Pensieve header: Testing and implementing lemmas 1,2,3 of the DoPeGDO handouts. Continues pensieve://2019-12/, continued pensieve://2020-03/.

$$[F:\mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad\text{and}\quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

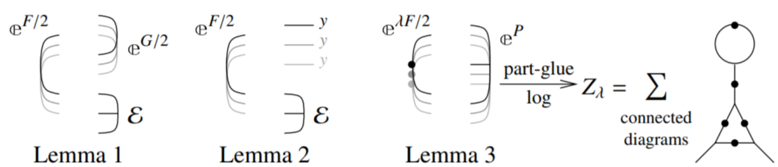
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B} y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Goals:

Implement the container $\mathbb{E}[\omega, Q, P] := \omega\,\mathbb{e}^{Q+P}$.

Implement the containers $|F,\mathcal{E}|_B := [F:\mathcal{E}]_B$ and $\langle F,\mathcal{E}\rangle_B := \langle F:\mathcal{E}\rangle_B$, their evaluator Ev_k as power series in $\hbar$ to degree k, and verify lemmas 1, 2, and 3. Inserting $\hbar$ in the appropriate places is user responsibility.

Implement DaGauss, DeLin, and a Lemma 3 evaluator, PEv.

Utilities

In[ ]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Generic Polynomials:

In[ ]:=
```
GenericPolynomial[d_Integer, vars_List, gc_] := Total[Map[
    gc_{Sequence@@#} Times @@ (vars^#) &,
    Join @@ (Permutations /@ IntegerPartitions[d + Length@vars, {Length@vars}]) - 1
  ]];
GenericPolynomial[specs_List, vars_List, gc_] :=
 Sum[GenericPolynomial[specs⟦1⟧, vars, gc], specs]
```

Preliminary Definitions

In[∘]:=
```
Unprotect[SeriesData];
Expand[sd_SeriesData] ^:= MapAt[Expand, sd, 3];
Protect[SeriesData];
```

In[∘]:=
```
CF[⟨F_, ℰ_⟩B_] := ⟨Simplify@F, Simplify@ℰ⟩B;
```

Variables and their duals:

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)i; (B_List)* := #* & /@ B;
```

Act and Contract:

In[∘]:=
```
Evk_@⦃F_, ℰ_⦄B_ := Expand[Total[
     CoefficientRules[Normal@Series[e^(B*.F.B*/2), {ℏ, 0, k}], B*] /.
      (ps_ → c_) ⧴ c D[ℰ, Sequence @@ Thread[{B, ps}]]
    ] + O[ℏ]^(k+1)];
Evk_@⟨F_, ℰ_⟩B_ := Evk@⦃F, ℰ⦄B /. Alternatives @@ B → 0
```

In[∘]:=
```
{Ev2@⦃ℏ (0 1; 1 0), e^(x y)⦄{x,y}, Ev3@⟨ℏ (0 1; 1 0), e^(3 x y)⟩{x,y}}
```

Out[∘]=
$$\left\{e^{x y}+\left(e^{x y}+e^{x y} x y\right) \hbar+\left(e^{x y}+2 e^{x y} x y+\frac{1}{2} e^{x y} x^2 y^2\right) \hbar^2+O[\hbar]^3,\ 1+3 \hbar+9 \hbar^2+27 \hbar^3+O[\hbar]^4\right\}$$

Testing Lemma 1

Lemma 1. With convergences left to the reader,

$$\left\langle F\colon \mathcal{E}\,e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}\colon \mathcal{E}\right\rangle_B$$

In[∘]:=
```
{p = 10, B = {x}, I = IdentityMatrix@Length@B,
  F = ℏ {{f}}, G = {{g}}, ℰ = GenericPolynomial[{d, 0, 4}, B, c]}
lhs = Ev_p@⟨F, ℰ e^(B.G.B/2)⟩_B
rhs = Ev_p@⟨F.Inverse[I - G.F], Det[I - G.F]^(-1/2) ℰ⟩_B;
HL[lhs == rhs]
```

Out[∘]= $\{10, \{x\}, \{\{1\}\}, \{\{f\,\hbar\}\}, \{\{g\}\}, c_0 + x\,c_1 + x^2\,c_2 + x^3\,c_3 + x^4\,c_4\}$

Out[∘]=
$$c_0 + \left(\frac{1}{2} f\,g\,c_0 + f\,c_2\right)\hbar + \left(\frac{3}{8} f^2 g^2 c_0 + \frac{3}{2} f^2 g\,c_2 + 3 f^2 c_4\right)\hbar^2 +$$
$$\left(\frac{5}{16} f^3 g^3 c_0 + \frac{15}{8} f^3 g^2 c_2 + \frac{15}{2} f^3 g\,c_4\right)\hbar^3 + \left(\frac{35}{128} f^4 g^4 c_0 + \frac{35}{16} f^4 g^3 c_2 + \frac{105}{8} f^4 g^2 c_4\right)\hbar^4 +$$
$$\left(\frac{63}{256} f^5 g^5 c_0 + \frac{315}{128} f^5 g^4 c_2 + \frac{315}{16} f^5 g^3 c_4\right)\hbar^5 + \left(\frac{231 f^6 g^6 c_0}{1024} + \frac{693}{256} f^6 g^5 c_2 + \frac{3465}{128} f^6 g^4 c_4\right)\hbar^6 +$$
$$\left(\frac{429 f^7 g^7 c_0}{2048} + \frac{3003 f^7 g^6 c_2}{1024} + \frac{9009}{256} f^7 g^5 c_4\right)\hbar^7 + \left(\frac{6435 f^8 g^8 c_0}{32768} + \frac{6435 f^8 g^7 c_2}{2048} + \frac{45045 f^8 g^6 c_4}{1024}\right)\hbar^8 +$$
$$\left(\frac{12155 f^9 g^9 c_0}{65536} + \frac{109395 f^9 g^8 c_2}{32768} + \frac{109395 f^9 g^7 c_4}{2048}\right)\hbar^9 +$$
$$\left(\frac{46189 f^{10} g^{10} c_0}{262144} + \frac{230945 f^{10} g^9 c_2}{65536} + \frac{2078505 f^{10} g^8 c_4}{32768}\right)\hbar^{10} + O[\hbar]^{11}$$

Out[∘]= True

In[ ]:=
```
{p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), ℰ = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ e^(B.G.B/2)⟩_B]
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) ℰ⟩_B;
HL[lhs == rhs]
```

Out[ ]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[ ]= $$\Big\{0.0625,\ c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right)\hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right)\hbar^2 + O[\hbar]^3\Big\}$$

Out[ ]= True

```
In[◦]:= {p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ (f11 f12
         f12 f22), G = (g11 g12
                        g12 g22), ℰ = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ e^(B.G.B/2)⟩_B]
rhs = Expand[Series[Det[𝓘 - G.F]^(-1/2), {ℏ, 0, p}] Ev_p[⟨F.Inverse[𝓘 - G.F], ℰ⟩_B]]
HL[lhs == rhs]
```

Out[◦]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y c_{0,1} + y^2 c_{0,2} + x c_{1,0} + x y c_{1,1} + x^2 c_{2,0}\}$

Out[◦]= $$\Big\{0.0625,\ c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right) \hbar^2 + O[\hbar]^3\Big\}$$

Out[◦]= $$c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right) \hbar^2 + O[\hbar]^3$$

Out[◦]= True

$$\left[F\colon \mathcal{E}\, e^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j}\right]_B = \det(1-GF)^{-1/2} e^{\frac{1}{2}\sum_{i,j\in B}(G(I-FG)^{-1})_{ij} z_i z_j} \cdot \left(\left[F(1-GF)^{-1}\colon \mathcal{E}\right]_B\right)_{z_B\to(I-}$$

```
In[◦]:= {p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
          F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
        Timing[lhs = Ev_p@⌈F, 𝓔 e^(B.G.B/2)⌉_B];
        rhs = Series[Det[𝓘 - G.F]^(-1/2) e^(B.G.Inverse[𝓘-F.G].B/2), {ℏ, 0, p}]
            (Ev_p@⌈F.Inverse[𝓘 - G.F], 𝓔⌉_B /. Thread[B → Inverse[𝓘 - F.G].B]);
        HL@FullSimplify[Normal[lhs - rhs] == 0]
Out[◦]= {2, {x, y}, {{1, 0}, {0, 1}}, {{ℏ f11, ℏ f12}, {ℏ f12, ℏ f22}},
          {{g11, g12}, {g12, g22}}, c0,0 + y c0,1 + y^2 c0,2 + x c1,0 + x y c1,1 + x^2 c2,0}
Out[◦]= True
```

Variants of Lemma 1

```
In[◦]:= {p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
          F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
        Timing[lhs = Ev_p@⌈F, 𝓔 e^(B.G.B/2)⌉_B];
        rhs = Series[Det[𝓘 - F.G]^(-1/2) e^(B.G.Inverse[𝓘-F.G].B/2), {ℏ, 0, p}]
            (Ev_p@⌈F.Inverse[𝓘 - G.F], 𝓔⌉_B /. Thread[B → Inverse[𝓘 - F.G].B]);
        HL@FullSimplify[Normal[lhs - rhs] == 0]
Out[◦]= {2, {x, y}, {{1, 0}, {0, 1}}, {{ℏ f11, ℏ f12}, {ℏ f12, ℏ f22}},
          {{g11, g12}, {g12, g22}}, c0,0 + y c0,1 + y^2 c0,2 + x c1,0 + x y c1,1 + x^2 c2,0}
Out[◦]= True
```

```
In[◦]:= {p = 1, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
          F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), 𝓔 = GenericPolynomial[{d, 0, 2}, B, c]}
        Timing[lhs = Ev_p@⌈F, 𝓔 e^(B.G.B/2)⌉_B];
        rhs = Series[Det[𝓘 - G.F]^(-1/2) e^(B.F.Inverse[𝓘-G.F].B/2), {ℏ, 0, p}]
            (Ev_p@⌈F.Inverse[𝓘 - G.F], 𝓔⌉_B /. Thread[B → Inverse[𝓘 - F.G].B]);
        HL@FullSimplify[Normal[lhs - rhs] == 0]
Out[◦]= {1, {x, y}, {{1, 0}, {0, 1}}, {{ℏ f11, ℏ f12}, {ℏ f12, ℏ f22}},
          {{g11, g12}, {g12, g22}}, c0,0 + y c0,1 + y^2 c0,2 + x c1,0 + x y c1,1 + x^2 c2,0}
Out[◦]= $Aborted
```

In[∘]:=
```
{p = 1, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), ℰ = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ e^(B.G.B/2)⟩_B];
rhs = Series[Det[𝓘 - G.F]^(-1/2) e^(B.G.Inverse[𝓘-F.G].B/2), {ℏ, 0, p}]
   (Ev_p@⟨F.Inverse[𝓘 - G.F], ℰ⟩_B /. Thread[B → Inverse[𝓘 - G.F].B]);
HL@FullSimplify[Normal[lhs - rhs] == 0]
```

Out[∘]= $\{1, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[∘]= $e^{\frac{1}{2}(x^2 g_{11} + y(2 x g_{12} + y g_{22}))} \hbar\, ((-f_{11} + f_{22}) g_{12} + f_{12}(g_{11} - g_{22}))\, (x\, c_{0,1} + 2 x y\, c_{0,2} + x^2 c_{1,1} - y\,(c_{1,0} + y\, c_{1,1} + 2 x\, c_{2,0})) == 0$

Implementing Lemma 1

In[∘]:=
```
DeGauss@⟨F_, ℰ_⟩_B_ := Module[{𝓘, Q, G, M, Δ},
   𝓘 = IdentityMatrix@Length@B;
   Q = Log[Normal[ℰ] /. ϵ → 0];
   G = Table[∂_{i,j} Q, {i, B}, {j, B}];
   M = Inverse[𝓘 - G.F];
   Δ = Simplify@Det@M;
   CF@⟨F.M, Δ^(1/2) ℰ e^(-B.G.B/2)⟩_B
  ]
```

```
In[]:= {p = 2, B = {x, y}, F = ℏ (f11 f12; f12 f22),
   G = (g11 g12; g12 g22), ℰ = 1 + ∈ GenericPolynomial[{d, 0, 2}, B, c]};
  lhs = ⟨F, ℰ e^(B.G.B/2)⟩_B
  rhs = DeGauss@⟨F, ℰ e^(B.G.B/2)⟩_B
  HL[Ev_p@lhs == Ev_p@rhs]
```

Out[]= $\left\langle \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, e^{\frac{1}{2}(x(x g_{11}+y g_{12})+y(x g_{12}+y g_{22}))}\left(1+\in\left(c_{0,0}+y c_{0,1}+y^2 c_{0,2}+x c_{1,0}+x y c_{1,1}+x^2 c_{2,0}\right)\right)\right\rangle_{\{x,y\}}$

Out[]= $\left\langle\left\{\left\{\frac{\hbar\left(\hbar f_{12}^2 g_{22}+f_{11}(1-\hbar f_{22} g_{22})\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)}, \frac{\hbar\left(f_{12}-\hbar f_{12}^2 g_{12}+\hbar f_{11} f_{22} g_{12}\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)}\right\}, \left\{\frac{\hbar\left(f_{12}-\hbar f_{12}^2 g_{12}+\hbar f_{11} f_{22} g_{12}\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)}, \frac{\hbar\left(\hbar f_{12}^2 g_{11}+f_{22}(1-\hbar f_{11} g_{11})\right)}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)}\right\}\right\}, \sqrt{\frac{1}{1-2\hbar f_{12} g_{12}-\hbar f_{22} g_{22}+\hbar^2 f_{12}^2\left(g_{12}^2-g_{11} g_{22}\right)+\hbar f_{11}\left(-\hbar f_{22} g_{12}^2+g_{11}(-1+\hbar f_{22} g_{22})\right)}}\left(1+\in c_{0,0}+y\in c_{0,1}+y^2\in c_{0,2}+x\in c_{1,0}+x y\in c_{1,1}+x^2\in c_{2,0}\right)\right\rangle_{\{x,y\}}$

Out[]= True

Implementing / Testing Lemma 2

Lemma 2. $\left\langle F: \mathcal{E}\, e^{\sum_{i\in B} y_i z_i}\right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+F y_B}\right\rangle_B.$

```
In[]:= {n = 2, p = 2, B = Table[z_i, {i, n}], I = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
   F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ∈ GenericPolynomial[{d, 0, 3}, B, c]}
  Timing[lhs = Ev_p@⟨F, ℰ e^(Y.B)⟩_B]
  Timing[rhs = Ev_p@⟨F, e^(Y.F.Y/2) ℰ /. Thread[B → B + F.Y]⟩_B]
  HL@Expand[lhs == rhs]
```

Out[]= $\{2, 2, \{z_1, z_2\}, \{\{1, 0\}, \{0, 1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, 1+\in\left(c_{0,0}+z_2 c_{0,1}+z_2^2 c_{0,2}+z_2^3 c_{0,3}+z_1 c_{1,0}+z_1 z_2 c_{1,1}+z_1 z_2^2 c_{1,2}+z_1^2 c_{2,0}+z_1^2 z_2 c_{2,1}+z_1^3 c_{3,0}\right)\}$

Out[•]=

$$\Big\{0.015625,$$

$$(1+\epsilon\, c_{0,0}) + \Big(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \epsilon f_{11} y_1^2 c_{0,0} + \epsilon f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \epsilon f_{22} y_2^2 c_{0,0} + \epsilon f_{12} y_1 c_{0,1} + \epsilon f_{22} y_2 c_{0,1} + \epsilon f_{22} c_{0,2} + \epsilon f_{11} y_1 c_{1,0} + \epsilon f_{12} y_2 c_{1,0} + \epsilon f_{12} c_{1,1} + \epsilon f_{11} c_{2,0}\Big) \hbar +$$

$$\Big(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 +$$

$$\frac{1}{8} \epsilon f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \epsilon f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \epsilon f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \epsilon f_{11} f_{22} y_1^2 y_2^2 c_{0,0} +$$

$$\frac{1}{2} \epsilon f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \epsilon f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \epsilon f_{11} f_{12} y_1^3 c_{0,1} + \epsilon f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \epsilon f_{11} f_{22} y_1^2 y_2 c_{0,1} +$$

$$\frac{3}{2} \epsilon f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \epsilon f_{22}^2 y_2^3 c_{0,1} + \epsilon f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \epsilon f_{11} f_{22} y_1^2 c_{0,2} + 3 \epsilon f_{12} f_{22} y_1 y_2 c_{0,2} +$$

$$\frac{3}{2} \epsilon f_{22}^2 y_2^2 c_{0,2} + 3 \epsilon f_{12} f_{22} y_1 c_{0,3} + 3 \epsilon f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \epsilon f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \epsilon f_{11} f_{12} y_1^2 y_2 c_{1,0} +$$

$$\epsilon f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \epsilon f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \epsilon f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \epsilon f_{11} f_{12} y_1^2 c_{1,1} + 2 \epsilon f_{12}^2 y_1 y_2 c_{1,1} +$$

$$\epsilon f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \epsilon f_{12} f_{22} y_2^2 c_{1,1} + 2 \epsilon f_{12}^2 y_1 c_{1,2} + \epsilon f_{11} f_{22} y_1 c_{1,2} + 3 \epsilon f_{12} f_{22} y_2 c_{1,2} +$$

$$\frac{3}{2} \epsilon f_{11}^2 y_1^2 c_{2,0} + 3 \epsilon f_{11} f_{12} y_1 y_2 c_{2,0} + \epsilon f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \epsilon f_{11} f_{22} y_2^2 c_{2,0} + 3 \epsilon f_{11} f_{12} y_1 c_{2,1} +$$

$$2 \epsilon f_{12}^2 y_2 c_{2,1} + \epsilon f_{11} f_{22} y_2 c_{2,1} + 3 \epsilon f_{11}^2 y_1 c_{3,0} + 3 \epsilon f_{11} f_{12} y_2 c_{3,0}\Big) \hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]= $\{0.015625, (1+\in c_{0,0}) + \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2}\in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2}\in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} + \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0}\right)\hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \frac{1}{8}\in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2}\in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2}\in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4}\in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2}\in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8}\in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2}\in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2}\in f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2}\in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2}\in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2}\in f_{11} f_{22} y_1^2 c_{0,2} + 3\in f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2}\in f_{22}^2 y_2^2 c_{0,2} + 3\in f_{12} f_{22} y_1 c_{0,3} + 3\in f_{22}^2 y_2 c_{0,3} + \frac{1}{2}\in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2}\in f_{11} f_{12} y_1^2 y_2 c_{1,0} + \in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2}\in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2}\in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2}\in f_{11} f_{12} y_1^2 c_{1,1} + 2\in f_{12}^2 y_1 y_2 c_{1,1} + \in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2}\in f_{12} f_{22} y_2^2 c_{1,1} + 2\in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3\in f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2}\in f_{11}^2 y_1^2 c_{2,0} + 3\in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2}\in f_{11} f_{22} y_2^2 c_{2,0} + 3\in f_{11} f_{12} y_1 c_{2,1} + 2\in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3\in f_{11}^2 y_1 c_{3,0} + 3\in f_{11} f_{12} y_2 c_{3,0}\right)\hbar^2 + O[\hbar]^3\}$

Out[∘]= True

$$\left[F\colon \mathcal{E}\, e^{\sum_{i\in B} y_i z_i}\right]_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j + \sum_{i\in B} y_i z_i}\left[F\colon\ \mathcal{E}\big|_{z_B\to z_B+F y_B}\right]_B$$

In[∘]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], 𝓘 = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
  F = ℏ Table[f_{{10,1}.Sort[{i,j}]}, {i, n}, {j, n}], 𝓔 = 1 + ∈ GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Ev_p@|F, 𝓔 e^(Y.B)|_B] // Short
Timing[rhs = e^(Y.B) Ev_p@|F, e^(Y.F.Y/2) 𝓔 /. Thread[B → B + F.Y]|_B];
HL@Expand[lhs == rhs]
```

Out[∘]= $\{2, 2, \{z_1, z_2\}, \{\{1,0\},\{0,1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, 1 + \in (c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0})\}$

Out[∘]//Short= $\{0.453125, (e^{y_1 z_1 + y_2 z_2} + e^{y_1 z_1 + y_2 z_2}\in c_{0,0} + \ll 8\gg + e^{y_1 z_1 + y_2 z_2}\in z_1^3 c_{3,0}) + \left(\frac{1}{2} e^{\ll 1\gg} f_{11} y_1^2 + \ll 64\gg + \ll 1\gg\right)\hbar + \ll 1\gg + O[\hbar]^3\}$

Out[∘]= True

```
In[•]:= {n = 2, p = 2, B = Table[z_i, {i, n}], 𝓘 = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
         F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ϵ GenericPolynomial[{d, 0, 3}, B, c]}
        Timing[lhs = Ev_p@|F, ℰ e^(Y.B)|_B] // Short
        Timing[rhs = e^(Y.B) (Ev_p@|F, e^(Y.F.Y/2) ℰ|_B /. Thread[B → B + F.Y])];
        HL@Expand[lhs == rhs]
```

Out[•]= $\{2, 2, \{z_1, z_2\}, \{\{1, 0\}, \{0, 1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$1 + \epsilon\,(c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0})\}$

Out[•]//Short= $\{0.015625, (e^{y_1 z_1 + y_2 z_2} + e^{y_1 z_1 + y_2 z_2} \epsilon\, c_{0,0} + \ll 8 \gg + e^{y_1 z_1 + y_2 z_2} \epsilon\, z_1^3 c_{3,0}) +$
$\left(\frac{1}{2} e^{\ll 1 \gg} f_{11} y_1^2 + \ll 64 \gg + \ll 1 \gg\right) \hbar + \ll 1 \gg + O[\hbar]^3\}$

Out[•]= True

```
In[•]:= DeLin@⟨F_, ℰ_⟩_B_ := Module[{L, Y},
          L = PowerExpand@Log[Normal[ℰ] /. ϵ → 0];
          Y = Table[∂_i L, {i, B}];
          CF@⟨F, e^(Y.F.Y/2) (e^(-B.Y) ℰ /. Thread[B → B + F.Y])⟩_B
        ]
```

```
In[•]:= {n = 2, p = 2, B = Table[z_i, {i, n}], Y = Table[y_i, {i, n}],
         F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ϵ GenericPolynomial[{d, 0, 3}, B, c]}
        Timing[lhs = Ev_p@⟨F, ℰ e^(Y.B)⟩_B]
        Timing[rhs = Ev_p@DeLin@⟨F, ℰ e^(Y.B)⟩_B]
        HL@Simplify[lhs == rhs]
```

Out[•]= $\{2, 2, \{z_1, z_2\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$1 + \epsilon\,(c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0})\}$

Out[∘]=

$$\Big\{0.03125,\ (1+\epsilon\, c_{0,0}) + \Big(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2}\epsilon f_{11} y_1^2 c_{0,0} + \epsilon f_{12} y_1 y_2 c_{0,0} + \frac{1}{2}\epsilon f_{22} y_2^2 c_{0,0} + \epsilon f_{12} y_1 c_{0,1} + \epsilon f_{22} y_2 c_{0,1} + \epsilon f_{22} c_{0,2} + \epsilon f_{11} y_1 c_{1,0} + \epsilon f_{12} y_2 c_{1,0} + \epsilon f_{12} c_{1,1} + \epsilon f_{11} c_{2,0}\Big)\hbar +$$

$$\Big(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 +$$

$$\frac{1}{8}\epsilon f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2}\epsilon f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2}\epsilon f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4}\epsilon f_{11} f_{22} y_1^2 y_2^2 c_{0,0} +$$

$$\frac{1}{2}\epsilon f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8}\epsilon f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2}\epsilon f_{11} f_{12} y_1^3 c_{0,1} + \epsilon f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2}\epsilon f_{11} f_{22} y_1^2 y_2 c_{0,1} +$$

$$\frac{3}{2}\epsilon f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2}\epsilon f_{22}^2 y_2^3 c_{0,1} + \epsilon f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2}\epsilon f_{11} f_{22} y_1^2 c_{0,2} + 3\epsilon f_{12} f_{22} y_1 y_2 c_{0,2} +$$

$$\frac{3}{2}\epsilon f_{22}^2 y_2^2 c_{0,2} + 3\epsilon f_{12} f_{22} y_1 c_{0,3} + 3\epsilon f_{22}^2 y_2 c_{0,3} + \frac{1}{2}\epsilon f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2}\epsilon f_{11} f_{12} y_1^2 y_2 c_{1,0} +$$

$$\epsilon f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2}\epsilon f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2}\epsilon f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2}\epsilon f_{11} f_{12} y_1^2 c_{1,1} + 2\epsilon f_{12}^2 y_1 y_2 c_{1,1} +$$

$$\epsilon f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2}\epsilon f_{12} f_{22} y_2^2 c_{1,1} + 2\epsilon f_{12}^2 y_1 c_{1,2} + \epsilon f_{11} f_{22} y_1 c_{1,2} + 3\epsilon f_{12} f_{22} y_2 c_{1,2} +$$

$$\frac{3}{2}\epsilon f_{11}^2 y_1^2 c_{2,0} + 3\epsilon f_{11} f_{12} y_1 y_2 c_{2,0} + \epsilon f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2}\epsilon f_{11} f_{22} y_2^2 c_{2,0} + 3\epsilon f_{11} f_{12} y_1 c_{2,1} +$$

$$2\epsilon f_{12}^2 y_2 c_{2,1} + \epsilon f_{11} f_{22} y_2 c_{2,1} + 3\epsilon f_{11}^2 y_1 c_{3,0} + 3\epsilon f_{11} f_{12} y_2 c_{3,0}\Big)\hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]= $\{0., (1+\in c_{0,0}) + \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2}\in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2}\in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} + \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0}\right)\hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \frac{1}{8}\in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2}\in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2}\in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4}\in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2}\in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8}\in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2}\in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2}\in f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2}\in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2}\in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2}\in f_{11} f_{22} y_1^2 c_{0,2} + 3\in f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2}\in f_{22}^2 y_2^2 c_{0,2} + 3\in f_{12} f_{22} y_1 c_{0,3} + 3\in f_{22}^2 y_2 c_{0,3} + \frac{1}{2}\in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2}\in f_{11} f_{12} y_1^2 y_2 c_{1,0} + \in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2}\in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2}\in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2}\in f_{11} f_{12} y_1^2 c_{1,1} + 2\in f_{12}^2 y_1 y_2 c_{1,1} + \in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2}\in f_{12} f_{22} y_2^2 c_{1,1} + 2\in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3\in f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2}\in f_{11}^2 y_1^2 c_{2,0} + 3\in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2}\in f_{11} f_{22} y_2^2 c_{2,0} + 3\in f_{11} f_{12} y_1 c_{2,1} + 2\in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3\in f_{11}^2 y_1 c_{3,0} + 3\in f_{11} f_{12} y_2 c_{3,0}\right)\hbar^2 + O[\hbar]^3\}$

Out[∘]= True

Testing Lemma 3

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda)\right).$$

```
In[∘]:= {n = 2, p = 2, B = Table[b_i, {i, n}],
  F = ℏ Table[f_{{10,1}.Sort[{i,j}]}, {i, n}, {j, n}], P = GenericPolynomial[{d, 0, 2}, B, c]}
```

Out[∘]= $\{2, 2, \{b_1, b_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}\}$

In[∘]:= **Z = PowerExpand@Expand@Log[Ev$_p$@ $\left|\lambda F, e^{P}\right|_B$]**

Out[∘]=

$$\left(c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}\right) +$$

$$\left(\frac{1}{2} \lambda f_{22} c_{0,1}^2 + \lambda f_{22} c_{0,2} + 2 \lambda b_2 f_{22} c_{0,1} c_{0,2} + 2 \lambda b_2^2 f_{22} c_{0,2}^2 + \lambda f_{12} c_{0,1} c_{1,0} + 2 \lambda b_2 f_{12} c_{0,2} c_{1,0} +\right.$$

$$\frac{1}{2} \lambda f_{11} c_{1,0}^2 + \lambda f_{12} c_{1,1} + \lambda b_2 f_{12} c_{0,1} c_{1,1} + \lambda b_1 f_{22} c_{0,1} c_{1,1} + 2 \lambda b_2^2 f_{12} c_{0,2} c_{1,1} +$$

$$2 \lambda b_1 b_2 f_{22} c_{0,2} c_{1,1} + \lambda b_2 f_{11} c_{1,0} c_{1,1} + \lambda b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2} \lambda b_2^2 f_{11} c_{1,1}^2 +$$

$$\lambda b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2} \lambda b_1^2 f_{22} c_{1,1}^2 + \lambda f_{11} c_{2,0} + 2 \lambda b_1 f_{12} c_{0,1} c_{2,0} + 4 \lambda b_1 b_2 f_{12} c_{0,2} c_{2,0} +$$

$$\left. 2 \lambda b_1 f_{11} c_{1,0} c_{2,0} + 2 \lambda b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 \lambda b_1^2 f_{12} c_{1,1} c_{2,0} + 2 \lambda b_1^2 f_{11} c_{2,0}^2\right) \hbar +$$

$$\left(\lambda^2 f_{22}^2 c_{0,1}^2 c_{0,2} + \lambda^2 f_{22}^2 c_{0,2}^2 + 4 \lambda^2 b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 4 \lambda^2 b_2^2 f_{22}^2 c_{0,2}^3 + 2 \lambda^2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} +\right.$$

$$4 \lambda^2 b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + \lambda^2 f_{12}^2 c_{0,2} c_{1,0}^2 + \lambda^2 f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 \lambda^2 f_{12} f_{22} c_{0,2} c_{1,1} +$$

$$6 \lambda^2 b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 2 \lambda^2 b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 8 \lambda^2 b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + 4 \lambda^2 b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} +$$

$$\lambda^2 f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + \lambda^2 f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 4 \lambda^2 b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + 2 \lambda^2 b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} +$$

$$2 \lambda^2 b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + \lambda^2 f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} \lambda^2 f_{12}^2 c_{1,1}^2 + \frac{1}{2} \lambda^2 f_{11} f_{22} c_{1,1}^2 + \lambda^2 b_2 f_{12}^2 c_{0,1} c_{1,1}^2 +$$

$$\lambda^2 b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 2 \lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + 3 \lambda^2 b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 2 \lambda^2 b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 +$$

$$6 \lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + \lambda^2 b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + 2 \lambda^2 b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + \lambda^2 b_1 f_{12}^2 c_{1,0} c_{1,1}^2 +$$

$$\lambda^2 b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + \lambda^2 b_2^2 f_{11} f_{12} c_{1,1}^3 + \lambda^2 b_1 b_2 f_{12}^2 c_{1,1}^3 + \lambda^2 b_1 b_2 f_{11} f_{22} c_{1,1}^3 + \lambda^2 b_1^2 f_{12} f_{22} c_{1,1}^3 +$$

$$\lambda^2 f_{12}^2 c_{0,1}^2 c_{2,0} + 2 \lambda^2 f_{12}^2 c_{0,2} c_{2,0} + 4 \lambda^2 b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 4 \lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} +$$

$$4 \lambda^2 b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 8 \lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + 2 \lambda^2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} +$$

$$4 \lambda^2 b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 4 \lambda^2 b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + \lambda^2 f_{11}^2 c_{1,0}^2 c_{2,0} + 2 \lambda^2 f_{11} f_{12} c_{1,1} c_{2,0} +$$

$$2 \lambda^2 b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 4 \lambda^2 b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 2 \lambda^2 b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} +$$

$$4 \lambda^2 b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 12 \lambda^2 b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 4 \lambda^2 b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} +$$

$$4 \lambda^2 b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 2 \lambda^2 b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 6 \lambda^2 b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} +$$

$$\lambda^2 b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 6 \lambda^2 b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 3 \lambda^2 b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 2 \lambda^2 b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} +$$

$$\lambda^2 f_{11}^2 c_{2,0}^2 + 4 \lambda^2 b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 8 \lambda^2 b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 4 \lambda^2 b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 +$$

$$\left. 4 \lambda^2 b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 4 \lambda^2 b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 8 \lambda^2 b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 4 \lambda^2 b_1^2 f_{11}^2 c_{2,0}^3\right) \hbar^2 + O[\hbar]^3$$

In[∘]:= **Z /. λ → 0**

Out[∘]= $\left(c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}\right) + O[\hbar]^3$

In[∘]:= **(Z /. λ → 0) - P**

Out[∘]= $O[\hbar]^3$

In[∘]:= **lhs = ∂$_\lambda$Z**

Out[∘]=

$$\Big(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + 2 b_2^2 f_{22} c_{0,2}^2 + f_{12} c_{0,1} c_{1,0} + 2 b_2 f_{12} c_{0,2} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 +$$
$$f_{12} c_{1,1} + b_2 f_{12} c_{0,1} c_{1,1} + b_1 f_{22} c_{0,1} c_{1,1} + 2 b_2^2 f_{12} c_{0,2} c_{1,1} + 2 b_1 b_2 f_{22} c_{0,2} c_{1,1} + b_2 f_{11} c_{1,0} c_{1,1} +$$
$$b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2} b_2^2 f_{11} c_{1,1}^2 + b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2} b_1^2 f_{22} c_{1,1}^2 + f_{11} c_{2,0} + 2 b_1 f_{12} c_{0,1} c_{2,0} +$$
$$4 b_1 b_2 f_{12} c_{0,2} c_{2,0} + 2 b_1 f_{11} c_{1,0} c_{2,0} + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2\Big) \hbar +$$
$$\Big(2 \lambda f_{22}^2 c_{0,1}^2 c_{0,2} + 2 \lambda f_{22}^2 c_{0,2}^2 + 8 \lambda b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 8 \lambda b_2^2 f_{22}^2 c_{0,2}^3 + 4 \lambda f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} +$$
$$8 \lambda b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + 2 \lambda f_{12}^2 c_{0,2} c_{1,0}^2 + 2 \lambda f_{12} f_{22} c_{0,1}^2 c_{1,1} + 4 \lambda f_{12} f_{22} c_{0,2} c_{1,1} +$$
$$12 \lambda b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 4 \lambda b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 16 \lambda b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} +$$
$$8 \lambda b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} + 2 \lambda f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 8 \lambda b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} +$$
$$4 \lambda b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{12} c_{1,0}^2 c_{1,1} + \lambda f_{12}^2 c_{1,1}^2 +$$
$$\lambda f_{11} f_{22} c_{1,1}^2 + 2 \lambda b_2 f_{12}^2 c_{0,1} c_{1,1}^2 + 2 \lambda b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 4 \lambda b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 +$$
$$6 \lambda b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 + 12 \lambda b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + 2 \lambda b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 +$$
$$4 \lambda b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{12}^2 c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + 2 \lambda b_2^2 f_{11} f_{12} c_{1,1}^3 +$$
$$2 \lambda b_1 b_2 f_{12}^2 c_{1,1}^3 + 2 \lambda b_1 b_2 f_{11} f_{22} c_{1,1}^3 + 2 \lambda b_1^2 f_{12} f_{22} c_{1,1}^3 + 2 \lambda f_{12}^2 c_{0,1}^2 c_{2,0} + 4 \lambda f_{12}^2 c_{0,2} c_{2,0} +$$
$$8 \lambda b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 16 \lambda b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} +$$
$$4 \lambda f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + 8 \lambda b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + 2 \lambda f_{11}^2 c_{1,0}^2 c_{2,0} +$$
$$4 \lambda f_{11} f_{12} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 4 \lambda b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} +$$
$$8 \lambda b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 24 \lambda b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} +$$
$$8 \lambda b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 12 \lambda b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} +$$
$$2 \lambda b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 12 \lambda b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 6 \lambda b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 4 \lambda b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} +$$
$$2 \lambda f_{11}^2 c_{2,0}^2 + 8 \lambda b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 16 \lambda b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 8 \lambda b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 +$$
$$8 \lambda b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 8 \lambda b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 16 \lambda b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 8 \lambda b_1^2 f_{11}^2 c_{2,0}^3\Big) \hbar^2 + O[\hbar]^3$$

In[∘]:= **Short[rhs = Expand@Sum[(∂$_{b1,b2}$(B.F.B)) (∂$_{b1,b2}$Z + (∂$_{b1}$Z) (∂$_{b2}$Z)) / 4, {b1, B}, {b2, B}]]**

Out[∘]//Short=

$$\Big(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + \ll 18 \gg + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2\Big)$$
$$\hbar + (\ll 1 \gg) \ll 1 \gg + \ll 1 \gg + O[\hbar]^4$$

In[∘]:= **HL[Normal[lhs − rhs] == 0]**

Out[∘]= True

In[∘]:= **Z /. λ → 1 /. Alternatives @@ B → 0**

Out[∘]=

$$c_{0,0} + \Big(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + f_{12} c_{0,1} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + f_{11} c_{2,0}\Big) \hbar +$$
$$\Big(f_{22}^2 c_{0,1}^2 c_{0,2} + f_{22}^2 c_{0,2}^2 + 2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + f_{12}^2 c_{0,2} c_{1,0}^2 + f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 f_{12} f_{22} c_{0,2} c_{1,1} +$$
$$f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} f_{12}^2 c_{1,1}^2 + \frac{1}{2} f_{11} f_{22} c_{1,1}^2 + f_{12}^2 c_{0,1}^2 c_{2,0} +$$
$$2 f_{12}^2 c_{0,2} c_{2,0} + 2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + f_{11}^2 c_{1,0}^2 c_{2,0} + 2 f_{11} f_{12} c_{1,1} c_{2,0} + f_{11}^2 c_{2,0}^2\Big) \hbar^2 + O[\hbar]^3$$