The Turbo-Gassner Representation

Pensieve header: The turbo Gassner representation. Continues pensieve://2016-06/.

(Alt) In[•]:=
```
Kδ /: Kδ_is__ := KroneckerDelta[1, Length[Union[{is}]]];
```

The Burau Representation

In[•]:=
```
Inverse[{{0, t}, {1, 1 - t}}] /. t → t^-1 // Simplify // Transpose // MatrixForm
```

Out[•]//MatrixForm=
$$\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$$

In[•]:=
```
B_i_,j_[ℰ_] := ℰ /. v_j :↦ (1 - t) v_i + t v_j;
B̄_i_,j_[ℰ_] := ℰ /. v_j :↦ (1 - t^-1) v_i + t^-1 v_j;
```

In[•]:=
```
{{v1, v2, v3} // B1,3, {v1, v2, v3} // B̄1,3}
```

Out[•]=
$$\left\{\{v_1, v_2, (1-t)\, v_1 + t\, v_3\}, \{v_1, v_2, \left(1 - \frac{1}{t}\right) v_1 + \frac{v_3}{t}\}\right\}$$

In[•]:=
```
{v1, v2, v3} // B1,3 // B̄1,3 // Expand
```

Out[•]=
$$\{v_1, v_2, v_3\}$$

In[•]:=
```
Column@{R3l = {v1, v2, v3} // B1,2 // B1,3 // B2,3,
  R3r = {v1, v2, v3} // B2,3 // B1,3 // B1,2 ,
  R3l - R3r // Expand}
```

Out[•]=
$$\{v_1, (1-t)\, v_1 + t\, v_2, (1-t)\, v_1 + t\, ((1-t)\, v_2 + t\, v_3)\}$$
$$\{v_1, (1-t)\, v_1 + t\, v_2, (1-t)\, ((1-t)\, v_1 + t\, v_2) + t\, ((1-t)\, v_1 + t\, v_3)\}$$
$$\{0, 0, 0\}$$

The Gassner Representation

In[•]:=
```
G_i_,j_[ℰ_] := ℰ /. v_j :↦ (1 - t_i) v_i + t_i v_j;
Ḡ_i_,j_[ℰ_] := ℰ /. v_j :↦ (1 - t_i^-1) v_i + t_i^-1 v_j;
```

In[•]:=
```
{v1, v2, v3} // G1,3 // Ḡ1,3 // Expand
```

Out[•]=
$$\{v_1, v_2, v_3\}$$

In[•]:=
```
Column@{R3l = {v1, v2, v3} // G1,2 // G1,3 // G2,3,
  R3r = {v1, v2, v3} // G2,3 // G1,3 // G1,2,
  R3l - R3r // Expand}
```

Out[•]=
$$\{v_1, (1-t_1)\, v_1 + t_1\, v_2, (1-t_1)\, v_1 + t_1\, ((1-t_2)\, v_2 + t_2\, v_3)\}$$
$$\{v_1, (1-t_1)\, v_1 + t_1\, v_2, (1-t_2)\, ((1-t_1)\, v_1 + t_1\, v_2) + t_2\, ((1-t_1)\, v_1 + t_1\, v_3)\}$$
$$\{0, 0, 0\}$$

```
In[•]:= Column@{OCl = {v1, v2, v3} // G1,2 // G1,3,
          OCr = {v1, v2, v3} // G1,3 // G1,2 ,
          OCl - OCr // Expand}
```

$$\{v_1,\ (1-t_1)\,v_1+t_1\,v_2,\ (1-t_1)\,v_1+t_1\,v_3\}$$

```
Out[•]= {v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 v3}
        {0, 0, 0}
```

```
In[•]:= Column@{UCl = {v1, v2, v3} // G1,3 // G2,3,
          UCr = {v1, v2, v3} // G2,3 // G1,3 ,
          UCl - UCr // Expand}
```

$$\{v_1,\ v_2,\ (1-t_1)\,v_1+t_1\,((1-t_2)\,v_2+t_2\,v_3)\}$$

```
Out[•]= {v1, v2, (1 - t2) v2 + t2 ((1 - t1) v1 + t1 v3)}
        {0, 0, v1 - t1 v1 - t2 v1 + t1 t2 v1 - v2 + t1 v2 + t2 v2 - t1 t2 v2}
```

The Gassner-Plus Representation

```
In[•]:= GPi_,j_[ℰ_] := Expand[ℰ /. {uj :> (1 - ti) ui + ti uj,
            f_. vj :> f (1 - ti) vi + f ti vj + (ti - 1) (ti ∂ti f - tj ∂tj f) ui + f ti ui }];
        GP̄i_,j_[ℰ_] := Expand[ℰ /. {uj :> (1 - ti^-1) ui + ti^-1 uj,
            f_. vj :> f (1 - ti^-1) vi + f ti^-1 vj + (ti^-1 - 1) (ti ∂ti f - tj ∂tj f) ui - f ti^-1 ui }];
```

```
In[•]:= GPchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3};
        GPchecks // GP1,3 // GP̄1,3
```

```
Out[•]= {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3}
```

```
In[•]:= R3l = GPchecks // GP1,2 // GP1,3 // GP2,3
```

$$\{f[t_1,t_2,t_3]\,v_1,\ f[t_1,t_2,t_3]\,t_1\,u_1+f[t_1,t_2,t_3]\,v_1-f[t_1,t_2,t_3]\,t_1\,v_1+f[t_1,t_2,t_3]\,t_1\,v_2+t_2\,u_1\,f^{(0,1,0)}[t_1,t_2,t_3]-t_1\,t_2\,u_1\,f^{(0,1,0)}[t_1,t_2,t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3],\ f[t_1,t_2,t_3]\,t_1\,u_1+f[t_1,t_2,t_3]\,t_1\,t_2\,u_2+f[t_1,t_2,t_3]\,v_1-f[t_1,t_2,t_3]\,t_1\,v_1+f[t_1,t_2,t_3]\,t_1\,v_2-f[t_1,t_2,t_3]\,t_1\,t_2\,v_2+f[t_1,t_2,t_3]\,t_1\,t_2\,v_3+t_3\,u_1\,f^{(0,0,1)}[t_1,t_2,t_3]-t_1\,t_3\,u_1\,f^{(0,0,1)}[t_1,t_2,t_3]+t_1\,t_3\,u_2\,f^{(0,0,1)}[t_1,t_2,t_3]-t_1\,t_2\,t_3\,u_2\,f^{(0,0,1)}[t_1,t_2,t_3]-t_1\,t_2\,u_2\,f^{(0,1,0)}[t_1,t_2,t_3]+t_1\,t_2^2\,u_2\,f^{(0,1,0)}[t_1,t_2,t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3],\ u_1,\ u_1-t_1\,u_1+t_1\,u_2,\ u_1-t_1\,u_1+t_1\,u_2-t_1\,t_2\,u_2+t_1\,t_2\,u_3\}$$

```
In[•]:= R3r = GPchecks // GP2,3 // GP1,3 // GP1,2; R3l - R3r
```

```
Out[•]= {0, 0, 0, 0, 0, 0}
```

```
In[•]:= OCl = GPchecks // GP1,2 // GP1,3
```

$$\{f[t_1,t_2,t_3]\,v_1,\ f[t_1,t_2,t_3]\,t_1\,u_1+f[t_1,t_2,t_3]\,v_1-f[t_1,t_2,t_3]\,t_1\,v_1+f[t_1,t_2,t_3]\,t_1\,v_2+t_2\,u_1\,f^{(0,1,0)}[t_1,t_2,t_3]-t_1\,t_2\,u_1\,f^{(0,1,0)}[t_1,t_2,t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3],\ f[t_1,t_2,t_3]\,t_1\,u_1+f[t_1,t_2,t_3]\,v_1-f[t_1,t_2,t_3]\,t_1\,v_1+f[t_1,t_2,t_3]\,t_1\,v_3+t_3\,u_1\,f^{(0,0,1)}[t_1,t_2,t_3]-t_1\,t_3\,u_1\,f^{(0,0,1)}[t_1,t_2,t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,t_2,t_3],\ u_1,\ u_1-t_1\,u_1+t_1\,u_2,\ u_1-t_1\,u_1+t_1\,u_3\}$$

```
In[•]:= OCr = GPchecks // GP1,3 // GP1,2; OCl - OCr
Out[•]= {0, 0, 0, 0, 0, 0}
```

Question. Does GP factor through G? How?

The End(G) Representation

```
In[•]:= EGi_,j_[ξ_] := Expand[ξ /. {uj → (1 - ti) ui + ti uj, wi → wi + (1 - ti^-1) wj, wj → ti^-1 wj}];
        EGi_,j_[ξ_] := Expand[ξ /. {uj → (1 - ti^-1) ui + ti^-1 uj, wi → wi + (1 - ti) wj, wj → ti wj}];
        EGchecks = Flatten@Table[ui wj, {i, 3}, {j, 3}];
        EGchecks // EG1,3 // EG1,3
Out[•]= {u1 w1, u1 w2, u1 w3, u2 w1, u2 w2, u2 w3, u3 w1, u3 w2, u3 w3}
```

```
Short[R3l = EGchecks // EG1,2 // EG1,3 // EG2,3, 10]
```

$$\Big\{u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1},\ \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},\ \frac{u_1 w_3}{t_1 t_2},$$
$$u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$
$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3,\ -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$
$$-\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$
$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$
$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$
$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$
$$\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3,\ -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\Big\}$$

```
R3r = EGchecks // EG2,3 // EG1,3 // EG1,2; R3l - R3r
{0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
(# → Collect[EGi,j[#], u_ w_, Simplify]) & /@ {uk wj, uk wi, uj wk, uj wi}
```

$$\Big\{u_k w_j \to \frac{u_k w_j}{t_i},\ u_k w_i \to u_k w_i + \left(1 - \frac{1}{t_i}\right) u_k w_j,\ u_j w_k \to (1 - t_i)\, u_i w_k + t_i u_j w_k,$$
$$u_j w_i \to (1 - t_i)\, u_i w_i + t_i u_j w_i - \frac{(-1 + t_i)^2 u_i w_j}{t_i} + (-1 + t_i)\, u_j w_j\Big\}$$

The End(G)+c Representation

Is there topology behind this representation?

In[•]:=
```
EGc_{i_,j_}[ξ_] := Expand[ξ /. {
      c_i → c_i - (1 - t_i^-1) u_i w_j, c_j → c_j + (1 - t_i^-1) u_i w_j,
      u_j → (1 - t_i) u_i + t_i u_j,
      w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
EGc̄_{i_,j_}[ξ_] := Expand[ξ /. {
      c_i → c_i + (t_i - 1) u_i w_j, c_j → c_j + (1 - t_i) u_i w_j,
      u_j → (1 - t_i^-1) u_i + t_i^-1 u_j,
      w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
EGcchecks = {c1, c2, c3, u1 w1, u1 w2, u1 w3, u2 w1, u2 w2, u2 w3, u3 w1, u3 w2, u3 w3};
EGcchecks // EGc_{1,3} // EGc̄_{1,3}
```

Out[•]=
$$\{c_1, c_2, c_3, u_1 w_1, u_1 w_2, u_1 w_3, u_2 w_1, u_2 w_2, u_2 w_3, u_3 w_1, u_3 w_2, u_3 w_3\}$$

```
(# → Collect[EGc_{1,2}[#], u_ w_, Simplify]) & /@ EGcchecks
```

$$\left\{c_1 \to c_1 + \left(-1 + \frac{1}{t_1}\right) u_1 w_2,\ c_2 \to c_2 + \left(1 - \frac{1}{t_1}\right) u_1 w_2,\right.$$
$$c_3 \to c_3,\ u_1 w_1 \to u_1 w_1 + \left(1 - \frac{1}{t_1}\right) u_1 w_2,\ u_1 w_2 \to \frac{u_1 w_2}{t_1},\ u_1 w_3 \to u_1 w_3,$$
$$u_2 w_1 \to \left(1 - t_1\right) u_1 w_1 + t_1 u_2 w_1 - \frac{\left(-1 + t_1\right)^2 u_1 w_2}{t_1} + \left(-1 + t_1\right) u_2 w_2,\ u_2 w_2 \to \left(-1 + \frac{1}{t_1}\right) u_1 w_2 + u_2 w_2,$$
$$\left.u_2 w_3 \to \left(1 - t_1\right) u_1 w_3 + t_1 u_2 w_3,\ u_3 w_1 \to u_3 w_1 + \left(1 - \frac{1}{t_1}\right) u_3 w_2,\ u_3 w_2 \to \frac{u_3 w_2}{t_1},\ u_3 w_3 \to u_3 w_3\right\}$$

```
u1 w2 + c2 // EGc_{1,2}
```

$$c_2 + u_1 w_2$$

```
(# → Simplify[EGc_{1,2}[#] /. {u_i_ w_i_ → 1, u_i_ w_j_ /; i ≠ j :> 0}]) & /@ EGcchecks
```

$$\{c_1 \to c_1,\ c_2 \to c_2,\ c_3 \to c_3,\ u_1 w_1 \to 1,\ u_1 w_2 \to 0,\ u_1 w_3 \to 0,$$
$$u_2 w_1 \to 0,\ u_2 w_2 \to 1,\ u_2 w_3 \to 0,\ u_3 w_1 \to 0,\ u_3 w_2 \to 0,\ u_3 w_3 \to 1\}$$

```
Short[R3l = EGcchecks // EGc1,2 // EGc1,3 // EGc2,3, 10]
```

$$\Big\{c_1 - u_1 w_2 + \frac{u_1 w_2}{t_1} - u_1 w_3 + \frac{u_1 w_3}{t_1},\ c_2 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2},$$
$$c_3 + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},\ u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1},\ \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},$$
$$\frac{u_1 w_3}{t_1 t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$
$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3,\ -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$
$$-\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$
$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$
$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$
$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$
$$\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3,\ -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\Big\}$$

```
R3r = EGcchecks // EGc2,3 // EGc1,3 // EGc1,2; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

The Turbo-Gassner Representation

```
In[•]:= TGi_,j_[ξ_] := Expand[ξ /. {
          f_. vk_ :> Plus[f vk /. vj → (1 - ti) vi + ti vj,
            (1 - ti^-1) (ti ∂ti f - tj ∂tj f) (uk /. uj → (1 - ti) ui + ti uj) ui wj,
            Kδk,i f (uj - ui) ui wj],
          uj → (1 - ti) ui + ti uj,
          wi → wi + (1 - ti^-1) wj, wj → ti^-1 wj}];
        TGi_,j_[ξ_] := Expand[ξ /. {
          f_. vk_ :> Plus[f vk /. vj → (1 - ti^-1) vi + ti^-1 vj,
            (1 - ti) (ti ∂ti f - tj ∂tj f) (uk /. uj → (1 - ti^-1) ui + ti^-1 uj) ui wj,
            Kδk,i f (ui - uj) ui wj],
          uj → (1 - ti^-1) ui + ti^-1 uj,
          wi → wi + (1 - ti) wj, wj → ti wj}];
        TGchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3};
        TGchecks // TG1,3 // TG1,3
Out[•]= {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3}
```

```
In[•]:= Short[R3l = TGchecks // TG1,2 // TG1,3 // TG2,3, 10]
```

Out[•]//Short=

$$\{f[t_1,t_2,t_3]\,v_1 - f[t_1,t_2,t_3]\,u_1^2\,w_2 + f[t_1,t_2,t_3]\,u_1\,u_2\,w_2 - f[t_1,t_2,t_3]\,u_1^2\,w_3 + f[t_1,t_2,t_3]\,u_1\,u_3\,w_3 - \frac{t_3\,u_1^2\,w_3\,f^{(0,0,1)}[t_1,t_2,t_3]}{t_2} + \text{<<15>>} + t_2\,u_1\,u_2\,w_3\,f^{(0,1,0)}[t_1,t_2,t_3] - u_1^2\,w_2\,f^{(1,0,0)}[t_1,t_2,t_3] + t_1\,u_1^2\,w_2\,f^{(1,0,0)}[t_1,t_2,t_3] - u_1^2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3] + t_1\,u_1^2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3],\ f[t_1,t_2,t_3]\,v_1 - \text{<<1>>}\,t_1\,v_1 + \text{<<63>>} + t_1^2\,u_1\,u_2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3],\ \text{<<5>>},\ \text{<<1>>},\ \frac{w_3}{t_1\,t_2}\}$$

```
In[•]:= R3r = TGchecks // TG2,3 // TG1,3 // TG1,2; R3l - R3r
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= Short[OCl = TGchecks // TG1,2 // TG1,3]
```

Out[•]//Short=

$$\{\text{<<18>>} + t_1\,\text{<<2>>}\,f^{\text{<<1>>}}[t_1,t_2,t_3],\ \text{<<1>>},\ \text{<<6>>},\ \frac{\text{<<1>>}}{\text{<<1>>}}\}$$

```
In[•]:= OCr = TGchecks // TG1,3 // TG1,2; OCl - OCr
```

Out[•]=

$$\{0, -f[t_1,t_2,t_3]\,u_1\,u_2\,w_3 + f[t_1,t_2,t_3]\,t_1\,u_1\,u_2\,w_3 + f[t_1,t_2,t_3]\,u_1\,u_3\,w_3 - f[t_1,t_2,t_3]\,t_1\,u_1\,u_3\,w_3, -f[t_1,t_2,t_3]\,u_1\,u_2\,w_2 + f[t_1,t_2,t_3]\,t_1\,u_1\,u_2\,w_2 + f[t_1,t_2,t_3]\,u_1\,u_3\,w_2 - f[t_1,t_2,t_3]\,t_1\,u_1\,u_3\,w_2, 0, 0, 0, 0, 0, 0\}$$

A Finite-Rank Turbo-Gassner Representation

```
(Alt) In[•]:= η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
FTG_{i_,j_}[ℱ_] := Expand[ℱ /. {
    f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i - η[i]) v_i + (t_i + η[i]) v_j,
      (1 - t_i^-1) (t_i ∂_η[i] f - t_j ∂_η[j] f) (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
      Kδ_{k,i} (f /. _η → 0) (u_j - u_i) u_i w_j],
    u_j → (1 - t_i) u_i + t_i u_j,
    w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
FTG‾_{i_,j_}[ℱ_] := Expand[ℱ /. {
    f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i^-1 + t_i^-2 η[i]) v_i + (t_i^-1 - t_i^-2 η[i]) v_j,
      (1 - t_i) (t_i ∂_η[i] f - t_j ∂_η[j] f) (u_k /. u_j → (1 - t_i^-1) u_i + t_i^-1 u_j) u_i w_j,
      Kδ_{k,i} (f /. _η → 0) (u_i - u_j) u_i w_j],
    u_j → (1 - t_i^-1) u_i + t_i^-1 u_j,
    w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
ff = f_0 + f_1 η[1] + f_2 η[2] + f_3 η[3];
FTGchecks = Expand@{ff v_1, ff v_2, ff v_3, u_1, u_2, u_3, w_1, w_2, w_3};
(FTGchecks // FTG1,3 // FTG‾1,3) - FTGchecks
(Alt) Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0}
```

(Alt) In[•]:=
```
({v1, v2, v3, v1,1, v1,2, v1,3, v2,1, v2,2, v2,3, v3,1, v3,2, v3,3, u1, u2, u3, w1, w2, w3} /.
    vl_,k_ :> η[l] vk) // FTG1,2 // Column
```

$$v_1 - u_1^2 w_2 + u_1 u_2 w_2$$
$$v_1 - t_1 v_1 + t_1 v_2 - v_1 \eta[1] + v_2 \eta[1]$$
$$v_3$$
$$-u_1^2 w_2 + t_1 u_1^2 w_2 + v_1 \eta[1]$$
$$-u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_2 w_2 + t_1^2 u_1 u_2 w_2 + v_1 \eta[1] - t_1 v_1 \eta[1] + t_1 v_2 \eta[1]$$
$$-u_1 u_3 w_2 + t_1 u_1 u_3 w_2 + v_3 \eta[1]$$
$$-t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + v_1 \eta[2]$$
$$-2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_2 w_2 - t_1 t_2 u_1 u_2 w_2 + v_1 \eta[2] - t_1 v_1 \eta[2] + t_1 v_2 \eta[2]$$

(Alt) Out[•]=
$$-t_2 u_1 u_3 w_2 + \frac{t_2 u_1 u_3 w_2}{t_1} + v_3 \eta[2]$$
$$v_1 \eta[3]$$
$$v_1 \eta[3] - t_1 v_1 \eta[3] + t_1 v_2 \eta[3]$$
$$v_3 \eta[3]$$
$$u_1$$
$$u_1 - t_1 u_1 + t_1 u_2$$
$$u_3$$
$$w_1 + w_2 - \frac{w_2}{t_1}$$
$$\frac{w_2}{t_1}$$
$$w_3$$

In[•]:=
```
Short[R3l = FTGchecks // FTG1,2 // FTG1,3 // FTG2,3, 10]
```

Out[•]//Short=
$$\left\{f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + f_1 t_1 u_1^2 w_2 - f_2 t_2 u_1^2 w_2 + \frac{f_2 t_2 u_1^2 w_2}{t_1} + \right.$$
$$f_0 u_1 u_2 w_2 - f_0 u_1^2 w_3 - f_1 u_1^2 w_3 + f_2 u_1^2 w_3 - \frac{f_2 u_1^2 w_3}{t_1} + f_1 t_1 u_1^2 w_3 - f_2 t_2 u_1^2 w_3 +$$
$$\frac{f_2 t_2 u_1^2 w_3}{t_1} - \frac{f_3 t_3 u_1^2 w_3}{t_2} + \frac{f_3 t_3 u_1^2 w_3}{t_1 t_2} - f_2 u_1 u_2 w_3 + f_2 t_2 u_1 u_2 w_3 - f_3 t_3 u_1 u_2 w_3 +$$
$$\frac{f_3 t_3 u_1 u_2 w_3}{t_2} + f_0 u_1 u_3 w_3 + f_1 v_1 \eta[1] + f_2 v_1 \eta[2] + f_3 v_1 \eta[3], \ll 7 \gg, \left. \frac{w_3}{t_1 t_2} \right\}$$

In[•]:=
```
R3r = FTGchecks // FTG2,3 // FTG1,3 // FTG1,2; R3l - R3r
```

Out[•]=
$$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

(Alt) In[•]:=
```
Short[OCl = FTGchecks // FTG1,2 // FTG1,3]
```

(Alt) Out[•]//Short=
$$\left\{\ll 20 \gg + \ll 1 \gg + f_3 v_1 \eta[3], \ll 1 \gg, \ll 5 \gg, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{w_3}{t_1}\right\}$$

(Alt) In[•]:=
```
Short[OCr = FTGchecks // FTG1,3 // FTG1,2]
```

(Alt) Out[•]//Short=
$$\left\{\ll 20 \gg + \ll 1 \gg + f_3 v_1 \eta[3], \ll 1 \gg, \ll 5 \gg, \frac{\ll 1 \gg}{\ll 1 \gg}, \frac{w_3}{t_1}\right\}$$

(Alt) In[•]:=
```
OCl - OCr
```

(Alt) Out[•]=
$$\{0, -f_0 u_1 u_2 w_3 + f_0 t_1 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - f_0 t_1 u_1 u_3 w_3,$$
$$-f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - f_0 t_1 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0\}$$

In[•]:=
```
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3] + f4 η[4];
FTGchecks4 = Expand@{ff v1, ff v2, ff v3, ff v4, u1, u2, u3, u4, w1, w2, w3, w4};
```

In[•]:=
```
Short[FOC1 = FTGchecks4 // FTG1,2 // FTG1,3 // FTG1,4]
```

Out[•]//Short=
$$\left\{\text{<<30>>} + f_3 v_1 \eta[3] + f_4 v_1 \eta[4], \text{<<1>>}, \text{<<1>>}, \text{<<6>>}, \frac{\text{<<1>>}}{\text{<<1>>}}, \frac{w_3}{t_1}, \frac{w_4}{t_1}\right\}$$

In[•]:=
```
Short[FOC2 = FTGchecks4 // FTG1,3 // FTG1,2 // FTG1,4]
```

Out[•]//Short=
$$\left\{\text{<<30>>} + f_3 v_1 \eta[3] + f_4 v_1 \eta[4], \text{<<1>>}, \text{<<1>>}, \text{<<6>>}, \frac{\text{<<1>>}}{\text{<<1>>}}, \frac{w_3}{t_1}, \frac{w_4}{t_1}\right\}$$

In[•]:=
```
Short[FOC3 = FTGchecks4 // FTG1,4 // FTG1,2 // FTG1,3]
```

Out[•]//Short=
$$\left\{\text{<<30>>} + f_3 v_1 \eta[3] + f_4 v_1 \eta[4], \text{<<1>>}, \text{<<1>>}, \text{<<6>>}, \frac{\text{<<1>>}}{\text{<<1>>}}, \frac{w_3}{t_1}, \frac{w_4}{t_1}\right\}$$

In[•]:=
```
Short[FOC4 = FTGchecks4 // FTG1,4 // FTG1,3 // FTG1,2]
```

Out[•]//Short=
$$\left\{\text{<<30>>} + f_3 v_1 \eta[3] + f_4 v_1 \eta[4], \text{<<10>>}, \frac{w_4}{t_1}\right\}$$

In[•]:=
```
FOC1 - FOC2 - FOC3 + FOC4
```

Out[•]=
$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

In[•]:=
```
FOC1 - FOC2
```

Out[•]=
$$\{0, -f_0 u_1 u_2 w_3 + f_0 t_1 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - f_0 t_1 u_1 u_3 w_3,$$
$$-f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - f_0 t_1 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0, 0, 0\}$$

In[•]:=
```
FOC2 - FOC3
```

Out[•]=
$$\{0, f_0 u_1 u_2 w_3 - f_0 t_1 u_1 u_2 w_3 - f_0 u_1 u_3 w_3 + f_0 t_1 u_1 u_3 w_3 -$$
$$f_0 u_1 u_2 w_4 + f_0 t_1 u_1 u_2 w_4 + f_0 u_1 u_4 w_4 - f_0 t_1 u_1 u_4 w_4, f_0 u_1 u_2 w_2 - f_0 t_1 u_1 u_2 w_2 -$$
$$f_0 u_1 u_3 w_2 + f_0 t_1 u_1 u_3 w_2 - f_0 u_1 u_3 w_4 + f_0 t_1 u_1 u_3 w_4 + f_0 u_1 u_4 w_4 - f_0 t_1 u_1 u_4 w_4,$$
$$-f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_4 w_2 - f_0 t_1 u_1 u_4 w_2 - f_0 u_1 u_3 w_3 +$$
$$f_0 t_1 u_1 u_3 w_3 + f_0 u_1 u_4 w_3 - f_0 t_1 u_1 u_4 w_3, 0, 0, 0, 0, 0, 0, 0, 0\}$$

In[•]:=
```
FOC1 - FOC3
```

Out[•]=
$$\{0, -f_0 u_1 u_2 w_4 + f_0 t_1 u_1 u_2 w_4 + f_0 u_1 u_4 w_4 - f_0 t_1 u_1 u_4 w_4,$$
$$-f_0 u_1 u_3 w_4 + f_0 t_1 u_1 u_3 w_4 + f_0 u_1 u_4 w_4 - f_0 t_1 u_1 u_4 w_4, -f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_4 w_2 -$$
$$f_0 t_1 u_1 u_4 w_2 - f_0 u_1 u_3 w_3 + f_0 t_1 u_1 u_3 w_3 + f_0 u_1 u_4 w_3 - f_0 t_1 u_1 u_4 w_3, 0, 0, 0, 0, 0, 0, 0, 0\}$$

In[•]:=
```
(FTGchecks4 // FTG1,2 // FTG1,3 // FTG̅1,2 // FTG̅1,3 // FTG1,4 // FTG1,3 // FTG1,2 // FTG̅1,3 //
    FTG̅1,2 // FTG̅1,4) - FTGchecks4
```

Out[•]=
$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

```
In[•]:= (FTGchecks4 // FTG1,2 // FTG1,3 // FTG̅1,2 // FTG̅1,3 // FTG1,3 // FTG1,3 // FTG1,2 // FTG̅1,3 //
          FTG̅1,2 // FTG̅1,3) - FTGchecks4
```

```
Out[•]= {0, f0 u1^2 w3 - 2 f0 t1 u1^2 w3 + f0 t1^2 u1^2 w3 - f0 u1 u2 w3 + 2 f0 t1 u1 u2 w3 - f0 t1^2 u1 u2 w3,
  f0 u1^2 w2 - 2 f0 t1 u1^2 w2 + f0 t1^2 u1^2 w2 - f0 u1 u2 w2 + 2 f0 t1 u1 u2 w2 - f0 t1^2 u1 u2 w2,
  0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
In[•]:= ff = f0 + f1 η[1] + f2 η[2] + f3 η[3] + f4 η[4] + f5 η[5];
FTGchecks5 = Expand@{ff v1, ff v2, ff v3, ff v4, ff v5 u1, u2, u3, u4, u5, w1, w2, w3, w4, w5};
(FTGchecks5 // FTG1,2 // FTG1,3 // FTG̅1,2 // FTG̅1,3 // FTG1,4 // FTG1,5 // FTG̅1,4 // FTG̅1,5) -
 FTGchecks5
```

```
Out[•]= {0, -f0 u1 u2 w3 + f0 t1 u1 u2 w3 + f0 u1 u3 w3 - f0 t1 u1 u3 w3,
  -f0 u1 u2 w2 + f0 t1 u1 u2 w2 + f0 u1 u3 w2 - f0 t1 u1 u3 w2,
  -f0 u1 u4 w5 + f0 t1 u1 u4 w5 + f0 u1 u5 w5 - f0 t1 u1 u5 w5,
  -f0 u1^2 u4 w4 + f0 t1 u1^2 u4 w4 + f0 u1^2 u5 w4 - f0 t1 u1^2 u5 w4, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

The Turbo-Burau Representation

```
In[•]:= η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
TBi_,j_[ℰ_] :=
  Expand[ℰ /. {
    f_. vk_ :> Plus[f vk /. vj → (1 - t - η[i]) vi + (t + η[i]) vj,
      (t - 1) (∂η[i] f - ∂η[j] f) (uk /. uj → (1 - t) ui + t uj) ui wj,
      Kδk,i (f /. _η → 0) (uj - ui) ui wj],
    uj → (1 - t) ui + t uj,
    wi → wi + (1 - t^-1) wj, wj → t^-1 wj}];
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
TBchecks = {ff v1, ff v2, ff v3, u1^2 w1, u1^2 w2, u1, u2, u3, w1, w2, w3};
```

```
In[•]:= Short[R3l = TBchecks // TB1,2 // TB1,3 // TB2,3, 10]
```

```
Out[•]//Short= {f0 v1 - f0 u1^2 w2 - f1 u1^2 w2 + t f1 u1^2 w2 + f2 u1^2 w2 - t f2 u1^2 w2 + f0 u1 u2 w2 - f0 u1^2 w3 - f1 u1^2 w3 +
    t f1 u1^2 w3 + 2 f2 u1^2 w3 - f2 u1^2 w3 / t - t f2 u1^2 w3 - f3 u1^2 w3 + f3 u1^2 w3 / t - f2 u1 u2 w3 + t f2 u1 u2 w3 +
    f3 u1 u2 w3 - t f3 u1 u2 w3 + f0 u1 u3 w3 + f1 v1 η[1] + f2 v1 η[2] + f3 v1 η[3], <<9>>, w3 / t^2}
```

```
In[•]:= R3r = TBchecks // TB2,3 // TB1,3 // TB1,2; R3l - R3r
```

```
Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
In[•]:= Short[OCl = TBchecks // TB1,2 // TB1,3]
```

```
Out[•]//Short= {f0 v1 - f0 u1^2 w2 - f1 u1^2 w2 + t f1 u1^2 w2 + <<12>> + f0 u1 u3 w3 + f1 v1 η[1] + f2 v1 η[2] + f3 v1 η[3],
    <<1>>, <<1>>, <<5>>, w1 + <<6>>, w2 / t, w3 / t}
```

```
In[•]:= OCr = TBchecks // TB1,3 // TB1,2; OCl - OCr
```

```
Out[•]= {0, -f0 u1 u2 w3 + t f0 u1 u2 w3 + f0 u1 u3 w3 - t f0 u1 u3 w3,
  -f0 u1 u2 w2 + t f0 u1 u2 w2 + f0 u1 u3 w2 - t f0 u1 u3 w2, 0, 0, 0, 0, 0, 0, 0, 0}
```

The Turbo@1 Burau (Non) Representation

```
In[•]:= TB1_{i_,j_}[ℰ_] :=
          Expand[ℰ /. {
            v_k_ :> (v_k /. v_j → (1 - t) v_i + t v_j + Kδ_{1,i} (v1_j - v1_1)) + Kδ_{k,i} (u_j - u_i) u_i w_j,
            v1_k_ :>
              (v1_k /. v1_j → (1 - t) v1_i + t v1_j) + (t - 1) (Kδ_{1,i} - Kδ_{1,j}) (u_k /. u_j → (1 - t) u_i + t u_j) u_i w_j,
            u_j → (1 - t) u_i + t u_j,
            w_i → w_i + (1 - t^-1) w_j,  w_j → t^-1 w_j}];
        TB1checks = {v_1, v_2, v_3, v1_1, v1_2, v1_3, u_1^2 w_1, u_1^2 w_2, u_1, u_2, u_3, w_1, w_2, w_3};

In[•]:= Short[R3l = TB1checks // TB1_{1,2} // TB1_{1,3} // TB1_{2,3}, 10]

Out[•]//Short= {v_1 - u_1^2 w_2 + u_1 u_2 w_2 - u_1^2 w_3 + u_1 u_3 w_3,
 v_1 - t v_1 + t v_2 - v1_1 + v1_2 - u_1 u_2 w_3 + t u_1 u_2 w_3 - t u_2^2 w_3 + u_1 u_3 w_3 - t u_1 u_3 w_3 + t u_2 u_3 w_3,
 v_1 - t v_1 + t v_2 - t^2 v_2 + t^2 v_3 - v1_1 + v1_2 - t v1_2 + t v1_3, v1_1 - u_1^2 w_2 + t u_1^2 w_2 - u_1^2 w_3 + t u_1^2 w_3,
 v1_1 - t v1_1 + t v1_2 - u_1^2 w_2 + 2 t u_1^2 w_2 - t^2 u_1^2 w_2 - t u_1 u_2 w_2 + t^2 u_1 u_2 w_2 - u_1^2 w_3 + 2 t u_1^2 w_3 -
  t^2 u_1^2 w_3 - t u_1 u_2 w_3 + t^2 u_1 u_2 w_3, v1_1 - t v1_1 + t v1_2 - t^2 v1_2 + t^2 v1_3 - u_1^2 w_2 + 2 t u_1^2 w_2 -
  t^2 u_1^2 w_2 - t u_1 u_2 w_2 + 2 t^2 u_1 u_2 w_2 - t^3 u_1 u_2 w_2 - t^2 u_1 u_3 w_2 + t^3 u_1 u_3 w_2 - u_1^2 w_3 +
  2 t u_1^2 w_3 - t^2 u_1^2 w_3 - t u_1 u_2 w_3 + 2 t^2 u_1 u_2 w_3 - t^3 u_1 u_2 w_3 - t^2 u_1 u_3 w_3 + t^3 u_1 u_3 w_3,
 u_1^2 w_1 + u_1^2 w_2 - u_1^2 w_2/t + u_1^2 w_3 - u_1^2 w_3/t, u_1^2 w_2/t - u_1^2 w_3/t^2 + u_1^2 w_3/t, u_1, u_1 - t u_1 + t u_2,
 u_1 - t u_1 + t u_2 - t^2 u_2 + t^2 u_3, w_1 + w_2 - w_2/t + w_3 - w_3/t, w_2/t - w_3/t^2 + w_3/t, w_3/t^2}

In[•]:= R3r = TB1checks // TB1_{2,3} // TB1_{1,3} // TB1_{1,2}; R3l - R3r

Out[•]= {0, 0, -t u_1 u_2 w_2 + t^2 u_1 u_2 w_2 + t u_1 u_3 w_2 - t^2 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```