

Pensieve header: Whirling matrices until they get inverted.

```
In[*]:= CF = ExpandNumerator@* ExpandDenominator@* Together
```

```
Out[*]:= ExpandNumerator@* ExpandDenominator@* Together
```

```
In[*]:= MatrixForm[A = Table[a10i+j, {i, 3}, {j, 3}]]
```

```
Out[*]//MatrixForm=

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

```

```
In[*]:= MatrixForm[Ξ = A[[1 ;; -2, 1 ;; -2]]]
```

```
Out[*]//MatrixForm=

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

```

```
In[*]:= Θ = A[[-1, 1 ;; -2]]
```

```
Out[*]:= {a31, a32}
```

```
In[*]:= Φ = A[[1 ;; -2, -1]]
```

```
Out[*]:= {a13, a23}
```

```
In[*]:= α = A[[-1, -1]]
```

```
Out[*]:= a33
```

```
In[*]:= Outer[Times, φ, Θ] // MatrixForm
```

```
Out[*]//MatrixForm=

$$\begin{pmatrix} a_{13} a_{31} & a_{13} a_{32} \\ a_{23} a_{31} & a_{23} a_{32} \end{pmatrix}$$

```

```
In[*]:= W[A_?MatrixQ] := Module[{
  Ξ = A[[1 ;; -2, 1 ;; -2],
  Θ = A[[-1, 1 ;; -2],
  φ = A[[1 ;; -2, -1],
  α = A[[-1, -1]
},
CF@ArrayFlatten[

$$\left( \begin{array}{c} \{\{\alpha^{-1}\}\} \\ \text{List} /@ (\alpha^{-1} \phi) \end{array} \quad \begin{array}{c} \{-\alpha^{-1} \Theta\} \\ \Xi - \alpha^{-1} \text{Outer}[\text{Times}, \phi, \Theta] \end{array} \right)$$

]
```

```
In[*]:= W[A] // MatrixForm
```

```
Out[*]//MatrixForm=

$$\begin{pmatrix} \frac{1}{a_{33}} & -\frac{a_{31}}{a_{33}} & -\frac{a_{32}}{a_{33}} \\ a_{13} & -a_{13} a_{31} + a_{11} a_{33} & -a_{13} a_{32} + a_{12} a_{33} \\ a_{23} & -a_{23} a_{31} + a_{21} a_{33} & -a_{23} a_{32} + a_{22} a_{33} \end{pmatrix}$$

```

In[ ]:= **A // W // W // MatrixForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{a_{33}}{a_{32}} & \frac{a_{23}}{a_{22}} & \frac{-a_{23} a_{31} + a_{21} a_{33}}{a_{23} a_{32} - a_{22} a_{33}} \\ -a_{23} a_{32} + a_{22} a_{33} & a_{23} a_{32} - a_{22} a_{33} & a_{23} a_{32} - a_{22} a_{33} \\ \frac{a_{32}}{a_{22}} & \frac{a_{22}}{a_{22}} & \frac{-a_{22} a_{31} + a_{21} a_{32}}{-a_{23} a_{32} + a_{22} a_{33}} \\ a_{23} a_{32} - a_{22} a_{33} & -a_{23} a_{32} + a_{22} a_{33} & -a_{23} a_{32} + a_{22} a_{33} \\ \frac{a_{13} a_{32} - a_{12} a_{33}}{a_{23} a_{32} - a_{22} a_{33}} & \frac{a_{13} a_{22} - a_{12} a_{23}}{-a_{23} a_{32} + a_{22} a_{33}} & \frac{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}}{-a_{23} a_{32} + a_{22} a_{33}} \end{pmatrix}$$

In[ ]:= **A // W // W // W // MatrixForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{a_{23} a_{32} - a_{22} a_{33}}{a_{13} a_{22} a_{31} - a_{12} a_{23} a_{31} - a_{13} a_{21} a_{32} + a_{11} a_{23} a_{32} + a_{12} a_{21} a_{33} - a_{11} a_{22} a_{33}} & \frac{a_{13} a_{32} - a_{12} a_{33}}{a_{23} a_{31} - a_{21} a_{33}} & \frac{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}}{a_{13} a_{22} a_{31} - a_{12} a_{23} a_{31} - a_{13} a_{21} a_{32} + a_{11} a_{23} a_{32} + a_{12} a_{21} a_{33} - a_{11} a_{22} a_{33}} & a_{13} a_{22} \\ -a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33} & \frac{-a_{13} a_{31} + a_{11} a_{33}}{-a_{13} a_{31} + a_{11} a_{33}} & \frac{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}}{-a_{13} a_{31} + a_{11} a_{33}} & a_{13} a_{22} \\ \frac{a_{22} a_{31} - a_{21} a_{32}}{a_{13} a_{22} a_{31} - a_{12} a_{23} a_{31} - a_{13} a_{21} a_{32} + a_{11} a_{23} a_{32} + a_{12} a_{21} a_{33} - a_{11} a_{22} a_{33}} & \frac{-a_{12} a_{31} + a_{11} a_{32}}{-a_{12} a_{31} + a_{11} a_{32}} & \frac{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}}{-a_{12} a_{31} + a_{11} a_{32}} & -a_{13} a_{22} \end{pmatrix}$$

In[ ]:= **Inverse[A] // MatrixForm**

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{-a_{23} a_{32} + a_{22} a_{33}}{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}} & \frac{a_{13} a_{32} - a_{12} a_{33}}{a_{23} a_{31} - a_{21} a_{33}} & \frac{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}}{-a_{13} a_{31} + a_{11} a_{33}} & -a_{13} a_{22} \\ \frac{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}}{-a_{13} a_{31} + a_{11} a_{33}} & \frac{-a_{13} a_{31} + a_{11} a_{33}}{-a_{13} a_{31} + a_{11} a_{33}} & \frac{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}}{-a_{13} a_{31} + a_{11} a_{33}} & -a_{13} a_{22} \\ \frac{-a_{22} a_{31} + a_{21} a_{32}}{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}} & \frac{a_{12} a_{31} - a_{11} a_{32}}{a_{12} a_{31} - a_{11} a_{32}} & \frac{-a_{13} a_{22} a_{31} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} + a_{11} a_{22} a_{33}}{a_{12} a_{31} - a_{11} a_{32}} & -a_{13} a_{22} \end{pmatrix}$$

In[ ]:= **CF [ (A // W // W // W) - Inverse[A] ]**

Out[ ]:= { {0, 0, 0}, {0, 0, 0}, {0, 0, 0} }

In[ ]:= **n = 5;**

**MatrixForm[B = Table[a<sub>10</sub><sub>i+j</sub>, {i, n}, {j, n}]]**

Out[ ]//MatrixForm=

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

In[ ]:= **CF [Nest[W, B, n] - Inverse[B] ]**

Out[ ]:= { {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0}, {0, 0, 0, 0, 0} }

In[ ]:= **n = 10;**

**MatrixForm[B = Table[RandomInteger[{-1, 1} 10^5], {i, n}, {j, n}]]**

Out[ ]//MatrixForm=

$$\begin{pmatrix} -4143 & 87899 & 87054 & 98271 & -70574 & -13162 & -56434 & 61043 & 1674 & -75250 \\ 15058 & -42530 & 42424 & -93947 & -55293 & -50368 & -1688 & -60674 & 4508 & -36648 \\ 93947 & 56064 & -20232 & 59574 & 19143 & -96265 & 456 & 30209 & 46588 & 44152 \\ 54791 & -87839 & -74683 & 82052 & 9962 & 70247 & -7715 & -55861 & 16367 & -85179 \\ 85148 & -5924 & -93166 & -49896 & -74440 & -67517 & 31522 & -139 & -42226 & -67983 \\ -59229 & 11784 & 31778 & 99696 & -62313 & -20723 & -55338 & -49285 & 14221 & -42696 \\ -25827 & -65258 & -78811 & -98153 & 30283 & 28077 & -20722 & 43606 & -41353 & 3317 \\ 21377 & -59027 & -29901 & -27762 & -38732 & -67756 & 70216 & -60508 & -35837 & 2540 \\ 84579 & -75893 & 78563 & -12447 & -77222 & 22711 & 38654 & 25133 & -73457 & 74327 \\ -99062 & -94389 & 90564 & 50603 & 33905 & -63174 & -48489 & -54443 & -17932 & 44226 \end{pmatrix}$$

In[ ]:= **W[B]**

$$\text{Out[ ]} = \left\{ \left\{ \frac{1}{44226}, \frac{49531}{22113}, \frac{31463}{14742}, -\frac{15094}{7371}, -\frac{7229}{6318}, -\frac{33905}{44226}, \frac{10529}{7371}, \frac{2309}{2106}, \frac{54443}{44226}, \frac{8966}{22113} \right\}, \right. \\
 \left. \left\{ \frac{5375}{3159}, -\frac{545545987}{3159}, -\frac{76555978}{1053}, \frac{253928362}{1053}, \frac{582429214}{3159}, \right. \right. \\
 \left. \left. -\frac{40703891}{3159}, -\frac{127046336}{1053}, -\frac{146301127}{1053}, -\frac{99796288}{3159}, -\frac{91096334}{3159} \right\}, \right. \\
 \left. \left\{ \frac{2036}{2457}, -\frac{164692726}{2457}, -\frac{98890738}{819}, \frac{96208024}{819}, -\frac{18257153}{351}, -\frac{66824321}{2457}, -\frac{84125480}{819}, \right. \right. \\
 \left. \left. -\frac{4898620}{117}, -\frac{259921966}{2457}, -\frac{25433396}{2457} \right\}, \left\{ \frac{22076}{22113}, \frac{4264342723}{22113}, \frac{1107824932}{7371}, -\frac{815560360}{7371}, \right. \\
 \left. \frac{28606862}{22113}, -\frac{325177621}{7371}, -\frac{244692907}{1053}, \frac{51453652}{22113}, \frac{1869895285}{22113}, \frac{1426067276}{22113} \right\}, \\
 \left. \left\{ \frac{28393}{14742}, -\frac{1002469222}{7371}, -\frac{1324969805}{4914}, \frac{245067811}{2457}, \frac{378054509}{2106}, \right. \right. \\
 \left. \left. \frac{1109524469}{14742}, -\frac{126353018}{2457}, -\frac{70975367}{702}, -\frac{2369302961}{14742}, -\frac{133930481}{7371} \right\}, \right. \\
 \left. \left\{ \frac{22661}{14742}, -\frac{494796083}{7371}, -\frac{742093579}{4914}, \frac{113136272}{2457}, \frac{58735393}{2106}, -\frac{329073275}{14742}, -\frac{404486938}{2457}, \right. \right. \\
 \left. \left. -\frac{30195805}{702}, -\frac{1235781961}{14742}, -\frac{514426372}{7371} \right\}, \left\{ -\frac{2372}{2457}, -\frac{380500717}{2457}, -\frac{64979140}{819}, \right. \\
 \left. \frac{97632118}{819}, \frac{52140484}{351}, -\frac{72680381}{2457}, -\frac{66921713}{819}, -\frac{11951494}{117}, -\frac{250232041}{2457}, -\frac{7593707}{2457} \right\}, \\
 \left. \left\{ \frac{3317}{44226}, -\frac{406818124}{22113}, -\frac{857670665}{14742}, -\frac{630982679}{7371}, -\frac{644109247}{6318}, \right. \right. \\
 \left. \left. \frac{1226833073}{44226}, \frac{241880260}{7371}, -\frac{35981579}{2106}, \frac{2109106387}{44226}, -\frac{884698667}{22113} \right\}, \right. \\
 \left. \left\{ \frac{1270}{22113}, \frac{598518341}{22113}, -\frac{395130007}{7371}, -\frac{258739031}{7371}, -\frac{96880988}{3159}, \right. \right. \\
 \left. \left. -\frac{899540066}{22113}, -\frac{472685816}{7371}, \frac{76869878}{1053}, -\frac{1268870794}{22113}, -\frac{769689941}{22113} \right\}, \right. \\
 \left. \left\{ \frac{74327}{44226}, -\frac{5551786064}{22113}, \frac{1219735795}{14742}, -\frac{542803865}{7371}, -\frac{615950029}{6318}, \right. \right. \\
 \left. \left. -\frac{5935277107}{44226}, \frac{949991764}{7371}, \frac{253026367}{2106}, \frac{5158116919}{44226}, -\frac{957938759}{22113} \right\} \right\}$$

In[ ]:= **CF[Nest[W, B, n] - Inverse[B]]**

$$\text{Out[ ]} = \left\{ \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \right. \\
 \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
 \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \\
 \left. \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\} \right\}$$

In[\*]:= **Inverse[B] // N // MatrixForm**

Out[\*]//MatrixForm=

$$\begin{pmatrix} -6.31403 \times 10^{-6} & -2.82206 \times 10^{-7} & -1.38457 \times 10^{-6} & 1.25013 \times 10^{-6} & 0.0000131471 & -4.62976 \times 10^{-6} \\ -9.86413 \times 10^{-6} & -5.77493 \times 10^{-6} & -7.65508 \times 10^{-6} & -5.57444 \times 10^{-6} & 0.0000183677 & -1.4279 \times 10^{-6} \\ 3.25348 \times 10^{-6} & 2.71256 \times 10^{-6} & -2.14721 \times 10^{-6} & 3.1982 \times 10^{-7} & -4.4742 \times 10^{-7} & -6.38718 \times 10^{-6} \\ 2.19493 \times 10^{-6} & -4.22913 \times 10^{-6} & 1.38406 \times 10^{-6} & 1.98801 \times 10^{-6} & -1.81323 \times 10^{-6} & 1.47985 \times 10^{-6} \\ -4.48703 \times 10^{-6} & -4.55307 \times 10^{-6} & -5.65801 \times 10^{-6} & 5.88079 \times 10^{-7} & 0.0000136995 & -0.000012472 \\ -3.75328 \times 10^{-6} & -1.30195 \times 10^{-6} & -4.1979 \times 10^{-6} & 9.03903 \times 10^{-7} & 7.99735 \times 10^{-7} & 1.42522 \times 10^{-6} \\ 0.0000114219 & 5.55664 \times 10^{-7} & 2.57932 \times 10^{-6} & 2.05572 \times 10^{-6} & -0.0000186107 & -4.19326 \times 10^{-6} \\ 0.0000167415 & 3.34875 \times 10^{-6} & 9.52171 \times 10^{-6} & 2.60143 \times 10^{-6} & -0.0000223396 & -5.35339 \times 10^{-7} \\ 0.0000135816 & 0.0000130298 & 0.000015718 & 5.15334 \times 10^{-6} & -0.0000357039 & 7.77473 \times 10^{-6} \\ -7.65167 \times 10^{-6} & -2.02747 \times 10^{-6} & 2.63783 \times 10^{-6} & -3.6405 \times 10^{-6} & -1.01403 \times 10^{-7} & 7.46159 \times 10^{-6} \end{pmatrix}$$

In[\*]:= **N@Log[10, Sqrt@Tr[Numerator[#].Transpose@Numerator[#]]] & /@ NestList[W, B, n]**

Out[\*]= {5.75717, 10.0861, 15.2185, 20.195, 25.1154,  
30.1879, 34.6256, 39.7111, 45.3524, 49.9933, 46.1434}

In[\*]:= **N@Log[10, Sqrt@Tr[Denominator[#].Transpose@Denominator[#]]] & /@ NestList[W, B, n]**

Out[\*]= {1., 5.22833, 10.0545, 15.2021, 20.2118,  
24.9757, 30.2623, 35.2257, 40.7494, 46.0526, 51.1484}

In[\*]:=

```
Y[A_?MatrixQ] := Module[{
  ε = A[[1 ;; -2, 1 ;; -2]],
  θ = A[[-1, 1 ;; -2]],
  φ = A[[1 ;; -2, -1]],
  α = A[[-1, -1]]
},
CF@ArrayFlatten[({{1}}
List /@ (φ) α ε - Outer[Times, φ, θ] )]]
]
```

In[\*]:= **n = 4;**

**MatrixForm[B = Table[RandomInteger[{-1, 1} n], {i, n}, {j, n}]]**

Out[\*]//MatrixForm=

$$\begin{pmatrix} -1 & 2 & -2 & 2 \\ 4 & 3 & -3 & -1 \\ 0 & -1 & -1 & -3 \\ -2 & 2 & 0 & -1 \end{pmatrix}$$

In[\*]:= **Nest[Y, B, n].B**

Out[\*]= {{47, 110, -46, 14}, {-3244, 20, 552, -2312},  
{-1184, 80, -740, -3352}, {-1476, 1016, 920, 1060}}