_{0,3} + 3 \in f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0} + \in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1} + 2 \in f_{12}^2 y_1 y_2 c_{1,1} + \in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1} + 2 \in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3 \in f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 c_{2,1} + 2 \in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3 \in f_{11}^2 y_1 c_{3,0} + 3 \in f_{11} f_{12} y_2 c_{3,0}\right) \hbar^2 + O[\hbar]^3\}$

Out[∘]= True

Testing Lemma 3

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

In[∘]:=
```
{n = 2, p = 2, B = Table[b_i, {i, n}],
 F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], P = GenericPolynomial[{d, 0, 2}, B, c]}
```

Out[∘]= $\{2, 2, \{b_1, b_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}\}$

In[∘]:=
```
Z = PowerExpand@Expand@Log[Ev_p@ |λ F, 𝕖^P|_B]
```

Out[∘]=

$$\left(c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}\right) +$$

$$\left(\frac{1}{2}\lambda f_{22} c_{0,1}^2 + \lambda f_{22} c_{0,2} + 2\lambda b_2 f_{22} c_{0,1} c_{0,2} + 2\lambda b_2^2 f_{22} c_{0,2}^2 + \lambda f_{12} c_{0,1} c_{1,0} + 2\lambda b_2 f_{12} c_{0,2} c_{1,0} +\right.$$
$$\frac{1}{2}\lambda f_{11} c_{1,0}^2 + \lambda f_{12} c_{1,1} + \lambda b_2 f_{12} c_{0,1} c_{1,1} + \lambda b_1 f_{22} c_{0,1} c_{1,1} + 2\lambda b_2^2 f_{12} c_{0,2} c_{1,1} +$$
$$2\lambda b_1 b_2 f_{22} c_{0,2} c_{1,1} + \lambda b_2 f_{11} c_{1,0} c_{1,1} + \lambda b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2}\lambda b_2^2 f_{11} c_{1,1}^2 +$$
$$\lambda b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2}\lambda b_1^2 f_{22} c_{1,1}^2 + \lambda f_{11} c_{2,0} + 2\lambda b_1 f_{12} c_{0,1} c_{2,0} + 4\lambda b_1 b_2 f_{12} c_{0,2} c_{2,0} +$$
$$\left.2\lambda b_1 f_{11} c_{1,0} c_{2,0} + 2\lambda b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2\lambda b_1^2 f_{12} c_{1,1} c_{2,0} + 2\lambda b_1^2 f_{11} c_{2,0}^2\right)\hbar +$$

$$\left(\lambda^2 f_{22}^2 c_{0,1}^2 c_{0,2} + \lambda^2 f_{22}^2 c_{0,2}^2 + 4\lambda^2 b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 4\lambda^2 b_2^2 f_{22}^2 c_{0,2}^3 + 2\lambda^2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} +\right.$$
$$4\lambda^2 b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + \lambda^2 f_{12}^2 c_{0,2} c_{1,0}^2 + \lambda^2 f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2\lambda^2 f_{12} f_{22} c_{0,2} c_{1,1} +$$
$$6\lambda^2 b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 2\lambda^2 b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 8\lambda^2 b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + 4\lambda^2 b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} +$$
$$\lambda^2 f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + \lambda^2 f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 4\lambda^2 b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + 2\lambda^2 b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} +$$
$$2\lambda^2 b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + \lambda^2 f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2}\lambda^2 f_{12}^2 c_{1,1}^2 + \frac{1}{2}\lambda^2 f_{11} f_{22} c_{1,1}^2 + \lambda^2 b_2 f_{12}^2 c_{0,1} c_{1,1}^2 +$$
$$\lambda^2 b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 2\lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + 3\lambda^2 b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 2\lambda^2 b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 +$$
$$6\lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + \lambda^2 b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + 2\lambda^2 b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + \lambda^2 b_1 f_{12}^2 c_{1,0} c_{1,1}^2 +$$
$$\lambda^2 b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + \lambda^2 b_2^2 f_{11} f_{12} c_{1,1}^3 + \lambda^2 b_1 b_2 f_{12}^2 c_{1,1}^3 + \lambda^2 b_1 b_2 f_{11} f_{22} c_{1,1}^3 + \lambda^2 b_1^2 f_{12} f_{22} c_{1,1}^3 +$$
$$\lambda^2 f_{12}^2 c_{0,1}^2 c_{2,0} + 2\lambda^2 f_{12}^2 c_{0,2} c_{2,0} + 4\lambda^2 b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 4\lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} +$$
$$4\lambda^2 b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 8\lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + 2\lambda^2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} +$$
$$4\lambda^2 b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 4\lambda^2 b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + \lambda^2 f_{11}^2 c_{1,0}^2 c_{2,0} + 2\lambda^2 f_{11} f_{12} c_{1,1} c_{2,0} +$$
$$2\lambda^2 b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 4\lambda^2 b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 2\lambda^2 b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} +$$
$$4\lambda^2 b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 12\lambda^2 b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 4\lambda^2 b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} +$$
$$4\lambda^2 b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 2\lambda^2 b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 6\lambda^2 b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} +$$
$$\lambda^2 b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 6\lambda^2 b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 3\lambda^2 b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 2\lambda^2 b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} +$$
$$\lambda^2 f_{11}^2 c_{2,0}^2 + 4\lambda^2 b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 8\lambda^2 b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 4\lambda^2 b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 +$$
$$\left.4\lambda^2 b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 4\lambda^2 b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 8\lambda^2 b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 4\lambda^2 b_1^2 f_{11}^2 c_{2,0}^3\right)\hbar^2 + O[\hbar]^3$$

In[∘]:= **Z /. λ → 0**

Out[∘]= $\left(c_{0,0}+b_2\,c_{0,1}+b_2^2\,c_{0,2}+b_1\,c_{1,0}+b_1\,b_2\,c_{1,1}+b_1^2\,c_{2,0}\right)+O[\hbar]^3$

In[∘]:= **(Z /. λ → 0) − P**

Out[∘]= $O[\hbar]^3$

In[∘]:= **lhs = ∂$_\lambda$Z**

Out[∘]=

$$\Bigl(\frac{1}{2}\,f_{22}\,c_{0,1}^2+f_{22}\,c_{0,2}+2\,b_2\,f_{22}\,c_{0,1}\,c_{0,2}+2\,b_2^2\,f_{22}\,c_{0,2}^2+f_{12}\,c_{0,1}\,c_{1,0}+2\,b_2\,f_{12}\,c_{0,2}\,c_{1,0}+\frac{1}{2}\,f_{11}\,c_{1,0}^2+$$
$$f_{12}\,c_{1,1}+b_2\,f_{12}\,c_{0,1}\,c_{1,1}+b_1\,f_{22}\,c_{0,1}\,c_{1,1}+2\,b_2^2\,f_{12}\,c_{0,2}\,c_{1,1}+2\,b_1\,b_2\,f_{22}\,c_{0,2}\,c_{1,1}+b_2\,f_{11}\,c_{1,0}\,c_{1,1}+$$
$$b_1\,f_{12}\,c_{1,0}\,c_{1,1}+\frac{1}{2}\,b_2^2\,f_{11}\,c_{1,1}^2+b_1\,b_2\,f_{12}\,c_{1,1}^2+\frac{1}{2}\,b_1^2\,f_{22}\,c_{1,1}^2+f_{11}\,c_{2,0}+2\,b_1\,f_{12}\,c_{0,1}\,c_{2,0}+$$
$$4\,b_1\,b_2\,f_{12}\,c_{0,2}\,c_{2,0}+2\,b_1\,f_{11}\,c_{1,0}\,c_{2,0}+2\,b_1\,b_2\,f_{11}\,c_{1,1}\,c_{2,0}+2\,b_1^2\,f_{12}\,c_{1,1}\,c_{2,0}+2\,b_1^2\,f_{11}\,c_{2,0}^2\Bigr)\,\hbar+$$
$$\bigl(2\,\lambda\,f_{22}^2\,c_{0,1}^2\,c_{0,2}+2\,\lambda\,f_{22}^2\,c_{0,2}^2+8\,\lambda\,b_2\,f_{22}^2\,c_{0,1}\,c_{0,2}^2+8\,\lambda\,b_2^2\,f_{22}^2\,c_{0,2}^3+4\,\lambda\,f_{12}\,f_{22}\,c_{0,1}\,c_{0,2}\,c_{1,0}+$$
$$8\,\lambda\,b_2\,f_{12}\,f_{22}\,c_{0,2}^2\,c_{1,0}+2\,\lambda\,f_{12}^2\,c_{0,2}\,c_{1,0}^2+2\,\lambda\,f_{12}\,f_{22}\,c_{0,1}^2\,c_{1,1}+4\,\lambda\,f_{12}\,f_{22}\,c_{0,2}\,c_{1,1}+$$
$$12\,\lambda\,b_2\,f_{12}\,f_{22}\,c_{0,1}\,c_{0,2}\,c_{1,1}+4\,\lambda\,b_1\,f_{22}^2\,c_{0,1}\,c_{0,2}\,c_{1,1}+16\,\lambda\,b_2^2\,f_{12}\,f_{22}\,c_{0,2}^2\,c_{1,1}+$$
$$8\,\lambda\,b_1\,b_2\,f_{22}^2\,c_{0,2}^2\,c_{1,1}+2\,\lambda\,f_{12}^2\,c_{0,1}\,c_{1,0}\,c_{1,1}+2\,\lambda\,f_{11}\,f_{22}\,c_{0,1}\,c_{1,0}\,c_{1,1}+8\,\lambda\,b_2\,f_{12}^2\,c_{0,2}\,c_{1,0}\,c_{1,1}+$$
$$4\,\lambda\,b_2\,f_{11}\,f_{22}\,c_{0,2}\,c_{1,0}\,c_{1,1}+4\,\lambda\,b_1\,f_{12}\,f_{22}\,c_{0,2}\,c_{1,0}\,c_{1,1}+2\,\lambda\,f_{11}\,f_{12}\,c_{1,0}^2\,c_{1,1}+\lambda\,f_{12}^2\,c_{1,1}^2+$$
$$\lambda\,f_{11}\,f_{22}\,c_{1,1}^2+2\,\lambda\,b_2\,f_{12}^2\,c_{0,1}\,c_{1,1}^2+2\,\lambda\,b_2\,f_{11}\,f_{22}\,c_{0,1}\,c_{1,1}^2+4\,\lambda\,b_1\,f_{12}\,f_{22}\,c_{0,1}\,c_{1,1}^2+$$
$$6\,\lambda\,b_2^2\,f_{12}^2\,c_{0,2}\,c_{1,1}^2+4\,\lambda\,b_2^2\,f_{11}\,f_{22}\,c_{0,2}\,c_{1,1}^2+12\,\lambda\,b_1\,b_2\,f_{12}\,f_{22}\,c_{0,2}\,c_{1,1}^2+2\,\lambda\,b_1^2\,f_{22}^2\,c_{0,2}\,c_{1,1}^2+$$
$$4\,\lambda\,b_2\,f_{11}\,f_{12}\,c_{1,0}\,c_{1,1}^2+2\,\lambda\,b_1\,f_{12}^2\,c_{1,0}\,c_{1,1}^2+2\,\lambda\,b_1\,f_{11}\,f_{22}\,c_{1,0}\,c_{1,1}^2+2\,\lambda\,b_2^2\,f_{11}\,f_{12}\,c_{1,1}^3+$$
$$2\,\lambda\,b_1\,b_2\,f_{12}^2\,c_{1,1}^3+2\,\lambda\,b_1\,b_2\,f_{11}\,f_{22}\,c_{1,1}^3+2\,\lambda\,b_1^2\,f_{12}\,f_{22}\,c_{1,1}^3+2\,\lambda\,f_{12}^2\,c_{0,1}^2\,c_{2,0}+4\,\lambda\,f_{12}^2\,c_{0,2}\,c_{2,0}+$$
$$8\,\lambda\,b_2\,f_{12}^2\,c_{0,1}\,c_{0,2}\,c_{2,0}+8\,\lambda\,b_1\,f_{12}\,f_{22}\,c_{0,1}\,c_{0,2}\,c_{2,0}+8\,\lambda\,b_2^2\,f_{12}^2\,c_{0,2}^2\,c_{2,0}+16\,\lambda\,b_1\,b_2\,f_{12}\,f_{22}\,c_{0,2}^2\,c_{2,0}+$$
$$4\,\lambda\,f_{11}\,f_{12}\,c_{0,1}\,c_{1,0}\,c_{2,0}+8\,\lambda\,b_2\,f_{11}\,f_{12}\,c_{0,2}\,c_{1,0}\,c_{2,0}+8\,\lambda\,b_1\,f_{12}^2\,c_{0,2}\,c_{1,0}\,c_{2,0}+2\,\lambda\,f_{11}^2\,c_{1,0}^2\,c_{2,0}+$$
$$4\,\lambda\,f_{11}\,f_{12}\,c_{1,1}\,c_{2,0}+4\,\lambda\,b_2\,f_{11}\,f_{12}\,c_{0,1}\,c_{1,1}\,c_{2,0}+8\,\lambda\,b_1\,f_{12}^2\,c_{0,1}\,c_{1,1}\,c_{2,0}+4\,\lambda\,b_1\,f_{11}\,f_{22}\,c_{0,1}\,c_{1,1}\,c_{2,0}+$$
$$8\,\lambda\,b_2^2\,f_{11}\,f_{12}\,c_{0,2}\,c_{1,1}\,c_{2,0}+24\,\lambda\,b_1\,b_2\,f_{12}^2\,c_{0,2}\,c_{1,1}\,c_{2,0}+8\,\lambda\,b_1\,b_2\,f_{11}\,f_{22}\,c_{0,2}\,c_{1,1}\,c_{2,0}+$$
$$8\,\lambda\,b_1^2\,f_{12}\,f_{22}\,c_{0,2}\,c_{1,1}\,c_{2,0}+4\,\lambda\,b_2\,f_{11}^2\,c_{1,0}\,c_{1,1}\,c_{2,0}+12\,\lambda\,b_1\,f_{11}\,f_{12}\,c_{1,0}\,c_{1,1}\,c_{2,0}+$$
$$2\,\lambda\,b_2^2\,f_{11}^2\,c_{1,1}^2\,c_{2,0}+12\,\lambda\,b_1\,b_2\,f_{11}\,f_{12}\,c_{1,1}^2\,c_{2,0}+6\,\lambda\,b_1^2\,f_{12}^2\,c_{1,1}^2\,c_{2,0}+4\,\lambda\,b_1^2\,f_{11}\,f_{22}\,c_{1,1}^2\,c_{2,0}+$$
$$2\,\lambda\,f_{11}^2\,c_{2,0}^2+8\,\lambda\,b_1\,f_{11}\,f_{12}\,c_{0,1}\,c_{2,0}^2+16\,\lambda\,b_1\,b_2\,f_{11}\,f_{12}\,c_{0,2}\,c_{2,0}^2+8\,\lambda\,b_1^2\,f_{12}^2\,c_{0,2}\,c_{2,0}^2+$$
$$8\,\lambda\,b_1\,f_{11}^2\,c_{1,0}\,c_{2,0}^2+8\,\lambda\,b_1\,b_2\,f_{11}^2\,c_{1,1}\,c_{2,0}^2+16\,\lambda\,b_1^2\,f_{11}\,f_{12}\,c_{1,1}\,c_{2,0}^2+8\,\lambda\,b_1^2\,f_{11}^2\,c_{2,0}^3\bigr)\,\hbar^2+O[\hbar]^3$$

In[∘]:= **Short[rhs = Expand@Sum[(∂$_{b1,b2}$(B.F.B)) (∂$_{b1,b2}$Z + (∂$_{b1}$Z) (∂$_{b2}$Z)) / 4, {b1, B}, {b2, B}]]**

Out[∘]//Short=

$$\Bigl(\frac{1}{2}\,f_{22}\,c_{0,1}^2+f_{22}\,c_{0,2}+2\,b_2\,f_{22}\,c_{0,1}\,c_{0,2}+\ll 18\gg+2\,b_1\,b_2\,f_{11}\,c_{1,1}\,c_{2,0}+2\,b_1^2\,f_{12}\,c_{1,1}\,c_{2,0}+2\,b_1^2\,f_{11}\,c_{2,0}^2\Bigr)$$
$$\hbar+\bigl(\ll 1\gg\bigr)\ \ll 1\gg+\ll 1\gg+O[\hbar]^4$$

In[∘]:= **HL[Normal[lhs − rhs] == 0]**

Out[∘]= True

In[ ]:= **Z /. λ → 1 /. Alternatives @@ B → 0**

Out[ ]= $c_{0,0} + \left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + f_{12} c_{0,1} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \left(f_{22}^2 c_{0,1}^2 c_{0,2} + f_{22}^2 c_{0,2}^2 + 2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + f_{12}^2 c_{0,2} c_{1,0}^2 + f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 f_{12} f_{22} c_{0,2} c_{1,1} + f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} f_{12}^2 c_{1,1}^2 + \frac{1}{2} f_{11} f_{22} c_{1,1}^2 + f_{12}^2 c_{0,1}^2 c_{2,0} + 2 f_{12}^2 c_{0,2} c_{2,0} + 2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + f_{11}^2 c_{1,0}^2 c_{2,0} + 2 f_{11} f_{12} c_{1,1} c_{2,0} + f_{11}^2 c_{2,0}^2\right) \hbar^2 + O[\hbar]^3$