

Pensieve header: Testing and implementing lemmas 1,2,3 of the DoPeGDO handouts. Continues pensieve://2019-11/, continued pensieve://2020-02/.

$$[F : \mathcal{E}]_B := e^{\frac{1}{2} \sum_{i,j \in B} F_{ij} \partial_{z_i} \partial_{z_j}} \mathcal{E} \quad \text{and} \quad \langle F : \mathcal{E} \rangle_B := [F : \mathcal{E}]_B|_{z_B \rightarrow 0},$$

where \mathcal{E} is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where \mathcal{E} has no B - B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} e^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B.$$

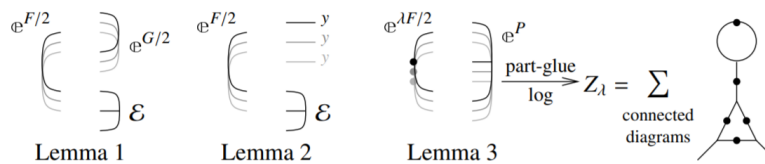
The next lemma dispatches the case where \mathcal{E} has a B -linear part:

Lemma 2. $\left\langle F : \mathcal{E} e^{\sum_{i \in B} y_i z_i} \right\rangle_B = e^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E}|_{z_B \rightarrow z_B + F y_B} \right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : e^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$



Goals:

Implement the container $\mathbb{E}[\omega, Q, P] := \omega e^{Q+P}$.

Implement the containers $|F, \mathcal{E}|_B := [F : \mathcal{E}]_B$ and $\langle F, \mathcal{E} \rangle_B := \langle F : \mathcal{E} \rangle_B$, their evaluator Ev_k as power series in \hbar to degree k , and verify lemmas 1, 2, and 3. Inserting \hbar in the appropriate places is user responsibility.

Implement DaGauss, DeLin, and a Lemma 3 evaluator, PEv.

Utilities

```
In[ ]:= HL[ε_] := Style[ε, Background → If[TrueQ@ε, Green, Red]];
```

Generic Polynomials:

```
In[ ]:= GenericPolynomial[d_Integer, vars_List, gc_] := Total[Map[
  gcSequence@# Times @@ (vars^#) &,
  Join @@ (Permutations /@ IntegerPartitions[d + Length@vars, {Length@vars}]) - 1
]];
GenericPolynomial[specs_List, vars_List, gc_] :=
  Sum[GenericPolynomial[specs[[1]], vars, gc], specs]
```

Preliminary Definitions

```
In[ ]:= Unprotect[SeriesData];
Expand[sd_SeriesData] ^= MapAt[Expand, sd, 3];
Protect[SeriesData];
```

```
In[ ]:= CF[⟨F_, E_⟩B_] := ⟨Simplify@F, Simplify@E⟩B;
```

Variables and their duals:

```
In[ ]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_{i_})^* := (u^*)_i; (B_List)^* := #* & /@B;
```

Act and Contract:

```
In[ ]:= EV_k_@|F_, E_|B_ := Expand[Total[
  CoefficientRules[Normal@Series[e^{B*.F.B*/2}, {ħ, 0, k}], B*] /.
  (ps_ -> c_) :-> c D[E, Sequence@@Thread[{B, ps}]]
] + O[ħ]^{k+1}];
EV_k_@⟨F_, E_⟩B_ := EV_k_@|F, E|B /. Alternatives@@B -> 0
```

```
In[ ]:= {EV_2@|ħ ( 0 1 ) , e^{xy} |_{x,y} , EV_3@|ħ ( 0 1 ) , e^{3xy} |_{x,y} }
```

```
Out[ ]:= {e^{xy} + (e^{xy} + e^{xy} x y) ħ + (e^{xy} + 2 e^{xy} x y + 1/2 e^{xy} x^2 y^2) ħ^2 + O[ħ]^3, 1 + 3 ħ + 9 ħ^2 + 27 ħ^3 + O[ħ]^4}
```

Implementing / Testing Lemma 1

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \otimes \frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

```
In[*]:= {p = 10, B = {x}, I = IdentityMatrix@Length@B,
  F = h {{f}}, G = {{g}}, ε = GenericPolynomial[{d, 0, 4}, B, c]}
lhs = Evp@{F, ε e^{B.G.B/2}}_B
rhs = Evp@{F.Inverse[I - G.F], Det[I - G.F]^{-1/2} ε}_B;
HL[lhs == rhs]
```

```
Out[*]:= {10, {x}, {{1}}, {{f h}}, {{g}}, c_0 + x c_1 + x^2 c_2 + x^3 c_3 + x^4 c_4}
```

$$\begin{aligned} \text{Out[*]} = & c_0 + \left(\frac{1}{2} f g c_0 + f c_2 \right) h + \left(\frac{3}{8} f^2 g^2 c_0 + \frac{3}{2} f^2 g c_2 + 3 f^2 c_4 \right) h^2 + \\ & \left(\frac{5}{16} f^3 g^3 c_0 + \frac{15}{8} f^3 g^2 c_2 + \frac{15}{2} f^3 g c_4 \right) h^3 + \left(\frac{35}{128} f^4 g^4 c_0 + \frac{35}{16} f^4 g^3 c_2 + \frac{105}{8} f^4 g^2 c_4 \right) h^4 + \\ & \left(\frac{63}{256} f^5 g^5 c_0 + \frac{315}{128} f^5 g^4 c_2 + \frac{315}{16} f^5 g^3 c_4 \right) h^5 + \left(\frac{231 f^6 g^6 c_0}{1024} + \frac{693}{256} f^6 g^5 c_2 + \frac{3465}{128} f^6 g^4 c_4 \right) h^6 + \\ & \left(\frac{429 f^7 g^7 c_0}{2048} + \frac{3003 f^7 g^6 c_2}{1024} + \frac{9009}{256} f^7 g^5 c_4 \right) h^7 + \left(\frac{6435 f^8 g^8 c_0}{32768} + \frac{6435 f^8 g^7 c_2}{2048} + \frac{45045 f^8 g^6 c_4}{1024} \right) h^8 + \\ & \left(\frac{12155 f^9 g^9 c_0}{65536} + \frac{109395 f^9 g^8 c_2}{32768} + \frac{109395 f^9 g^7 c_4}{2048} \right) h^9 + \\ & \left(\frac{46189 f^{10} g^{10} c_0}{262144} + \frac{230945 f^{10} g^9 c_2}{65536} + \frac{2078505 f^{10} g^8 c_4}{32768} \right) h^{10} + O[h]^{11} \end{aligned}$$

```
Out[*]:= True
```

```
In[*]:= {p = 2, B = {x, y}, I = IdentityMatrix@Length@B,
  F = ħ  $\begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$ , G =  $\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$ , ε = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Evp@⟨F, ε eB.G.B/2⟩B]
rhs = Evp@⟨F.Inverse[I - G.F], Det[I - G.F]-1/2 ε⟩B;
HL[lhs == rhs]
```

```
Out[*]:= {2, {x, y}, {{1, 0}, {0, 1}}, {{ħ f11, ħ f12}, {ħ f12, ħ f22}},
  {{g11, g12}, {g12, g22}}, c0,0 + y c0,1 + y2 c0,2 + x c1,0 + x y c1,1 + x2 c2,0}
```

```
Out[*]:= {0.0625, c0,0 +  $\left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar +$ 
 $\left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} +$ 
 $\frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} +$ 
 $3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} +$ 
 $\frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right) \hbar^2 + O[\hbar]^3}$ 
```

```
Out[*]:= True
```

```
In[*]:= DeGauss@⟨F_, ε_⟩B_ := Module[{I, Q, G, M, Δ},
  I = IdentityMatrix@Length@B;
  Q = Log[Normal[ε] /. e → 0];
  G = Table[∂i,jQ, {i, B}, {j, B}];
  M = Inverse[I - G.F];
  Δ = Simplify@Det@M;
  CF@⟨F.M, Δ1/2 ε e-B.G.B/2⟩B
]
```

```

In[ ]:= {p = 2, B = {x, y}, F = ħ ( f11 f12
      f12 f22 ),
  G = ( g11 g12
      g12 g22 ), ε = 1 + ε GenericPolynomial[{d, 0, 2}, B, c]};
lhs = ⟨F, ε e^{B.G.B/2}⟩_B
rhs = DeGauss@⟨F, ε e^{B.G.B/2}⟩_B
HL[Ev_p@lhs == Ev_p@rhs]

Out[ ]:= { {ħ f11, ħ f12}, {ħ f12, ħ f22} },
  e^{1/2 (x (x g11 + y g12) + y (x g12 + y g22))} (1 + ε (c_{0,0} + y c_{0,1} + y^2 c_{0,2} + x c_{1,0} + x y c_{1,1} + x^2 c_{2,0})) }_{x,y}

Out[ ]:= { { {
  ħ (ħ f12^2 g22 + f11 (1 - ħ f22 g22))
  1 - 2 ħ f12 g12 - ħ f22 g22 + ħ^2 f12^2 (g12^2 - g11 g22) + ħ f11 (-ħ f22 g12^2 + g11 (-1 + ħ f22 g22))
  ħ (f12 - ħ f12^2 g12 + ħ f11 f22 g12)
  1 - 2 ħ f12 g12 - ħ f22 g22 + ħ^2 f12^2 (g12^2 - g11 g22) + ħ f11 (-ħ f22 g12^2 + g11 (-1 + ħ f22 g22))
  ħ (f12 - ħ f12^2 g12 + ħ f11 f22 g12)
  1 - 2 ħ f12 g12 - ħ f22 g22 + ħ^2 f12^2 (g12^2 - g11 g22) + ħ f11 (-ħ f22 g12^2 + g11 (-1 + ħ f22 g22))
  ħ (ħ f12^2 g11 + f22 (1 - ħ f11 g11))
  1 - 2 ħ f12 g12 - ħ f22 g22 + ħ^2 f12^2 (g12^2 - g11 g22) + ħ f11 (-ħ f22 g12^2 + g11 (-1 + ħ f22 g22))
} } },
  1
  1 - 2 ħ f12 g12 - ħ f22 g22 + ħ^2 f12^2 (g12^2 - g11 g22) + ħ f11 (-ħ f22 g12^2 + g11 (-1 + ħ f22 g22))
  (1 + ε c_{0,0} + y ε c_{0,1} + y^2 ε c_{0,2} + x ε c_{1,0} + x y ε c_{1,1} + x^2 ε c_{2,0}) }_{x,y}

```

Out[]:= **True**

Implementing / Testing Lemma 2

Lemma 2. $\langle F: \mathcal{E}_{\mathbb{C}^{\sum_{i \in B} y_i z_i}} \rangle_B = \mathbb{C}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \langle F: \mathcal{E}_{|z_B \rightarrow z_B + F y_B} \rangle_B$.

```

In[ ]:= {n = 2, p = 2, B = Table[z_i, {i, n}], I = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
  F = ħ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ε = 1 + ε GenericPolynomial[{d, 0, 3}, B, c]
Timing[lhs = Ev_p@⟨F, ε e^{Y.B}⟩_B]
Timing[rhs = Ev_p@⟨F, e^{Y.F.Y/2} ε /. Thread[B → B + F.Y]⟩_B]
HL[lhs == rhs]

Out[ ]:= {2, 2, {z1, z2}, {{1, 0}, {0, 1}}, {y1, y2}, {{ħ f11, ħ f12}, {ħ f12, ħ f22}},
  1 + ε (c_{0,0} + z2 c_{0,1} + z2^2 c_{0,2} + z2^3 c_{0,3} + z1 c_{1,0} + z1 z2 c_{1,1} + z1 z2^2 c_{1,2} + z1^2 c_{2,0} + z1^2 z2 c_{2,1} + z1^3 c_{3,0}) }

```

$$\begin{aligned}
 \text{Out[*]} = & \{0.4375, (1 + \in c_{0,0}) + \\
 & \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} + \right. \\
 & \left. \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0} \right) \hbar + \\
 & \left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \right. \\
 & \frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \\
 & \frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1} + \\
 & \frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 y_2 c_{0,2} + \\
 & \frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 c_{0,3} + 3 \in f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0} + \\
 & \in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1} + 2 \in f_{12}^2 y_1 y_2 c_{1,1} + \\
 & \in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1} + 2 \in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3 \in f_{12} f_{22} y_2 c_{1,2} + \\
 & \frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 c_{2,1} + \\
 & \left. 2 \in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3 \in f_{11}^2 y_1 c_{3,0} + 3 \in f_{11} f_{12} y_2 c_{3,0} \right) \hbar^2 + O[\hbar]^3 \}
 \end{aligned}$$

$$\begin{aligned}
 \text{Out[*]} = & \{0.125, (1 + \in c_{0,0}) + \\
 & \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} + \right. \\
 & \left. \in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0} \right) \hbar + \\
 & \left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \right. \\
 & \frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \\
 & \frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1} + \\
 & \frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 y_2 c_{0,2} + \\
 & \frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 c_{0,3} + 3 \in f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0} + \\
 & \in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1} + 2 \in f_{12}^2 y_1 y_2 c_{1,1} + \\
 & \in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1} + 2 \in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3 \in f_{12} f_{22} y_2 c_{1,2} + \\
 & \frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 c_{2,1} + \\
 & \left. 2 \in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3 \in f_{11}^2 y_1 c_{3,0} + 3 \in f_{11} f_{12} y_2 c_{3,0} \right) \hbar^2 + O[\hbar]^3 \}
 \end{aligned}$$

Out[*]= True

```
In[*]:= DeLin@<F_, ε_>_B := Module[{L, Y},
  L = PowerExpand@Log[Normal[ε] /. ε → 0];
  Y = Table[∂iL, {i, B}];
  CF@<F, eY.F.Y/2 (e-B.Y ε /. Thread[B → B + F.Y])>_B
]
```

```
In[*]:= {n = 2, p = 2, B = Table[zi, {i, n}], Y = Table[yi, {i, n}],
  F = ħ Table[f{10,1}.Sort[{i,j}], {i, n}, {j, n}], ε = 1 + ε GenericPolynomial[{d, 0, 3}, B, c]}
Timing[lhs = Evp@<F, ε eY.B>_B]
Timing[rhs = Evp@DeLin@<F, ε eY.B>_B]
HL@Simplify[lhs == rhs]
```

```
Out[*]= {2, 2, {z1, z2}, {y1, y2}, {{ħ f11, ħ f12}, {ħ f12, ħ f22}},
  1 + ε (c0,0 + z2 c0,1 + z22 c0,2 + z23 c0,3 + z1 c1,0 + z1 z2 c1,1 + z1 z22 c1,2 + z12 c2,0 + z12 z2 c2,1 + z13 c3,0) }
```

```
Out[*]= {0.03125, (1 + ε c0,0) +
  (  $\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{0,0} + \in f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \in f_{22} y_2^2 c_{0,0} + \in f_{12} y_1 c_{0,1} +$ 
 $\in f_{22} y_2 c_{0,1} + \in f_{22} c_{0,2} + \in f_{11} y_1 c_{1,0} + \in f_{12} y_2 c_{1,0} + \in f_{12} c_{1,1} + \in f_{11} c_{2,0}$  ) ħ +
  (  $\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 +$ 
 $\frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0} +$ 
 $\frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1} + \in f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1} +$ 
 $\frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1} + \in f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 y_2 c_{0,2} +$ 
 $\frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2} + 3 \in f_{12} f_{22} y_1 c_{0,3} + 3 \in f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \in f_{11} y_1^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0} +$ 
 $\in f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1} + 2 \in f_{12}^2 y_1 y_2 c_{1,1} +$ 
 $\in f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1} + 2 \in f_{12}^2 y_1 c_{1,2} + \in f_{11} f_{22} y_1 c_{1,2} + 3 \in f_{12} f_{22} y_2 c_{1,2} +$ 
 $\frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 y_2 c_{2,0} + \in f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0} + 3 \in f_{11} f_{12} y_1 c_{2,1} +$ 
 $2 \in f_{12}^2 y_2 c_{2,1} + \in f_{11} f_{22} y_2 c_{2,1} + 3 \in f_{11}^2 y_1 c_{3,0} + 3 \in f_{11} f_{12} y_2 c_{3,0}$  ) ħ2 + 0 [ħ]3 }
```

$$\begin{aligned}
 \text{Out[*]} = & \left\{ \mathbf{0.}, \left(\mathbf{1} + \in c_{\mathbf{0},\mathbf{0}} \right) + \right. \\
 & \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \in f_{11} y_1^2 c_{\mathbf{0},\mathbf{0}} + \in f_{12} y_1 y_2 c_{\mathbf{0},\mathbf{0}} + \frac{1}{2} \in f_{22} y_2^2 c_{\mathbf{0},\mathbf{0}} + \in f_{12} y_1 c_{\mathbf{0},\mathbf{1}} + \right. \\
 & \left. \in f_{22} y_2 c_{\mathbf{0},\mathbf{1}} + \in f_{22} c_{\mathbf{0},\mathbf{2}} + \in f_{11} y_1 c_{\mathbf{1},\mathbf{0}} + \in f_{12} y_2 c_{\mathbf{1},\mathbf{0}} + \in f_{12} c_{\mathbf{1},\mathbf{1}} + \in f_{11} c_{\mathbf{2},\mathbf{0}} \right) \hbar + \\
 & \left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \right. \\
 & \frac{1}{8} \in f_{11}^2 y_1^4 c_{\mathbf{0},\mathbf{0}} + \frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{\mathbf{0},\mathbf{0}} + \frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{\mathbf{0},\mathbf{0}} + \frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{\mathbf{0},\mathbf{0}} + \\
 & \frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{\mathbf{0},\mathbf{0}} + \frac{1}{8} \in f_{22}^2 y_2^4 c_{\mathbf{0},\mathbf{0}} + \frac{1}{2} \in f_{11} f_{12} y_1^3 c_{\mathbf{0},\mathbf{1}} + \in f_{12}^2 y_1^2 y_2 c_{\mathbf{0},\mathbf{1}} + \frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{\mathbf{0},\mathbf{1}} + \\
 & \frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{\mathbf{0},\mathbf{1}} + \frac{1}{2} \in f_{22}^2 y_2^3 c_{\mathbf{0},\mathbf{1}} + \in f_{12}^2 y_1^2 c_{\mathbf{0},\mathbf{2}} + \frac{1}{2} \in f_{11} f_{22} y_1^2 c_{\mathbf{0},\mathbf{2}} + 3 \in f_{12} f_{22} y_1 y_2 c_{\mathbf{0},\mathbf{2}} + \\
 & \frac{3}{2} \in f_{22}^2 y_2^2 c_{\mathbf{0},\mathbf{2}} + 3 \in f_{12} f_{22} y_1 c_{\mathbf{0},\mathbf{3}} + 3 \in f_{22}^2 y_2 c_{\mathbf{0},\mathbf{3}} + \frac{1}{2} \in f_{11}^2 y_1^3 c_{\mathbf{1},\mathbf{0}} + \frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{\mathbf{1},\mathbf{0}} + \\
 & \in f_{12}^2 y_1 y_2^2 c_{\mathbf{1},\mathbf{0}} + \frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{\mathbf{1},\mathbf{0}} + \frac{1}{2} \in f_{12} f_{22} y_2^3 c_{\mathbf{1},\mathbf{0}} + \frac{3}{2} \in f_{11} f_{12} y_1^2 c_{\mathbf{1},\mathbf{1}} + 2 \in f_{12}^2 y_1 y_2 c_{\mathbf{1},\mathbf{1}} + \\
 & \in f_{11} f_{22} y_1 y_2 c_{\mathbf{1},\mathbf{1}} + \frac{3}{2} \in f_{12} f_{22} y_2^2 c_{\mathbf{1},\mathbf{1}} + 2 \in f_{12}^2 y_1 c_{\mathbf{1},\mathbf{2}} + \in f_{11} f_{22} y_1 c_{\mathbf{1},\mathbf{2}} + 3 \in f_{12} f_{22} y_2 c_{\mathbf{1},\mathbf{2}} + \\
 & \frac{3}{2} \in f_{11}^2 y_1^2 c_{\mathbf{2},\mathbf{0}} + 3 \in f_{11} f_{12} y_1 y_2 c_{\mathbf{2},\mathbf{0}} + \in f_{12}^2 y_2^2 c_{\mathbf{2},\mathbf{0}} + \frac{1}{2} \in f_{11} f_{22} y_2^2 c_{\mathbf{2},\mathbf{0}} + 3 \in f_{11} f_{12} y_1 c_{\mathbf{2},\mathbf{1}} + \\
 & \left. 2 \in f_{12}^2 y_2 c_{\mathbf{2},\mathbf{1}} + \in f_{11} f_{22} y_2 c_{\mathbf{2},\mathbf{1}} + 3 \in f_{11}^2 y_1 c_{\mathbf{3},\mathbf{0}} + 3 \in f_{11} f_{12} y_2 c_{\mathbf{3},\mathbf{0}} \right) \hbar^2 + \mathbf{O}[\hbar]^3 \}
 \end{aligned}$$

Out[*]= True

Testing Lemma 3

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathbb{C}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

In[*]= {n = 2, p = 2, B = Table[b_i, {i, n}],

F = ħ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], P = GenericPolynomial[{d, 0, 2}, B, c]}

Out[*]= {2, 2, {b₁, b₂}, {{ħ f₁₁, ħ f₁₂}, {ħ f₁₂, ħ f₂₂}}, c_{0,0} + b₂ c_{0,1} + b₂² c_{0,2} + b₁ c_{1,0} + b₁ b₂ c_{1,1} + b₁² c_{2,0}}

In[]:= **Z = PowerExpand@Expand@Log[Evp,@[λ F, e^P |_B]]**

$$\begin{aligned}
 \text{Out[]} = & \left(c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0} \right) + \\
 & \left(\frac{1}{2} \lambda f_{22} c_{0,1}^2 + \lambda f_{22} c_{0,2} + 2 \lambda b_2 f_{22} c_{0,1} c_{0,2} + 2 \lambda b_2^2 f_{22} c_{0,2}^2 + \lambda f_{12} c_{0,1} c_{1,0} + 2 \lambda b_2 f_{12} c_{0,2} c_{1,0} + \right. \\
 & \frac{1}{2} \lambda f_{11} c_{1,0}^2 + \lambda f_{12} c_{1,1} + \lambda b_2 f_{12} c_{0,1} c_{1,1} + \lambda b_1 f_{22} c_{0,1} c_{1,1} + 2 \lambda b_2^2 f_{12} c_{0,2} c_{1,1} + \\
 & 2 \lambda b_1 b_2 f_{22} c_{0,2} c_{1,1} + \lambda b_2 f_{11} c_{1,0} c_{1,1} + \lambda b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2} \lambda b_2^2 f_{11} c_{1,1}^2 + \\
 & \lambda b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2} \lambda b_1^2 f_{22} c_{1,1}^2 + \lambda f_{11} c_{2,0} + 2 \lambda b_1 f_{12} c_{0,1} c_{2,0} + 4 \lambda b_1 b_2 f_{12} c_{0,2} c_{2,0} + \\
 & \left. 2 \lambda b_1 f_{11} c_{1,0} c_{2,0} + 2 \lambda b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 \lambda b_1^2 f_{12} c_{1,1} c_{2,0} + 2 \lambda b_1^2 f_{11} c_{2,0}^2 \right) \hbar + \\
 & \left(\lambda^2 f_{22}^2 c_{0,1}^2 c_{0,2} + \lambda^2 f_{22}^2 c_{0,2}^2 + 4 \lambda^2 b_2 f_{22}^2 c_{0,1} c_{0,2} + 4 \lambda^2 b_2^2 f_{22}^2 c_{0,2}^3 + 2 \lambda^2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + \right. \\
 & 4 \lambda^2 b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + \lambda^2 f_{12}^2 c_{0,2} c_{1,0}^2 + \lambda^2 f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 \lambda^2 f_{12} f_{22} c_{0,2} c_{1,1} + \\
 & 6 \lambda^2 b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 2 \lambda^2 b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 8 \lambda^2 b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + 4 \lambda^2 b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} + \\
 & \lambda^2 f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + \lambda^2 f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 4 \lambda^2 b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + 2 \lambda^2 b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} + \\
 & 2 \lambda^2 b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + \lambda^2 f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} \lambda^2 f_{12}^2 c_{1,1}^2 + \frac{1}{2} \lambda^2 f_{11} f_{22} c_{1,1}^2 + \lambda^2 b_2 f_{12}^2 c_{0,1} c_{1,1}^2 + \\
 & \lambda^2 b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 2 \lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + 3 \lambda^2 b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 2 \lambda^2 b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 + \\
 & 6 \lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + \lambda^2 b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + 2 \lambda^2 b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + \lambda^2 b_1 f_{12}^2 c_{1,0} c_{1,1}^2 + \\
 & \lambda^2 b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + \lambda^2 b_2^2 f_{11} f_{12} c_{1,1}^3 + \lambda^2 b_1 b_2 f_{12}^2 c_{1,1}^3 + \lambda^2 b_1 b_2 f_{11} f_{22} c_{1,1}^3 + \lambda^2 b_1^2 f_{12} f_{22} c_{1,1}^3 + \\
 & \lambda^2 f_{12}^2 c_{0,1}^2 c_{2,0} + 2 \lambda^2 f_{12}^2 c_{0,2} c_{2,0} + 4 \lambda^2 b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 4 \lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} + \\
 & 4 \lambda^2 b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 8 \lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + 2 \lambda^2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + \\
 & 4 \lambda^2 b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 4 \lambda^2 b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + \lambda^2 f_{11}^2 c_{1,0}^2 c_{2,0} + 2 \lambda^2 f_{11} f_{12} c_{1,1} c_{2,0} + \\
 & 2 \lambda^2 b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 4 \lambda^2 b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 2 \lambda^2 b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} + \\
 & 4 \lambda^2 b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 12 \lambda^2 b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 4 \lambda^2 b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} + \\
 & 4 \lambda^2 b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 2 \lambda^2 b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 6 \lambda^2 b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} + \\
 & \lambda^2 b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 6 \lambda^2 b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 3 \lambda^2 b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 2 \lambda^2 b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} + \\
 & \lambda^2 f_{11}^2 c_{2,0}^2 + 4 \lambda^2 b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 8 \lambda^2 b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 4 \lambda^2 b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 + \\
 & \left. 4 \lambda^2 b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 4 \lambda^2 b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 8 \lambda^2 b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 4 \lambda^2 b_1^2 f_{11}^2 c_{2,0}^3 \right) \hbar^2 + O[\hbar]^3
 \end{aligned}$$

In[]:= **Z /. λ → 0**

$$\text{Out[]} = \left(c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0} \right) + O[\hbar]^3$$

In[]:= **(Z /. λ → 0) - P**

$$\text{Out[]} = O[\hbar]^3$$

In[*]:= **lhs = $\partial_\lambda Z$**

$$\begin{aligned} \text{Out[*]} = & \left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + 2 b_2^2 f_{22} c_{0,2}^2 + f_{12} c_{0,1} c_{1,0} + 2 b_2 f_{12} c_{0,2} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + \right. \\ & f_{12} c_{1,1} + b_2 f_{12} c_{0,1} c_{1,1} + b_1 f_{22} c_{0,1} c_{1,1} + 2 b_2^2 f_{12} c_{0,2} c_{1,1} + 2 b_1 b_2 f_{22} c_{0,2} c_{1,1} + b_2 f_{11} c_{1,0} c_{1,1} + \\ & b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2} b_2^2 f_{11} c_{1,1}^2 + b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2} b_1^2 f_{22} c_{1,1}^2 + f_{11} c_{2,0} + 2 b_1 f_{12} c_{0,1} c_{2,0} + \\ & \left. 4 b_1 b_2 f_{12} c_{0,2} c_{2,0} + 2 b_1 f_{11} c_{1,0} c_{2,0} + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2 \right) \hbar + \\ & \left(2 \lambda f_{22}^2 c_{0,1}^2 c_{0,2} + 2 \lambda f_{22}^2 c_{0,2}^2 + 8 \lambda b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 8 \lambda b_2^2 f_{22}^2 c_{0,2}^3 + 4 \lambda f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + \right. \\ & 8 \lambda b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + 2 \lambda f_{12}^2 c_{0,2} c_{1,0}^2 + 2 \lambda f_{12} f_{22} c_{0,1}^2 c_{1,1} + 4 \lambda f_{12} f_{22} c_{0,2} c_{1,1} + \\ & 12 \lambda b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 4 \lambda b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 16 \lambda b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + \\ & 8 \lambda b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} + 2 \lambda f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 8 \lambda b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + \\ & 4 \lambda b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{12} c_{1,0}^2 c_{1,1} + \lambda f_{12}^2 c_{1,1}^2 + \\ & \lambda f_{11} f_{22} c_{1,1}^2 + 2 \lambda b_2 f_{12}^2 c_{0,1} c_{1,1}^2 + 2 \lambda b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 4 \lambda b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + \\ & 6 \lambda b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 + 12 \lambda b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + 2 \lambda b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + \\ & 4 \lambda b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{12}^2 c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + 2 \lambda b_2^2 f_{11} f_{12} c_{1,1}^3 + \\ & 2 \lambda b_1 b_2 f_{12}^2 c_{1,1}^3 + 2 \lambda b_1 b_2 f_{11} f_{22} c_{1,1}^3 + 2 \lambda b_1^2 f_{12} f_{22} c_{1,1}^3 + 2 \lambda f_{12}^2 c_{0,1}^2 c_{2,0} + 4 \lambda f_{12}^2 c_{0,2} c_{2,0} + \\ & 8 \lambda b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 16 \lambda b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + \\ & 4 \lambda f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + 8 \lambda b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + 2 \lambda f_{11}^2 c_{1,0}^2 c_{2,0} + \\ & 4 \lambda f_{11} f_{12} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 4 \lambda b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} + \\ & 8 \lambda b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 24 \lambda b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} + \\ & 8 \lambda b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 12 \lambda b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} + \\ & 2 \lambda b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 12 \lambda b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 6 \lambda b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 4 \lambda b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} + \\ & 2 \lambda f_{11}^2 c_{2,0}^2 + 8 \lambda b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 16 \lambda b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 8 \lambda b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 + \\ & \left. 8 \lambda b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 8 \lambda b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 16 \lambda b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 8 \lambda b_1^2 f_{11}^2 c_{2,0}^3 \right) \hbar^2 + O[\hbar]^3 \end{aligned}$$

In[*]:= **Short[rhs = Expand@Sum[($\partial_{b_1, b_2}(\mathbf{B} \cdot \mathbf{F} \cdot \mathbf{B})$) ($\partial_{b_1, b_2} \mathbf{Z} + (\partial_{b_1} \mathbf{Z}) (\partial_{b_2} \mathbf{Z})$) / 4, {b1, B}, {b2, B}]]**

$$\begin{aligned} \text{Out[*]//Short} = & \left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + \ll 18 \gg + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2 \right) \\ & \hbar + (\ll 1 \gg) \ll 1 \gg + \ll 1 \gg + O[\hbar]^4 \end{aligned}$$

In[*]:= **HL[Normal[lhs - rhs] == 0]**

Out[*]:= **True**

In[*]:= **Z /. $\lambda \rightarrow 1$ /. Alternatives@@B \rightarrow 0**

$$\begin{aligned} \text{Out[*]} = & c_{0,0} + \left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + f_{12} c_{0,1} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + f_{11} c_{2,0} \right) \hbar + \\ & \left(f_{22}^2 c_{0,1}^2 c_{0,2} + f_{22}^2 c_{0,2}^2 + 2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + f_{12}^2 c_{0,2} c_{1,0}^2 + f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 f_{12} f_{22} c_{0,2} c_{1,1} + \right. \\ & f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} f_{12}^2 c_{1,1}^2 + \frac{1}{2} f_{11} f_{22} c_{1,1}^2 + f_{12}^2 c_{0,1}^2 c_{2,0} + \\ & \left. 2 f_{12}^2 c_{0,2} c_{2,0} + 2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + f_{11}^2 c_{1,0}^2 c_{2,0} + 2 f_{11} f_{12} c_{1,1} c_{2,0} + f_{11}^2 c_{2,0}^2 \right) \hbar^2 + O[\hbar]^3 \end{aligned}$$