Pensieve header: Testing and implementing lemmas 1,2,3 of the DoPeGDO handouts. Continues pensieve://2019-10/, continued pensieve://2019-12/.

$$[F:\mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad\text{and}\quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

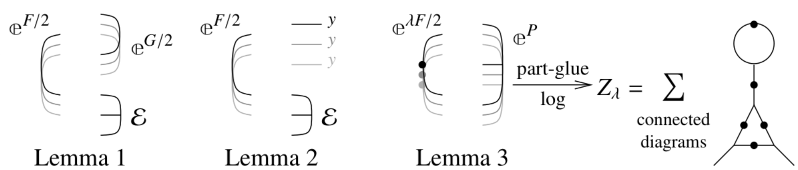
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B} y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Goals:

Implement the containers $|F, \mathcal{E}|_B := [F:\mathcal{E}]_B$ and $\langle F, \mathcal{E}\rangle_B := \langle F:\mathcal{E}\rangle_B$, their evaluator Ev_k as power series in $\hbar$ to degree k, and verify lemmas 1, 2, and 3. Inserting $\hbar$ in the appropriate places is user responsibility.

Implement DaGauss, DeLin, and a Lemma 3 evaluator, PEv.

Utilities

In[ ]:=
```
HL[ℰ_] := Style[ℰ, Background → If[TrueQ@ℰ, ■, ■]];
```

Generic Polynomials:

In[ ]:=
```
GenericPolynomial[d_Integer, vars_List, gc_] := Total[Map[
    gc_Sequence@@# Times @@ (vars^#) &,
    Join @@ (Permutations /@ IntegerPartitions[d + Length@vars, {Length@vars}]) - 1
  ]];
GenericPolynomial[specs_List, vars_List, gc_] :=
 Sum[GenericPolynomial[specs〚1〛, vars, gc], specs]
```

Preliminary Definitions

In[∘]:=
```
Unprotect[SeriesData];
Expand[sd_SeriesData] ^:= MapAt[Expand, sd, 3];
Protect[SeriesData];
```

In[∘]:=
```
CF[⟨F_, ℰ_⟩B_] := ⟨Simplify@F, Simplify@ℰ⟩B;
```

Variables and their duals:

In[∘]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)i; (B_List)* := #* & /@B;
```

Act and Contract:

In[∘]:=
```
Evk_ @ ¦F_, ℰ_¦B_ := Expand[Total[
     CoefficientRules[Normal@Series[e^(B*.F.B*/2), {ℏ, 0, k}], B*] /.
      (ps_ → c_) :→ c D[ℰ, Sequence @@ Thread[{B, ps}]]
    ] + O[ℏ]^(k+1)];
Evk_ @⟨F_, ℰ_⟩B_ := Evk@¦F, ℰ¦B /. Alternatives @@ B → 0
```

In[∘]:=
```
{Ev2@|ℏ (0 1
        1 0), e^(x y)|{x,y}, Ev3@⟨ℏ (0 1
                                  1 0), e^(3 x y)⟩{x,y}}
```

Out[∘]=
$$\left\{e^{x y}+\left(e^{x y}+e^{x y}\, x\, y\right)\hbar+\left(e^{x y}+2\, e^{x y}\, x\, y+\frac{1}{2}\, e^{x y}\, x^2\, y^2\right)\hbar^2+O[\hbar]^3,\ 1+3\,\hbar+9\,\hbar^2+27\,\hbar^3+O[\hbar]^4\right\}$$

Implementing / Testing Lemma 1

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\,\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij} z_i z_j} \right\rangle_B = \det(1-GF)^{-1/2} \left\langle F(1-GF)^{-1} : \mathcal{E} \right\rangle_B$$

In[ ]:=
```
{p = 10, B = {x}, 𝓘 = IdentityMatrix@Length@B,
  F = ℏ {{f}}, G = {{g}}, ℰ = GenericPolynomial[{d, 0, 4}, B, c]}
lhs = Ev_p@⟨F, ℰ 𝕖^(B.G.B/2)⟩_B
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) ℰ⟩_B;
HL[lhs == rhs]
```

Out[ ]= $\{10, \{x\}, \{\{1\}\}, \{\{f\,\hbar\}\}, \{\{g\}\}, c_0 + x\,c_1 + x^2 c_2 + x^3 c_3 + x^4 c_4\}$

Out[ ]= $c_0 + \left(\frac{1}{2} f\,g\,c_0 + f\,c_2\right)\hbar + \left(\frac{3}{8} f^2 g^2 c_0 + \frac{3}{2} f^2 g\,c_2 + 3 f^2 c_4\right)\hbar^2 + \left(\frac{5}{16} f^3 g^3 c_0 + \frac{15}{8} f^3 g^2 c_2 + \frac{15}{2} f^3 g\,c_4\right)\hbar^3 + \left(\frac{35}{128} f^4 g^4 c_0 + \frac{35}{16} f^4 g^3 c_2 + \frac{105}{8} f^4 g^2 c_4\right)\hbar^4 + \left(\frac{63}{256} f^5 g^5 c_0 + \frac{315}{128} f^5 g^4 c_2 + \frac{315}{16} f^5 g^3 c_4\right)\hbar^5 + \left(\frac{231 f^6 g^6 c_0}{1024} + \frac{693}{256} f^6 g^5 c_2 + \frac{3465}{128} f^6 g^4 c_4\right)\hbar^6 + \left(\frac{429 f^7 g^7 c_0}{2048} + \frac{3003 f^7 g^6 c_2}{1024} + \frac{9009}{256} f^7 g^5 c_4\right)\hbar^7 + \left(\frac{6435 f^8 g^8 c_0}{32768} + \frac{6435 f^8 g^7 c_2}{2048} + \frac{45045 f^8 g^6 c_4}{1024}\right)\hbar^8 + \left(\frac{12155 f^9 g^9 c_0}{65536} + \frac{109395 f^9 g^8 c_2}{32768} + \frac{109395 f^9 g^7 c_4}{2048}\right)\hbar^9 + \left(\frac{46189 f^{10} g^{10} c_0}{262144} + \frac{230945 f^{10} g^9 c_2}{65536} + \frac{2078505 f^{10} g^8 c_4}{32768}\right)\hbar^{10} + O[\hbar]^{11}$

Out[ ]= True

In[∘]:=
```
{p = 2, B = {x, y}, 𝓘 = IdentityMatrix@Length@B,
 F = ℏ (f11 f12; f12 f22), G = (g11 g12; g12 g22), ℰ = GenericPolynomial[{d, 0, 2}, B, c]}
Timing[lhs = Ev_p@⟨F, ℰ e^(B.G.B/2)⟩_B]
rhs = Ev_p@⟨F.Inverse[𝓘 - G.F], Det[𝓘 - G.F]^(-1/2) ℰ⟩_B;
HL[lhs == rhs]
```

Out[∘]= $\{2, \{x, y\}, \{\{1, 0\}, \{0, 1\}\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$

Out[∘]= $$\Big\{0.0625,\ c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0}\right) \hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]= True

In[∘]:=
```
DeGauss@⟨F_, ℰ_⟩_B_ := Module[{𝓘, Q, G, M, Δ},
  𝓘 = IdentityMatrix@Length@B;
  Q = Log[Normal[ℰ] /. ϵ → 0];
  G = Table[∂_{i,j} Q, {i, B}, {j, B}];
  M = Inverse[𝓘 - G.F];
  Δ = Simplify@Det@M;
  CF@⟨F.M, Δ^(1/2) ℰ e^(-B.G.B/2)⟩_B
 ]
```

In[∘]:=
$$\left\{p = 2,\ B = \{x, y\},\ F = \hbar \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix},\ G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix},\ \mathcal{E} = 1 + \epsilon\, \mathtt{GenericPolynomial}[\{d, 0, 2\}, B, c]\right\};$$
$$\mathtt{lhs} = \left\langle F,\ \mathcal{E}\, e^{B.G.B/2} \right\rangle_B$$
$$\mathtt{rhs} = \mathtt{DeGauss}@\left\langle F,\ \mathcal{E}\, e^{B.G.B/2} \right\rangle_B$$
$$\mathtt{HL}[\mathtt{Ev}_p@\mathtt{lhs} == \mathtt{Ev}_p@\mathtt{rhs}]$$

Out[∘]=
$$\left\langle \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},\ e^{\frac{1}{2}(x(x g_{11}+y g_{12})+y(x g_{12}+y g_{22}))}\left(1 + \epsilon\left(c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\right)\right)\right\rangle_{\{x,y\}}$$

Out[∘]=
$$\Bigg\langle \Bigg\{\Bigg\{\frac{\hbar\left(\hbar f_{12}^2 g_{22} + f_{11}\left(1 - \hbar f_{22} g_{22}\right)\right)}{1 - 2\hbar f_{12} g_{12} - \hbar f_{22} g_{22} + \hbar^2 f_{12}^2\left(g_{12}^2 - g_{11} g_{22}\right) + \hbar f_{11}\left(-\hbar f_{22} g_{12}^2 + g_{11}\left(-1 + \hbar f_{22} g_{22}\right)\right)},$$
$$\frac{\hbar\left(f_{12} - \hbar f_{12}^2 g_{12} + \hbar f_{11} f_{22} g_{12}\right)}{1 - 2\hbar f_{12} g_{12} - \hbar f_{22} g_{22} + \hbar^2 f_{12}^2\left(g_{12}^2 - g_{11} g_{22}\right) + \hbar f_{11}\left(-\hbar f_{22} g_{12}^2 + g_{11}\left(-1 + \hbar f_{22} g_{22}\right)\right)}\Bigg\},$$
$$\Bigg\{\frac{\hbar\left(f_{12} - \hbar f_{12}^2 g_{12} + \hbar f_{11} f_{22} g_{12}\right)}{1 - 2\hbar f_{12} g_{12} - \hbar f_{22} g_{22} + \hbar^2 f_{12}^2\left(g_{12}^2 - g_{11} g_{22}\right) + \hbar f_{11}\left(-\hbar f_{22} g_{12}^2 + g_{11}\left(-1 + \hbar f_{22} g_{22}\right)\right)},$$
$$\frac{\hbar\left(\hbar f_{12}^2 g_{11} + f_{22}\left(1 - \hbar f_{11} g_{11}\right)\right)}{1 - 2\hbar f_{12} g_{12} - \hbar f_{22} g_{22} + \hbar^2 f_{12}^2\left(g_{12}^2 - g_{11} g_{22}\right) + \hbar f_{11}\left(-\hbar f_{22} g_{12}^2 + g_{11}\left(-1 + \hbar f_{22} g_{22}\right)\right)}\Bigg\}\Bigg\},$$
$$\sqrt{\frac{1}{1 - 2\hbar f_{12} g_{12} - \hbar f_{22} g_{22} + \hbar^2 f_{12}^2\left(g_{12}^2 - g_{11} g_{22}\right) + \hbar f_{11}\left(-\hbar f_{22} g_{12}^2 + g_{11}\left(-1 + \hbar f_{22} g_{22}\right)\right)}}$$
$$\left(1 + \epsilon\, c_{0,0} + y\, \epsilon\, c_{0,1} + y^2 \epsilon\, c_{0,2} + x\, \epsilon\, c_{1,0} + x\, y\, \epsilon\, c_{1,1} + x^2 \epsilon\, c_{2,0}\right)\Bigg\rangle_{\{x,y\}}$$

Out[∘]= True

Implementing / Testing Lemma 2

Lemma 2. $\left\langle F: \mathcal{E}\, e^{\sum_{i\in B} y_i z_i}\right\rangle_B = e^{\frac{1}{2}\sum_{i,j\in B} F_{ij} y_i y_j}\left\langle F:\ \mathcal{E}|_{z_B \to z_B + F y_B}\right\rangle_B.$

In[∘]:=
$$\{n = 2,\ p = 2,\ B = \mathtt{Table}[z_i, \{i, n\}],\ \mathcal{I} = \mathtt{IdentityMatrix}@\mathtt{Length}@B,\ Y = \mathtt{Table}[y_i, \{i, n\}],$$
$$F = \hbar\, \mathtt{Table}[f_{\{10,1\}.\mathtt{Sort}[\{i,j\}]}, \{i, n\}, \{j, n\}],\ \mathcal{E} = 1 + \epsilon\, \mathtt{GenericPolynomial}[\{d, 0, 3\}, B, c]\}$$
$$\mathtt{Timing}\left[\mathtt{lhs} = \mathtt{Ev}_p@\left\langle F, \mathcal{E}\, e^{Y.B}\right\rangle_B\right]$$
$$\mathtt{Timing}\left[\mathtt{rhs} = \mathtt{Ev}_p@\left\langle F, e^{Y.F.Y/2}\, \mathcal{E}\ /.\ \mathtt{Thread}[B \to B + F.Y]\right\rangle_B\right]$$
$$\mathtt{HL}[\mathtt{lhs} == \mathtt{rhs}]$$

Out[∘]=
$$\{2, 2, \{z_1, z_2\}, \{\{1, 0\}, \{0, 1\}\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$$
$$1 + \epsilon\left(c_{0,0} + z_2\, c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0}\right)\}$$

Out[ ]=

$$\Big\{0.4375,\ \left(1+\in c_{0,0}\right)+$$
$$\Big(\frac{1}{2} f_{11} y_1^2+f_{12} y_1 y_2+\frac{1}{2} f_{22} y_2^2+\frac{1}{2} \in f_{11} y_1^2 c_{0,0}+\in f_{12} y_1 y_2 c_{0,0}+\frac{1}{2} \in f_{22} y_2^2 c_{0,0}+\in f_{12} y_1 c_{0,1}+$$
$$\in f_{22} y_2 c_{0,1}+\in f_{22} c_{0,2}+\in f_{11} y_1 c_{1,0}+\in f_{12} y_2 c_{1,0}+\in f_{12} c_{1,1}+\in f_{11} c_{2,0}\Big)\hbar+$$
$$\Big(\frac{1}{8} f_{11}^2 y_1^4+\frac{1}{2} f_{11} f_{12} y_1^3 y_2+\frac{1}{2} f_{12}^2 y_1^2 y_2^2+\frac{1}{4} f_{11} f_{22} y_1^2 y_2^2+\frac{1}{2} f_{12} f_{22} y_1 y_2^3+\frac{1}{8} f_{22}^2 y_2^4+$$
$$\frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0}+\frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0}+\frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0}+\frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0}+$$
$$\frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0}+\frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0}+\frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1}+\in f_{12}^2 y_1^2 y_2 c_{0,1}+\frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1}+$$
$$\frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1}+\frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1}+\in f_{12}^2 y_1^2 c_{0,2}+\frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2}+3 \in f_{12} f_{22} y_1 y_2 c_{0,2}+$$
$$\frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2}+3 \in f_{12} f_{22} y_1 c_{0,3}+3 \in f_{22}^2 y_2 c_{0,3}+\frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0}+\frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0}+$$
$$\in f_{12}^2 y_1 y_2^2 c_{1,0}+\frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0}+\frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0}+\frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1}+2 \in f_{12}^2 y_1 y_2 c_{1,1}+$$
$$\in f_{11} f_{22} y_1 y_2 c_{1,1}+\frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1}+2 \in f_{12}^2 y_1 c_{1,2}+\in f_{11} f_{22} y_1 c_{1,2}+3 \in f_{12} f_{22} y_2 c_{1,2}+$$
$$\frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0}+3 \in f_{11} f_{12} y_1 y_2 c_{2,0}+\in f_{12}^2 y_2^2 c_{2,0}+\frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0}+3 \in f_{11} f_{12} y_1 c_{2,1}+$$
$$2 \in f_{12}^2 y_2 c_{2,1}+\in f_{11} f_{22} y_2 c_{2,1}+3 \in f_{11}^2 y_1 c_{3,0}+3 \in f_{11} f_{12} y_2 c_{3,0}\Big)\hbar^2+O[\hbar]^3\Big\}$$

Out[ ]=

$$\Big\{0.125,\ \left(1+\in c_{0,0}\right)+$$
$$\Big(\frac{1}{2} f_{11} y_1^2+f_{12} y_1 y_2+\frac{1}{2} f_{22} y_2^2+\frac{1}{2} \in f_{11} y_1^2 c_{0,0}+\in f_{12} y_1 y_2 c_{0,0}+\frac{1}{2} \in f_{22} y_2^2 c_{0,0}+\in f_{12} y_1 c_{0,1}+$$
$$\in f_{22} y_2 c_{0,1}+\in f_{22} c_{0,2}+\in f_{11} y_1 c_{1,0}+\in f_{12} y_2 c_{1,0}+\in f_{12} c_{1,1}+\in f_{11} c_{2,0}\Big)\hbar+$$
$$\Big(\frac{1}{8} f_{11}^2 y_1^4+\frac{1}{2} f_{11} f_{12} y_1^3 y_2+\frac{1}{2} f_{12}^2 y_1^2 y_2^2+\frac{1}{4} f_{11} f_{22} y_1^2 y_2^2+\frac{1}{2} f_{12} f_{22} y_1 y_2^3+\frac{1}{8} f_{22}^2 y_2^4+$$
$$\frac{1}{8} \in f_{11}^2 y_1^4 c_{0,0}+\frac{1}{2} \in f_{11} f_{12} y_1^3 y_2 c_{0,0}+\frac{1}{2} \in f_{12}^2 y_1^2 y_2^2 c_{0,0}+\frac{1}{4} \in f_{11} f_{22} y_1^2 y_2^2 c_{0,0}+$$
$$\frac{1}{2} \in f_{12} f_{22} y_1 y_2^3 c_{0,0}+\frac{1}{8} \in f_{22}^2 y_2^4 c_{0,0}+\frac{1}{2} \in f_{11} f_{12} y_1^3 c_{0,1}+\in f_{12}^2 y_1^2 y_2 c_{0,1}+\frac{1}{2} \in f_{11} f_{22} y_1^2 y_2 c_{0,1}+$$
$$\frac{3}{2} \in f_{12} f_{22} y_1 y_2^2 c_{0,1}+\frac{1}{2} \in f_{22}^2 y_2^3 c_{0,1}+\in f_{12}^2 y_1^2 c_{0,2}+\frac{1}{2} \in f_{11} f_{22} y_1^2 c_{0,2}+3 \in f_{12} f_{22} y_1 y_2 c_{0,2}+$$
$$\frac{3}{2} \in f_{22}^2 y_2^2 c_{0,2}+3 \in f_{12} f_{22} y_1 c_{0,3}+3 \in f_{22}^2 y_2 c_{0,3}+\frac{1}{2} \in f_{11}^2 y_1^3 c_{1,0}+\frac{3}{2} \in f_{11} f_{12} y_1^2 y_2 c_{1,0}+$$
$$\in f_{12}^2 y_1 y_2^2 c_{1,0}+\frac{1}{2} \in f_{11} f_{22} y_1 y_2^2 c_{1,0}+\frac{1}{2} \in f_{12} f_{22} y_2^3 c_{1,0}+\frac{3}{2} \in f_{11} f_{12} y_1^2 c_{1,1}+2 \in f_{12}^2 y_1 y_2 c_{1,1}+$$
$$\in f_{11} f_{22} y_1 y_2 c_{1,1}+\frac{3}{2} \in f_{12} f_{22} y_2^2 c_{1,1}+2 \in f_{12}^2 y_1 c_{1,2}+\in f_{11} f_{22} y_1 c_{1,2}+3 \in f_{12} f_{22} y_2 c_{1,2}+$$
$$\frac{3}{2} \in f_{11}^2 y_1^2 c_{2,0}+3 \in f_{11} f_{12} y_1 y_2 c_{2,0}+\in f_{12}^2 y_2^2 c_{2,0}+\frac{1}{2} \in f_{11} f_{22} y_2^2 c_{2,0}+3 \in f_{11} f_{12} y_1 c_{2,1}+$$
$$2 \in f_{12}^2 y_2 c_{2,1}+\in f_{11} f_{22} y_2 c_{2,1}+3 \in f_{11}^2 y_1 c_{3,0}+3 \in f_{11} f_{12} y_2 c_{3,0}\Big)\hbar^2+O[\hbar]^3\Big\}$$

Out[∘]= True

```
In[∘]:= DeLin@⟨F_, ℰ_⟩B_ := Module[{L, Y},
          L = PowerExpand@Log[Normal[ℰ] /. ϵ → 0];
          Y = Table[∂i L, {i, B}];
          CF@⟨F, e^(Y.F.Y/2) (e^(-B.Y) ℰ /. Thread[B → B + F.Y])⟩B
        ]
```

```
In[∘]:= {n = 2, p = 2, B = Table[zi, {i, n}], Y = Table[yi, {i, n}],
          F = ℏ Table[f{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = 1 + ϵ GenericPolynomial[{d, 0, 3}, B, c]}
        Timing[lhs = Evp@⟨F, ℰ e^(Y.B)⟩B]
        Timing[rhs = Evp@DeLin@⟨F, ℰ e^(Y.B)⟩B]
        HL@Simplify[lhs == rhs]
```

Out[∘]= $\{2, 2, \{z_1, z_2\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, 1 + \epsilon\, (c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0})\}$

Out[∘]=
$$\Big\{0.03125,\ (1 + \epsilon c_{0,0}) + \Big(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2}\epsilon f_{11} y_1^2 c_{0,0} + \epsilon f_{12} y_1 y_2 c_{0,0} + \frac{1}{2}\epsilon f_{22} y_2^2 c_{0,0} + \epsilon f_{12} y_1 c_{0,1} + \epsilon f_{22} y_2 c_{0,1} + \epsilon f_{22} c_{0,2} + \epsilon f_{11} y_1 c_{1,0} + \epsilon f_{12} y_2 c_{1,0} + \epsilon f_{12} c_{1,1} + \epsilon f_{11} c_{2,0}\Big)\hbar +$$
$$\Big(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 +$$
$$\frac{1}{8}\epsilon f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2}\epsilon f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2}\epsilon f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4}\epsilon f_{11} f_{22} y_1^2 y_2^2 c_{0,0} +$$
$$\frac{1}{2}\epsilon f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8}\epsilon f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2}\epsilon f_{11} f_{12} y_1^3 c_{0,1} + \epsilon f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2}\epsilon f_{11} f_{22} y_1^2 y_2 c_{0,1} +$$
$$\frac{3}{2}\epsilon f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2}\epsilon f_{22}^2 y_2^3 c_{0,1} + \epsilon f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2}\epsilon f_{11} f_{22} y_1^2 c_{0,2} + 3\epsilon f_{12} f_{22} y_1 y_2 c_{0,2} +$$
$$\frac{3}{2}\epsilon f_{22}^2 y_2^2 c_{0,2} + 3\epsilon f_{12} f_{22} y_1 c_{0,3} + 3\epsilon f_{22}^2 y_2 c_{0,3} + \frac{1}{2}\epsilon f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2}\epsilon f_{11} f_{12} y_1^2 y_2 c_{1,0} +$$
$$\epsilon f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2}\epsilon f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2}\epsilon f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2}\epsilon f_{11} f_{12} y_1^2 c_{1,1} + 2\epsilon f_{12}^2 y_1 y_2 c_{1,1} +$$
$$\epsilon f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2}\epsilon f_{12} f_{22} y_2^2 c_{1,1} + 2\epsilon f_{12}^2 y_1 c_{1,2} + \epsilon f_{11} f_{22} y_1 c_{1,2} + 3\epsilon f_{12} f_{22} y_2 c_{1,2} +$$
$$\frac{3}{2}\epsilon f_{11}^2 y_1^2 c_{2,0} + 3\epsilon f_{11} f_{12} y_1 y_2 c_{2,0} + \epsilon f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2}\epsilon f_{11} f_{22} y_2^2 c_{2,0} + 3\epsilon f_{11} f_{12} y_1 c_{2,1} +$$
$$2\epsilon f_{12}^2 y_2 c_{2,1} + \epsilon f_{11} f_{22} y_2 c_{2,1} + 3\epsilon f_{11}^2 y_1 c_{3,0} + 3\epsilon f_{11} f_{12} y_2 c_{3,0}\Big)\hbar^2 + O[\hbar]^3\Big\}$$

Out[∘]= $\{0., (1+\epsilon c_{0,0}) + \left(\frac{1}{2} f_{11} y_1^2 + f_{12} y_1 y_2 + \frac{1}{2} f_{22} y_2^2 + \frac{1}{2} \epsilon f_{11} y_1^2 c_{0,0} + \epsilon f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} \epsilon f_{22} y_2^2 c_{0,0} + \epsilon f_{12} y_1 c_{0,1} + \epsilon f_{22} y_2 c_{0,1} + \epsilon f_{22} c_{0,2} + \epsilon f_{11} y_1 c_{1,0} + \epsilon f_{12} y_2 c_{1,0} + \epsilon f_{12} c_{1,1} + \epsilon f_{11} c_{2,0}\right) \hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 + \frac{1}{8} f_{22}^2 y_2^4 + \frac{1}{8} \epsilon f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} \epsilon f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} \epsilon f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} \epsilon f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} \epsilon f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} \epsilon f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} \epsilon f_{11} f_{12} y_1^3 c_{0,1} + \epsilon f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} \epsilon f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} \epsilon f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} \epsilon f_{22}^2 y_2^3 c_{0,1} + \epsilon f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} \epsilon f_{11} f_{22} y_1^2 c_{0,2} + 3 \epsilon f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} \epsilon f_{22}^2 y_2^2 c_{0,2} + 3 \epsilon f_{12} f_{22} y_1 c_{0,3} + 3 \epsilon f_{22}^2 y_2 c_{0,3} + \frac{1}{2} \epsilon f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} \epsilon f_{11} f_{12} y_1^2 y_2 c_{1,0} + \epsilon f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} \epsilon f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} \epsilon f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} \epsilon f_{11} f_{12} y_1^2 c_{1,1} + 2 \epsilon f_{12}^2 y_1 y_2 c_{1,1} + \epsilon f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} \epsilon f_{12} f_{22} y_2^2 c_{1,1} + 2 \epsilon f_{12}^2 y_1 c_{1,2} + \epsilon f_{11} f_{22} y_1 c_{1,2} + 3 \epsilon f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2} \epsilon f_{11}^2 y_1^2 c_{2,0} + 3 \epsilon f_{11} f_{12} y_1 y_2 c_{2,0} + \epsilon f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} \epsilon f_{11} f_{22} y_2^2 c_{2,0} + 3 \epsilon f_{11} f_{12} y_1 c_{2,1} + 2 \epsilon f_{12}^2 y_2 c_{2,1} + \epsilon f_{11} f_{22} y_2 c_{2,1} + 3 \epsilon f_{11}^2 y_1 c_{3,0} + 3 \epsilon f_{11} f_{12} y_2 c_{3,0}\right) \hbar^2 + O[\hbar]^3\}$

Out[∘]= True

Testing Lemma 3

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

In[∘]:=
```
{n = 2, p = 2, B = Table[b_i, {i, n}],
 F = ℏ Table[f_{{10,1}.Sort[{i,j}]}, {i, n}, {j, n}], P = GenericPolynomial[{d, 0, 2}, B, c]}
```

Out[∘]= $\{2, 2, \{b_1, b_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}\}$

In[∘]:= $Z = \text{PowerExpand@Expand@Log}\left[\text{Ev}_p\text{@}\left|\lambda F, e^P\right|_B\right]$

Out[∘]=
$$\left(c_{0,0} + b_2\, c_{0,1} + b_2^2\, c_{0,2} + b_1\, c_{1,0} + b_1\, b_2\, c_{1,1} + b_1^2\, c_{2,0}\right) +$$
$$\left(\frac{1}{2}\lambda f_{22}\, c_{0,1}^2 + \lambda f_{22}\, c_{0,2} + 2\lambda\, b_2\, f_{22}\, c_{0,1}\, c_{0,2} + 2\lambda\, b_2^2\, f_{22}\, c_{0,2}^2 + \lambda f_{12}\, c_{0,1}\, c_{1,0} + 2\lambda\, b_2\, f_{12}\, c_{0,2}\, c_{1,0} +\right.$$
$$\frac{1}{2}\lambda f_{11}\, c_{1,0}^2 + \lambda f_{12}\, c_{1,1} + \lambda\, b_2\, f_{12}\, c_{0,1}\, c_{1,1} + \lambda\, b_1\, f_{22}\, c_{0,1}\, c_{1,1} + 2\lambda\, b_2^2\, f_{12}\, c_{0,2}\, c_{1,1} +$$
$$2\lambda\, b_1\, b_2\, f_{22}\, c_{0,2}\, c_{1,1} + \lambda\, b_2\, f_{11}\, c_{1,0}\, c_{1,1} + \lambda\, b_1\, f_{12}\, c_{1,0}\, c_{1,1} + \frac{1}{2}\lambda\, b_2^2\, f_{11}\, c_{1,1}^2 +$$
$$\lambda\, b_1\, b_2\, f_{12}\, c_{1,1}^2 + \frac{1}{2}\lambda\, b_1^2\, f_{22}\, c_{1,1}^2 + \lambda f_{11}\, c_{2,0} + 2\lambda\, b_1\, f_{12}\, c_{0,1}\, c_{2,0} + 4\lambda\, b_1\, b_2\, f_{12}\, c_{0,2}\, c_{2,0} +$$
$$\left.2\lambda\, b_1\, f_{11}\, c_{1,0}\, c_{2,0} + 2\lambda\, b_1\, b_2\, f_{11}\, c_{1,1}\, c_{2,0} + 2\lambda\, b_1^2\, f_{12}\, c_{1,1}\, c_{2,0} + 2\lambda\, b_1^2\, f_{11}\, c_{2,0}^2\right)\hbar +$$
$$\left(\lambda^2 f_{22}^2\, c_{0,1}^2\, c_{0,2} + \lambda^2 f_{22}^2\, c_{0,2}^2 + 4\lambda^2 b_2\, f_{22}^2\, c_{0,1}\, c_{0,2}^2 + 4\lambda^2 b_2^2\, f_{22}^2\, c_{0,2}^3 + 2\lambda^2 f_{12}\, f_{22}\, c_{0,1}\, c_{0,2}\, c_{1,0} +\right.$$
$$4\lambda^2 b_2\, f_{12}\, f_{22}\, c_{0,2}^2\, c_{1,0} + \lambda^2 f_{12}^2\, c_{0,2}\, c_{1,0}^2 + \lambda^2 f_{12}\, f_{22}\, c_{0,1}^2\, c_{1,1} + 2\lambda^2 f_{12}\, f_{22}\, c_{0,2}\, c_{1,1} +$$
$$6\lambda^2 b_2\, f_{12}\, f_{22}\, c_{0,1}\, c_{0,2}\, c_{1,1} + 2\lambda^2 b_1\, f_{22}^2\, c_{0,1}\, c_{0,2}\, c_{1,1} + 8\lambda^2 b_2^2\, f_{12}\, f_{22}\, c_{0,2}^2\, c_{1,1} + 4\lambda^2 b_1\, b_2\, f_{22}^2\, c_{0,2}^2\, c_{1,1} +$$
$$\lambda^2 f_{12}^2\, c_{0,1}\, c_{1,0}\, c_{1,1} + \lambda^2 f_{11}\, f_{22}\, c_{0,1}\, c_{1,0}\, c_{1,1} + 4\lambda^2 b_2\, f_{12}^2\, c_{0,2}\, c_{1,0}\, c_{1,1} + 2\lambda^2 b_2\, f_{11}\, f_{22}\, c_{0,2}\, c_{1,0}\, c_{1,1} +$$
$$2\lambda^2 b_1\, f_{12}\, f_{22}\, c_{0,2}\, c_{1,0}\, c_{1,1} + \lambda^2 f_{11}\, f_{12}\, c_{1,0}^2\, c_{1,1} + \frac{1}{2}\lambda^2 f_{12}^2\, c_{1,1}^2 + \frac{1}{2}\lambda^2 f_{11}\, f_{22}\, c_{1,1}^2 + \lambda^2 b_2\, f_{12}^2\, c_{0,1}\, c_{1,1}^2 +$$
$$\lambda^2 b_2\, f_{11}\, f_{22}\, c_{0,1}\, c_{1,1}^2 + 2\lambda^2 b_1\, f_{12}\, f_{22}\, c_{0,1}\, c_{1,1}^2 + 3\lambda^2 b_2^2\, f_{12}^2\, c_{0,2}\, c_{1,1}^2 + 2\lambda^2 b_2^2\, f_{11}\, f_{22}\, c_{0,2}\, c_{1,1}^2 +$$
$$6\lambda^2 b_1\, b_2\, f_{12}\, f_{22}\, c_{0,2}\, c_{1,1}^2 + \lambda^2 b_1^2\, f_{22}^2\, c_{0,2}\, c_{1,1}^2 + 2\lambda^2 b_2\, f_{11}\, f_{12}\, c_{1,0}\, c_{1,1}^2 + \lambda^2 b_1\, f_{12}^2\, c_{1,0}\, c_{1,1}^2 +$$
$$\lambda^2 b_1\, f_{11}\, f_{22}\, c_{1,0}\, c_{1,1}^2 + \lambda^2 b_2^2\, f_{11}\, f_{12}\, c_{1,1}^3 + \lambda^2 b_1\, b_2\, f_{12}^2\, c_{1,1}^3 + \lambda^2 b_1\, b_2\, f_{11}\, f_{22}\, c_{1,1}^3 + \lambda^2 b_1^2\, f_{12}\, f_{22}\, c_{1,1}^3 +$$
$$\lambda^2 f_{12}^2\, c_{0,1}^2\, c_{2,0} + 2\lambda^2 f_{12}^2\, c_{0,2}\, c_{2,0} + 4\lambda^2 b_2\, f_{12}^2\, c_{0,1}\, c_{0,2}\, c_{2,0} + 4\lambda^2 b_1\, f_{12}\, f_{22}\, c_{0,1}\, c_{0,2}\, c_{2,0} +$$
$$4\lambda^2 b_2^2\, f_{12}^2\, c_{0,2}^2\, c_{2,0} + 8\lambda^2 b_1\, b_2\, f_{12}\, f_{22}\, c_{0,2}^2\, c_{2,0} + 2\lambda^2 f_{11}\, f_{12}\, c_{0,1}\, c_{1,0}\, c_{2,0} +$$
$$4\lambda^2 b_2\, f_{11}\, f_{12}\, c_{0,2}\, c_{1,0}\, c_{2,0} + 4\lambda^2 b_1\, f_{12}^2\, c_{0,2}\, c_{1,0}\, c_{2,0} + \lambda^2 f_{11}^2\, c_{1,0}^2\, c_{2,0} + 2\lambda^2 f_{11}\, f_{12}\, c_{1,1}\, c_{2,0} +$$
$$2\lambda^2 b_2\, f_{11}\, f_{12}\, c_{0,1}\, c_{1,1}\, c_{2,0} + 4\lambda^2 b_1\, f_{12}^2\, c_{0,1}\, c_{1,1}\, c_{2,0} + 2\lambda^2 b_1\, f_{11}\, f_{22}\, c_{0,1}\, c_{1,1}\, c_{2,0} +$$
$$4\lambda^2 b_2^2\, f_{11}\, f_{12}\, c_{0,2}\, c_{1,1}\, c_{2,0} + 12\lambda^2 b_1\, b_2\, f_{12}^2\, c_{0,2}\, c_{1,1}\, c_{2,0} + 4\lambda^2 b_1\, b_2\, f_{11}\, f_{22}\, c_{0,2}\, c_{1,1}\, c_{2,0} +$$
$$4\lambda^2 b_1^2\, f_{12}\, f_{22}\, c_{0,2}\, c_{1,1}\, c_{2,0} + 2\lambda^2 b_2\, f_{11}^2\, c_{1,0}\, c_{1,1}\, c_{2,0} + 6\lambda^2 b_1\, f_{11}\, f_{12}\, c_{1,0}\, c_{1,1}\, c_{2,0} +$$
$$\lambda^2 b_2^2\, f_{11}^2\, c_{1,1}^2\, c_{2,0} + 6\lambda^2 b_1\, b_2\, f_{11}\, f_{12}\, c_{1,1}^2\, c_{2,0} + 3\lambda^2 b_1^2\, f_{12}^2\, c_{1,1}^2\, c_{2,0} + 2\lambda^2 b_1^2\, f_{11}\, f_{22}\, c_{1,1}^2\, c_{2,0} +$$
$$\lambda^2 f_{11}^2\, c_{2,0}^2 + 4\lambda^2 b_1\, f_{11}\, f_{12}\, c_{0,1}\, c_{2,0}^2 + 8\lambda^2 b_1\, b_2\, f_{11}\, f_{12}\, c_{0,2}\, c_{2,0}^2 + 4\lambda^2 b_1^2\, f_{12}^2\, c_{0,2}\, c_{2,0}^2 +$$
$$\left.4\lambda^2 b_1\, f_{11}^2\, c_{1,0}\, c_{2,0}^2 + 4\lambda^2 b_1\, b_2\, f_{11}^2\, c_{1,1}\, c_{2,0}^2 + 8\lambda^2 b_1^2\, f_{11}\, f_{12}\, c_{1,1}\, c_{2,0}^2 + 4\lambda^2 b_1^2\, f_{11}^2\, c_{2,0}^3\right)\hbar^2 + O[\hbar]^3$$

In[∘]:= $Z\ /.\ \lambda \to 0$

Out[∘]= $\left(c_{0,0} + b_2\, c_{0,1} + b_2^2\, c_{0,2} + b_1\, c_{1,0} + b_1\, b_2\, c_{1,1} + b_1^2\, c_{2,0}\right) + O[\hbar]^3$

In[∘]:= $(Z\ /.\ \lambda \to 0) - P$

Out[∘]= $O[\hbar]^3$

In[∘]:= **lhs = ∂$_\lambda$Z**

Out[∘]= $\left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + 2 b_2^2 f_{22} c_{0,2}^2 + f_{12} c_{0,1} c_{1,0} + 2 b_2 f_{12} c_{0,2} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + b_2 f_{12} c_{0,1} c_{1,1} + b_1 f_{22} c_{0,1} c_{1,1} + 2 b_2^2 f_{12} c_{0,2} c_{1,1} + 2 b_1 b_2 f_{22} c_{0,2} c_{1,1} + b_2 f_{11} c_{1,0} c_{1,1} + b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2} b_2^2 f_{11} c_{1,1}^2 + b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2} b_1^2 f_{22} c_{1,1}^2 + f_{11} c_{2,0} + 2 b_1 f_{12} c_{0,1} c_{2,0} + 4 b_1 b_2 f_{12} c_{0,2} c_{2,0} + 2 b_1 f_{11} c_{1,0} c_{2,0} + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2\right) \hbar +$

$\left(2 \lambda f_{22}^2 c_{0,1}^2 c_{0,2} + 2 \lambda f_{22}^2 c_{0,2}^2 + 8 \lambda b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 8 \lambda b_2^2 f_{22}^2 c_{0,2}^3 + 4 \lambda f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + 8 \lambda b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + 2 \lambda f_{12}^2 c_{0,2} c_{1,0}^2 + 2 \lambda f_{12} f_{22} c_{0,1}^2 c_{1,1} + 4 \lambda f_{12} f_{22} c_{0,2} c_{1,1} + 12 \lambda b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 4 \lambda b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 16 \lambda b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + 8 \lambda b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} + 2 \lambda f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 8 \lambda b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{12} c_{1,0}^2 c_{1,1} + \lambda f_{12}^2 c_{1,1}^2 + \lambda f_{11} f_{22} c_{1,1}^2 + 2 \lambda b_2 f_{12}^2 c_{0,1} c_{1,1}^2 + 2 \lambda b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 4 \lambda b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + 6 \lambda b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 + 12 \lambda b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + 2 \lambda b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{12}^2 c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + 2 \lambda b_2^2 f_{11} f_{12} c_{1,1}^3 + 2 \lambda b_1 b_2 f_{12}^2 c_{1,1}^3 + 2 \lambda b_1 b_2 f_{11} f_{22} c_{1,1}^3 + 2 \lambda b_1^2 f_{12} f_{22} c_{1,1}^3 + 2 \lambda f_{12}^2 c_{0,1}^2 c_{2,0} + 4 \lambda f_{12}^2 c_{0,2} c_{2,0} + 8 \lambda b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 16 \lambda b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + 4 \lambda f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + 8 \lambda b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + 2 \lambda f_{11}^2 c_{1,0}^2 c_{2,0} + 4 \lambda f_{11} f_{12} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 4 \lambda b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 24 \lambda b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 12 \lambda b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} + 2 \lambda b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 12 \lambda b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 6 \lambda b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 4 \lambda b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} + 2 \lambda f_{11}^2 c_{2,0}^2 + 8 \lambda b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 16 \lambda b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 8 \lambda b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 + 8 \lambda b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 8 \lambda b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 16 \lambda b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 8 \lambda b_1^2 f_{11}^2 c_{2,0}^3\right) \hbar^2 + O[\hbar]^3$

In[∘]:= **Short[rhs = Expand@Sum[(∂$_{b1,b2}$(B.F.B)) (∂$_{b1,b2}$Z + (∂$_{b1}$Z) (∂$_{b2}$Z)) / 4, {b1, B}, {b2, B}]]**

Out[∘]//Short= $\left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + \ll 18 \gg + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2\right) \hbar + (\ll 1 \gg) \ll 1 \gg + \ll 1 \gg + O[\hbar]^4$

In[∘]:= **HL[Normal[lhs − rhs] == 0]**

Out[∘]= True

In[∘]:= **Z /. λ → 1 /. Alternatives @@ B → 0**

Out[∘]= $c_{0,0} + \left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + f_{12} c_{0,1} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar +$

$\left(f_{22}^2 c_{0,1}^2 c_{0,2} + f_{22}^2 c_{0,2}^2 + 2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + f_{12}^2 c_{0,2} c_{1,0}^2 + f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 f_{12} f_{22} c_{0,2} c_{1,1} + f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} f_{12}^2 c_{1,1}^2 + \frac{1}{2} f_{11} f_{22} c_{1,1}^2 + f_{12}^2 c_{0,1}^2 c_{2,0} + 2 f_{12}^2 c_{0,2} c_{2,0} + 2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + f_{11}^2 c_{1,0}^2 c_{2,0} + 2 f_{11} f_{12} c_{1,1} c_{2,0} + f_{11}^2 c_{2,0}^2\right) \hbar^2 + O[\hbar]^3$