

Pensieve header: Testing lemmas 1,2,3 of the DoPeGDO handouts. Continued pensieve://2019-11/.

$[F : \mathcal{E}]_B := \mathbb{e}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} \partial_{z_i} \partial_{z_j}} \mathcal{E}$ and $\langle F : \mathcal{E} \rangle_B := [F : \mathcal{E}]_B|_{z_B \rightarrow 0}$, where \mathcal{E} is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where \mathcal{E} has no B - B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} \mathbb{e}^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B.$$

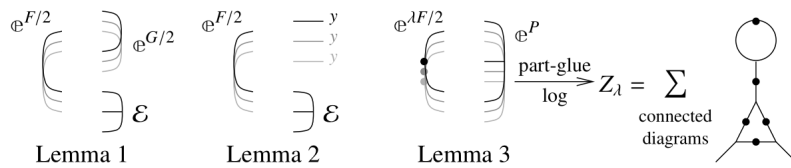
The next lemma dispatches the case where \mathcal{E} has a B -linear part:

Lemma 2. $\left\langle F : \mathcal{E} \mathbb{e}^{\sum_{i \in B} y_i z_i} \right\rangle_B = \mathbb{e}^{\frac{1}{2} \sum_{i,j \in B} F_{ij} y_i y_j} \left\langle F : \mathcal{E} \right\rangle_{z_B \rightarrow z_B + F y_B}$.

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$



Goals:

Implement the containers $|F, \mathcal{E}|_B := [F : \mathcal{E}]_B$ and $\langle F, \mathcal{E} \rangle_B := \langle F : \mathcal{E} \rangle_B$, their evaluator Ev_k as power series in \hbar to degree k , and verify lemmas 1, 2, and 3. Inserting \hbar in the appropriate places is user responsibility.

Expanding power series:

```
In[*]:= Unprotect[Expand]; Expand[sd_SeriesData] := MapAt[Expand, sd, 3];
Protect[Expand];
```

Generic Polynomials:

```
In[*]:= GenericPolynomial[d_Integer, vars_List, gc_] := Total[Map[
  gc Sequence @@ Times @@ (vars^#) &,
  Join @@ (Permutations /@ IntegerPartitions[d + Length@vars, {Length@vars}]) - 1
]];
GenericPolynomial[specs_List, vars_List, gc_] :=
  Sum[GenericPolynomial[specs[[1]], vars, gc], specs]
```

Variables and their duals:

```
In[*]:= {t*, b*, y*, a*, x*, z*} = {tau, beta, eta, alpha, xi, zeta};
{tau*, beta*, eta*, alpha*, xi*, zeta*} = {t, b, y, a, x, z}; (u_{-i})^* := (u^*)_i; (B_List)^* := #* & /@ B;
```

Act and Contract:

```
In[*]:= EV_k_@|F_ , ε_ |B_ := Expand[Total[
  CoefficientRules[Normal@Series[e^{B*.F.B*/2}, {ħ, 0, k}], B*] /.
  (ps_ -> c_) => c D[ε, Sequence@@Thread[{B, ps}]]
] + O[ħ]^{k+1};
EV_k_<F_ , ε_>B_ := EV_k_@|F, ε|B /. Alternatives@@B -> 0
```

```
In[*]:= EV_2@|ħ {{1}}, e^{xx}|_{x}
```

$$Out[*]:= e^{x^2} + (e^{x^2} + 2 e^{x^2} x^2) \hbar + \left(\frac{3 e^{x^2}}{2} + 6 e^{x^2} x^2 + 2 e^{x^2} x^4 \right) \hbar^2 + O[\hbar]^3$$

```
In[*]:= Log@EV_10@|ħ {{1}}, e^{xx}|_{x}
```

$$Out[*]:= \hbar + \hbar^2 + \frac{4 \hbar^3}{3} + 2 \hbar^4 + \frac{16 \hbar^5}{5} + \frac{16 \hbar^6}{3} + \frac{64 \hbar^7}{7} + 16 \hbar^8 + \frac{256 \hbar^9}{9} + \frac{256 \hbar^{10}}{5} + O[\hbar]^{11}$$

```
In[*]:= EV_2@|ħ ( 0 1 / 1 0 ), e^{xy}|_{x,y}
```

$$Out[*]:= e^{xy} + (e^{xy} + e^{xy} xy) \hbar + \left(e^{xy} + 2 e^{xy} xy + \frac{1}{2} e^{xy} x^2 y^2 \right) \hbar^2 + O[\hbar]^3$$

```
In[*]:= EV_3@|ħ ( 0 1 / 1 0 ), e^{3xy}|_{x,y}
```

$$Out[*]:= 1 + 3 \hbar + 9 \hbar^2 + 27 \hbar^3 + O[\hbar]^4$$

Testing Lemma 1.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E} e^{\frac{1}{2} \sum_{i,j \in B} G_{ij} z_i z_j} \right\rangle_B = \det(1 - GF)^{-1/2} \left\langle F(1 - GF)^{-1} : \mathcal{E} \right\rangle_B$$

```
In[*]:= {p = 10, B = {x}, I = IdentityMatrix@Length@B,
  F = ħ {{f}}, G = {{g}}, ε = GenericPolynomial[{d, 0, 4}, B, c]}
```

$$Out[*]:= \{10, \{x\}, \{\{1\}\}, \{\{f \hbar\}\}, \{\{g\}\}, c_0 + x c_1 + x^2 c_2 + x^3 c_3 + x^4 c_4\}$$

```
In[*]:= lhs = Ev_p@|F, ε e^{B.G.B/2}|_B
```

$$Out[*]:= c_0 + \left(\frac{1}{2} f g c_0 + f c_2 \right) \hbar + \left(\frac{3}{8} f^2 g^2 c_0 + \frac{3}{2} f^2 g c_2 + 3 f^2 c_4 \right) \hbar^2 +$$

$$\left(\frac{5}{16} f^3 g^3 c_0 + \frac{15}{8} f^3 g^2 c_2 + \frac{15}{2} f^3 g c_4 \right) \hbar^3 + \left(\frac{35}{128} f^4 g^4 c_0 + \frac{35}{16} f^4 g^3 c_2 + \frac{105}{8} f^4 g^2 c_4 \right) \hbar^4 +$$

$$\left(\frac{63}{256} f^5 g^5 c_0 + \frac{315}{128} f^5 g^4 c_2 + \frac{315}{16} f^5 g^3 c_4 \right) \hbar^5 + \left(\frac{231 f^6 g^6 c_0}{1024} + \frac{693}{256} f^6 g^5 c_2 + \frac{3465}{128} f^6 g^4 c_4 \right) \hbar^6 +$$

$$\left(\frac{429 f^7 g^7 c_0}{2048} + \frac{3003 f^7 g^6 c_2}{1024} + \frac{9009}{256} f^7 g^5 c_4 \right) \hbar^7 + \left(\frac{6435 f^8 g^8 c_0}{32768} + \frac{6435 f^8 g^7 c_2}{2048} + \frac{45045 f^8 g^6 c_4}{1024} \right) \hbar^8 +$$

$$\left(\frac{12155 f^9 g^9 c_0}{65536} + \frac{109395 f^9 g^8 c_2}{32768} + \frac{109395 f^9 g^7 c_4}{2048} \right) \hbar^9 +$$

$$\left(\frac{46189 f^{10} g^{10} c_0}{262144} + \frac{230945 f^{10} g^9 c_2}{65536} + \frac{2078505 f^{10} g^8 c_4}{32768} \right) \hbar^{10} + O[\hbar]^{11}$$

In[*]:= Timing[rhs = Evp@<F.Inverse[I - G.F], Det[I - G.F]^{-1/2} ε>_B]

$$\text{Out[*]} = \left\{ 0.015625, c_0 + \left(\frac{1}{2} f g c_0 + f c_2 \right) \hbar + \left(\frac{3}{8} f^2 g^2 c_0 + \frac{3}{2} f^2 g c_2 + 3 f^2 c_4 \right) \hbar^2 + \right. \\ \left(\frac{5}{16} f^3 g^3 c_0 + \frac{15}{8} f^3 g^2 c_2 + \frac{15}{2} f^3 g c_4 \right) \hbar^3 + \left(\frac{35}{128} f^4 g^4 c_0 + \frac{35}{16} f^4 g^3 c_2 + \frac{105}{8} f^4 g^2 c_4 \right) \hbar^4 + \\ \left(\frac{63}{256} f^5 g^5 c_0 + \frac{315}{128} f^5 g^4 c_2 + \frac{315}{16} f^5 g^3 c_4 \right) \hbar^5 + \left(\frac{231 f^6 g^6 c_0}{1024} + \frac{693}{256} f^6 g^5 c_2 + \frac{3465}{128} f^6 g^4 c_4 \right) \hbar^6 + \\ \left(\frac{429 f^7 g^7 c_0}{2048} + \frac{3003 f^7 g^6 c_2}{1024} + \frac{9009}{256} f^7 g^5 c_4 \right) \hbar^7 + \left(\frac{6435 f^8 g^8 c_0}{32768} + \frac{6435 f^8 g^7 c_2}{2048} + \frac{45045 f^8 g^6 c_4}{1024} \right) \hbar^8 + \\ \left(\frac{12155 f^9 g^9 c_0}{65536} + \frac{109395 f^9 g^8 c_2}{32768} + \frac{109395 f^9 g^7 c_4}{2048} \right) \hbar^9 + \\ \left. \left(\frac{46189 f^{10} g^{10} c_0}{262144} + \frac{230945 f^{10} g^9 c_2}{65536} + \frac{2078505 f^{10} g^8 c_4}{32768} \right) \hbar^{10} + O[\hbar]^{11} \right\}$$

In[*]:= lhs == rhs

Out[*]:= True

In[*]:= {p = 2, B = {x, y}, I = IdentityMatrix@Length@B, \\ F = \hbar \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}, G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}, \epsilon = GenericPolynomial[{d, \theta, 2}, B, c]}

Out[*]:= {2, {x, y}, {{1, \theta}, {\theta, 1}}, {{\hbar f_{11}, \hbar f_{12}}, {\hbar f_{12}, \hbar f_{22}}}, \\ {{g_{11}, g_{12}}, {g_{12}, g_{22}}}, c_{0,0} + y c_{0,1} + y^2 c_{0,2} + x c_{1,0} + x y c_{1,1} + x^2 c_{2,0}}

In[*]:= Timing[lhs = Evp@<F, ε e^{B.G.B/2}>_B]

$$\text{Out[*]} = \left\{ 0.0625, c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0} \right) \hbar + \right. \\ \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \right. \\ \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + \\ 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \\ \left. \left. \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0} \right) \hbar^2 + O[\hbar]^3 \right\}$$

In[*]:= Timing[rhs = Evp@<F.Inverse[I - G.F], Det[I - G.F]^{-1/2} ε>_B]

$$\text{Out[*]} = \left\{ 0.03125, c_{0,0} + \left(\frac{1}{2} f_{11} g_{11} c_{0,0} + f_{12} g_{12} c_{0,0} + \frac{1}{2} f_{22} g_{22} c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0} \right) \hbar + \right. \\ \left(\frac{3}{8} f_{11}^2 g_{11}^2 c_{0,0} + \frac{3}{2} f_{11} f_{12} g_{11} g_{12} c_{0,0} + f_{12}^2 g_{12}^2 c_{0,0} + \frac{1}{2} f_{11} f_{22} g_{12}^2 c_{0,0} + \frac{1}{2} f_{12}^2 g_{11} g_{22} c_{0,0} + \right. \\ \frac{1}{4} f_{11} f_{22} g_{11} g_{22} c_{0,0} + \frac{3}{2} f_{12} f_{22} g_{12} g_{22} c_{0,0} + \frac{3}{8} f_{22}^2 g_{22}^2 c_{0,0} + f_{12}^2 g_{11} c_{0,2} + \frac{1}{2} f_{11} f_{22} g_{11} c_{0,2} + \\ 3 f_{12} f_{22} g_{12} c_{0,2} + \frac{3}{2} f_{22}^2 g_{22} c_{0,2} + \frac{3}{2} f_{11} f_{12} g_{11} c_{1,1} + 2 f_{12}^2 g_{12} c_{1,1} + f_{11} f_{22} g_{12} c_{1,1} + \\ \left. \left. \frac{3}{2} f_{12} f_{22} g_{22} c_{1,1} + \frac{3}{2} f_{11}^2 g_{11} c_{2,0} + 3 f_{11} f_{12} g_{12} c_{2,0} + f_{12}^2 g_{22} c_{2,0} + \frac{1}{2} f_{11} f_{22} g_{22} c_{2,0} \right) \hbar^2 + O[\hbar]^3 \right\}$$

In[]:= lhs == rhs

Out[]:= True

Testing Lemma 2.

Lemma 2. $\langle F: \mathcal{E}^{\oplus \sum_{i \in B} Y_i z_i} \rangle_B = \mathbb{C}^{\frac{1}{2} \sum_{i, j \in B} F_{ij} Y_i Y_j} \langle F: \mathcal{E}|_{z_B \rightarrow z_B + F Y_B} \rangle_B$.

In[]:= {n = 2, p = 2, B = Table[z_i, {i, n}], I = IdentityMatrix@Length@B, Y = Table[y_i, {i, n}],
F = ħ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ε = GenericPolynomial[{d, 0, 3}, B, c]}

Out[]:= {2, 2, {z_1, z_2}, {{1, 0}, {0, 1}}, {y_1, y_2}, {{ħ f_11, ħ f_12}, {ħ f_12, ħ f_22}},
c_{0,0} + z_2 c_{0,1} + z_2^2 c_{0,2} + z_2^3 c_{0,3} + z_1 c_{1,0} + z_1 z_2 c_{1,1} + z_1 z_2^2 c_{1,2} + z_1^2 c_{2,0} + z_1^2 z_2 c_{2,1} + z_1^3 c_{3,0}}

In[]:= Timing[lhs = Ev_p@⟨F, ε e^{Y.B}⟩_B]

Out[]:= {0.015625,

$$\begin{aligned}
& c_{0,0} + \left(\frac{1}{2} f_{11} y_1^2 c_{0,0} + f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} f_{22} y_2^2 c_{0,0} + f_{12} y_1 c_{0,1} + f_{22} y_2 c_{0,1} + f_{22} c_{0,2} + f_{11} y_1 c_{1,0} + \right. \\
& \left. f_{12} y_2 c_{1,0} + f_{12} c_{1,1} + f_{11} c_{2,0} \right) \hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 c_{0,0} + \right. \\
& \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 c_{0,1} + f_{12}^2 y_1^2 y_2 c_{0,1} + \\
& \frac{1}{2} f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} f_{22}^2 y_2^3 c_{0,1} + f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} f_{11} f_{22} y_1^2 c_{0,2} + \\
& 3 f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} f_{22}^2 y_2^2 c_{0,2} + 3 f_{12} f_{22} y_1 c_{0,3} + 3 f_{22}^2 y_2 c_{0,3} + \frac{1}{2} f_{11}^2 y_1^3 c_{1,0} + \\
& \frac{3}{2} f_{11} f_{12} y_1^2 y_2 c_{1,0} + f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 c_{1,1} + \\
& 2 f_{12}^2 y_1 y_2 c_{1,1} + f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} f_{12} f_{22} y_2^2 c_{1,1} + 2 f_{12}^2 y_1 c_{1,2} + f_{11} f_{22} y_1 c_{1,2} + \\
& 3 f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2} f_{11}^2 y_1^2 c_{2,0} + 3 f_{11} f_{12} y_1 y_2 c_{2,0} + f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} f_{11} f_{22} y_2^2 c_{2,0} + \\
& \left. 3 f_{11} f_{12} y_1 c_{2,1} + 2 f_{12}^2 y_2 c_{2,1} + f_{11} f_{22} y_2 c_{2,1} + 3 f_{11}^2 y_1 c_{3,0} + 3 f_{11} f_{12} y_2 c_{3,0} \right) \hbar^2 + \mathcal{O}[\hbar^3]
\end{aligned}$$

In[]:= Timing[rhs = Expand[Series[e^{Y.F.Y/2} Ev_p@<F, ε / . Thread[B → B + F.Y]>_B, {ħ, 0, p}]]]

$$\begin{aligned}
 \text{Out[]} = & \left\{ 0., c_{0,0} + \left(\frac{1}{2} f_{11} y_1^2 c_{0,0} + f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} f_{22} y_2^2 c_{0,0} + \right. \right. \\
 & f_{12} y_1 c_{0,1} + f_{22} y_2 c_{0,1} + f_{22} c_{0,2} + f_{11} y_1 c_{1,0} + f_{12} y_2 c_{1,0} + f_{12} c_{1,1} + f_{11} c_{2,0} \Big) \hbar + \\
 & \left(\frac{1}{8} f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 c_{0,0} + \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 c_{0,0} + \right. \\
 & \frac{1}{8} f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 c_{0,1} + f_{12}^2 y_1^2 y_2 c_{0,1} + \frac{1}{2} f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} f_{12} f_{22} y_1 y_2^2 c_{0,1} + \\
 & \frac{1}{2} f_{22}^2 y_2^3 c_{0,1} + f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} f_{11} f_{22} y_1^2 c_{0,2} + 3 f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} f_{22}^2 y_2^2 c_{0,2} + \\
 & 3 f_{12} f_{22} y_1 c_{0,3} + 3 f_{22}^2 y_2 c_{0,3} + \frac{1}{2} f_{11}^2 y_1^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 y_2 c_{1,0} + f_{12}^2 y_1 y_2^2 c_{1,0} + \\
 & \frac{1}{2} f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 c_{1,1} + 2 f_{12}^2 y_1 y_2 c_{1,1} + \\
 & f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} f_{12} f_{22} y_2^2 c_{1,1} + 2 f_{12}^2 y_1 c_{1,2} + f_{11} f_{22} y_1 c_{1,2} + 3 f_{12} f_{22} y_2 c_{1,2} + \\
 & \frac{3}{2} f_{11}^2 y_1^2 c_{2,0} + 3 f_{11} f_{12} y_1 y_2 c_{2,0} + f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} f_{11} f_{22} y_2^2 c_{2,0} + 3 f_{11} f_{12} y_1 c_{2,1} + \\
 & \left. \left. 2 f_{12}^2 y_2 c_{2,1} + f_{11} f_{22} y_2 c_{2,1} + 3 f_{11}^2 y_1 c_{3,0} + 3 f_{11} f_{12} y_2 c_{3,0} \right) \hbar^2 + O[\hbar^3] \right\}
 \end{aligned}$$

In[]:= Timing[rhs = Ev_p@<F, e^{Y.F.Y/2} ε / . Thread[B → B + F.Y]>_B]

$$\begin{aligned}
 \text{Out[]} = & \left\{ 0.09375, \right. \\
 & c_{0,0} + \left(\frac{1}{2} f_{11} y_1^2 c_{0,0} + f_{12} y_1 y_2 c_{0,0} + \frac{1}{2} f_{22} y_2^2 c_{0,0} + f_{12} y_1 c_{0,1} + f_{22} y_2 c_{0,1} + f_{22} c_{0,2} + f_{11} y_1 c_{1,0} + \right. \\
 & f_{12} y_2 c_{1,0} + f_{12} c_{1,1} + f_{11} c_{2,0} \Big) \hbar + \left(\frac{1}{8} f_{11}^2 y_1^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 y_2 c_{0,0} + \frac{1}{2} f_{12}^2 y_1^2 y_2^2 c_{0,0} + \right. \\
 & \frac{1}{4} f_{11} f_{22} y_1^2 y_2^2 c_{0,0} + \frac{1}{2} f_{12} f_{22} y_1 y_2^3 c_{0,0} + \frac{1}{8} f_{22}^2 y_2^4 c_{0,0} + \frac{1}{2} f_{11} f_{12} y_1^3 c_{0,1} + f_{12}^2 y_1^2 y_2 c_{0,1} + \\
 & \frac{1}{2} f_{11} f_{22} y_1^2 y_2 c_{0,1} + \frac{3}{2} f_{12} f_{22} y_1 y_2^2 c_{0,1} + \frac{1}{2} f_{22}^2 y_2^3 c_{0,1} + f_{12}^2 y_1^2 c_{0,2} + \frac{1}{2} f_{11} f_{22} y_1^2 c_{0,2} + \\
 & 3 f_{12} f_{22} y_1 y_2 c_{0,2} + \frac{3}{2} f_{22}^2 y_2^2 c_{0,2} + 3 f_{12} f_{22} y_1 c_{0,3} + 3 f_{22}^2 y_2 c_{0,3} + \frac{1}{2} f_{11}^2 y_1^3 c_{1,0} + \\
 & \frac{3}{2} f_{11} f_{12} y_1^2 y_2 c_{1,0} + f_{12}^2 y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{11} f_{22} y_1 y_2^2 c_{1,0} + \frac{1}{2} f_{12} f_{22} y_2^3 c_{1,0} + \frac{3}{2} f_{11} f_{12} y_1^2 c_{1,1} + \\
 & 2 f_{12}^2 y_1 y_2 c_{1,1} + f_{11} f_{22} y_1 y_2 c_{1,1} + \frac{3}{2} f_{12} f_{22} y_2^2 c_{1,1} + 2 f_{12}^2 y_1 c_{1,2} + f_{11} f_{22} y_1 c_{1,2} + \\
 & 3 f_{12} f_{22} y_2 c_{1,2} + \frac{3}{2} f_{11}^2 y_1^2 c_{2,0} + 3 f_{11} f_{12} y_1 y_2 c_{2,0} + f_{12}^2 y_2^2 c_{2,0} + \frac{1}{2} f_{11} f_{22} y_2^2 c_{2,0} + \\
 & \left. \left. 3 f_{11} f_{12} y_1 c_{2,1} + 2 f_{12}^2 y_2 c_{2,1} + f_{11} f_{22} y_2 c_{2,1} + 3 f_{11}^2 y_1 c_{3,0} + 3 f_{11} f_{12} y_2 c_{3,0} \right) \hbar^2 + O[\hbar^3] \right\}
 \end{aligned}$$

In[]:= lhs == rhs

Out[]:= True

Testing Lemma 3.

Lemma 3. With an extra variable λ , $Z_\lambda := \log[\lambda F : \mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad \text{and} \quad \partial_\lambda Z_\lambda = \frac{1}{2} \sum_{i,j \in B} F_{ij} \left(\partial_{z_i} \partial_{z_j} Z_\lambda + (\partial_{z_i} Z_\lambda)(\partial_{z_j} Z_\lambda) \right).$$

In[]:= {n = 2, p = 2, B = Table[b_i, {i, n}],

F = ħ Table[f_{i0,1}.Sort[{i,j}], {i, n}, {j, n}], P = GenericPolynomial[{d, 0, 2}, B, c]}

Out[]:= {2, 2, {b_1, b_2}, {{ħ f_11, ħ f_12}, {ħ f_12, ħ f_22}}, c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}}

In[]:= Z = PowerExpand@Expand@Log[Evp@{λ F, e^P}|_B]

Out[]:= (c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}) +

$$\left(\frac{1}{2} \lambda f_{22} c_{0,1}^2 + \lambda f_{22} c_{0,2} + 2 \lambda b_2 f_{22} c_{0,1} c_{0,2} + 2 \lambda b_2^2 f_{22} c_{0,2}^2 + \lambda f_{12} c_{0,1} c_{1,0} + 2 \lambda b_2 f_{12} c_{0,2} c_{1,0} + \right.$$

$$\left. \frac{1}{2} \lambda f_{11} c_{1,0}^2 + \lambda f_{12} c_{1,1} + \lambda b_2 f_{12} c_{0,1} c_{1,1} + \lambda b_1 f_{22} c_{0,1} c_{1,1} + 2 \lambda b_2^2 f_{12} c_{0,2} c_{1,1} + \right.$$

$$\left. 2 \lambda b_1 b_2 f_{22} c_{0,2} c_{1,1} + \lambda b_2 f_{11} c_{1,0} c_{1,1} + \lambda b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2} \lambda b_2^2 f_{11} c_{1,1}^2 + \right.$$

$$\left. \lambda b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2} \lambda b_1^2 f_{22} c_{1,1}^2 + \lambda f_{11} c_{2,0} + 2 \lambda b_1 f_{12} c_{0,1} c_{2,0} + 4 \lambda b_1 b_2 f_{12} c_{0,2} c_{2,0} + \right.$$

$$\left. 2 \lambda b_1 f_{11} c_{1,0} c_{2,0} + 2 \lambda b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 \lambda b_1^2 f_{12} c_{1,1} c_{2,0} + 2 \lambda b_1^2 f_{11} c_{2,0}^2 \right) \hbar +$$

$$\left(\lambda^2 f_{22}^2 c_{0,1}^2 c_{0,2} + \lambda^2 f_{22}^2 c_{0,2}^2 + 4 \lambda^2 b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 4 \lambda^2 b_2^2 f_{22}^2 c_{0,2}^3 + 2 \lambda^2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + \right.$$

$$\left. 4 \lambda^2 b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + \lambda^2 f_{12}^2 c_{0,2} c_{1,0}^2 + \lambda^2 f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 \lambda^2 f_{12} f_{22} c_{0,2} c_{1,1} + \right.$$

$$\left. 6 \lambda^2 b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 2 \lambda^2 b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 8 \lambda^2 b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + 4 \lambda^2 b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} + \right.$$

$$\left. \lambda^2 f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + \lambda^2 f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 4 \lambda^2 b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + 2 \lambda^2 b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} + \right.$$

$$\left. 2 \lambda^2 b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + \lambda^2 f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} \lambda^2 f_{12}^2 c_{1,1}^2 + \frac{1}{2} \lambda^2 f_{11} f_{22} c_{1,1}^2 + \lambda^2 b_2 f_{12}^2 c_{0,1} c_{1,1}^2 + \right.$$

$$\left. \lambda^2 b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 2 \lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + 3 \lambda^2 b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 2 \lambda^2 b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 + \right.$$

$$\left. 6 \lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + \lambda^2 b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + 2 \lambda^2 b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + \lambda^2 b_1 f_{12}^2 c_{1,0} c_{1,1}^2 + \right.$$

$$\left. \lambda^2 b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + \lambda^2 b_2^2 f_{11} f_{12} c_{1,1}^3 + \lambda^2 b_1 b_2 f_{12}^2 c_{1,1}^3 + \lambda^2 b_1 b_2 f_{11} f_{22} c_{1,1}^3 + \lambda^2 b_1^2 f_{12} f_{22} c_{1,1}^3 + \right.$$

$$\left. \lambda^2 f_{12}^2 c_{0,1}^2 c_{2,0} + 2 \lambda^2 f_{12}^2 c_{0,2} c_{2,0} + 4 \lambda^2 b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 4 \lambda^2 b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} + \right.$$

$$\left. 4 \lambda^2 b_2^2 f_{12}^2 c_{0,2}^2 c_{2,0} + 8 \lambda^2 b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + 2 \lambda^2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + \right.$$

$$\left. 4 \lambda^2 b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 4 \lambda^2 b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + \lambda^2 f_{11}^2 c_{1,0}^2 c_{2,0} + 2 \lambda^2 f_{11} f_{12} c_{1,1} c_{2,0} + \right.$$

$$\left. 2 \lambda^2 b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 4 \lambda^2 b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 2 \lambda^2 b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} + \right.$$

$$\left. 4 \lambda^2 b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 12 \lambda^2 b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 4 \lambda^2 b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} + \right.$$

$$\left. 4 \lambda^2 b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 2 \lambda^2 b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 6 \lambda^2 b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} + \right.$$

$$\left. \lambda^2 b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 6 \lambda^2 b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 3 \lambda^2 b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 2 \lambda^2 b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} + \right.$$

$$\left. \lambda^2 f_{11}^2 c_{1,0}^2 c_{2,0} + 4 \lambda^2 b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 8 \lambda^2 b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 4 \lambda^2 b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 + \right.$$

$$\left. 4 \lambda^2 b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 4 \lambda^2 b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 8 \lambda^2 b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 4 \lambda^2 b_1^2 f_{11}^2 c_{2,0}^3 \right) \hbar^2 + O[\hbar]^3$$

In[]:= Z /. λ → 0

Out[]:= (c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}) + O[ħ]^3

In[*]:= $(Z /. \lambda \rightarrow \theta) - P$

Out[*]:= $O[\hbar]^3$

In[*]:= **lhs = $\partial_\lambda Z$**

Out[*]:=
$$\left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + 2 b_2^2 f_{22} c_{0,2}^2 + f_{12} c_{0,1} c_{1,0} + 2 b_2 f_{12} c_{0,2} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + b_2 f_{12} c_{0,1} c_{1,1} + b_1 f_{22} c_{0,1} c_{1,1} + 2 b_2^2 f_{12} c_{0,2} c_{1,1} + 2 b_1 b_2 f_{22} c_{0,2} c_{1,1} + b_2 f_{11} c_{1,0} c_{1,1} + b_1 f_{12} c_{1,0} c_{1,1} + \frac{1}{2} b_2^2 f_{11} c_{1,1}^2 + b_1 b_2 f_{12} c_{1,1}^2 + \frac{1}{2} b_1^2 f_{22} c_{1,1}^2 + f_{11} c_{2,0} + 2 b_1 f_{12} c_{0,1} c_{2,0} + 4 b_1 b_2 f_{12} c_{0,2} c_{2,0} + 2 b_1 f_{11} c_{1,0} c_{2,0} + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2 \right) \hbar +$$

$$\left(2 \lambda f_{22}^2 c_{0,1}^2 c_{0,2} + 2 \lambda f_{22}^2 c_{0,2}^2 + 8 \lambda b_2 f_{22}^2 c_{0,1} c_{0,2}^2 + 8 \lambda b_2^2 f_{22}^2 c_{0,2}^3 + 4 \lambda f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + 8 \lambda b_2 f_{12} f_{22} c_{0,2}^2 c_{1,0} + 2 \lambda f_{12}^2 c_{0,2} c_{1,0}^2 + 2 \lambda f_{12} f_{22} c_{0,1}^2 c_{1,1} + 4 \lambda f_{12} f_{22} c_{0,2} c_{1,1} + 12 \lambda b_2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,1} + 4 \lambda b_1 f_{22}^2 c_{0,1} c_{0,2} c_{1,1} + 16 \lambda b_2^2 f_{12} f_{22} c_{0,2}^2 c_{1,1} + 8 \lambda b_1 b_2 f_{22}^2 c_{0,2}^2 c_{1,1} + 2 \lambda f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + 8 \lambda b_2 f_{12}^2 c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_2 f_{11} f_{22} c_{0,2} c_{1,0} c_{1,1} + 4 \lambda b_1 f_{12} f_{22} c_{0,2} c_{1,0} c_{1,1} + 2 \lambda f_{11} f_{12} c_{1,0}^2 c_{1,1} + \lambda f_{12}^2 c_{1,1}^2 + \lambda f_{11} f_{22} c_{1,1}^2 + 2 \lambda b_2 f_{12}^2 c_{0,1} c_{1,1}^2 + 2 \lambda b_2 f_{11} f_{22} c_{0,1} c_{1,1}^2 + 4 \lambda b_1 f_{12} f_{22} c_{0,1} c_{1,1}^2 + 6 \lambda b_2^2 f_{12}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2^2 f_{11} f_{22} c_{0,2} c_{1,1}^2 + 12 \lambda b_1 b_2 f_{12} f_{22} c_{0,2} c_{1,1}^2 + 2 \lambda b_1^2 f_{22}^2 c_{0,2} c_{1,1}^2 + 4 \lambda b_2 f_{11} f_{12} c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{12}^2 c_{1,0} c_{1,1}^2 + 2 \lambda b_1 f_{11} f_{22} c_{1,0} c_{1,1}^2 + 2 \lambda b_2^2 f_{11} f_{12} c_{1,1}^3 + 2 \lambda b_1 b_2 f_{12}^2 c_{1,1}^3 + 2 \lambda b_1 b_2 f_{11} f_{22} c_{1,1}^3 + 2 \lambda b_1^2 f_{12} f_{22} c_{1,1}^3 + 2 \lambda f_{12}^2 c_{0,1} c_{2,0} + 4 \lambda f_{12}^2 c_{0,2} c_{2,0} + 8 \lambda b_2 f_{12}^2 c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_1 f_{12} f_{22} c_{0,1} c_{0,2} c_{2,0} + 8 \lambda b_2^2 f_{12}^2 c_{0,2} c_{2,0} + 16 \lambda b_1 b_2 f_{12} f_{22} c_{0,2}^2 c_{2,0} + 4 \lambda f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + 8 \lambda b_2 f_{11} f_{12} c_{0,2} c_{1,0} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,2} c_{1,0} c_{2,0} + 2 \lambda f_{11}^2 c_{1,0}^2 c_{2,0} + 4 \lambda f_{11} f_{12} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11} f_{12} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_1 f_{12}^2 c_{0,1} c_{1,1} c_{2,0} + 4 \lambda b_1 f_{11} f_{22} c_{0,1} c_{1,1} c_{2,0} + 8 \lambda b_2^2 f_{11} f_{12} c_{0,2} c_{1,1} c_{2,0} + 24 \lambda b_1 b_2 f_{12}^2 c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1 b_2 f_{11} f_{22} c_{0,2} c_{1,1} c_{2,0} + 8 \lambda b_1^2 f_{12} f_{22} c_{0,2} c_{1,1} c_{2,0} + 4 \lambda b_2 f_{11}^2 c_{1,0} c_{1,1} c_{2,0} + 12 \lambda b_1 f_{11} f_{12} c_{1,0} c_{1,1} c_{2,0} + 2 \lambda b_2^2 f_{11}^2 c_{1,1}^2 c_{2,0} + 12 \lambda b_1 b_2 f_{11} f_{12} c_{1,1}^2 c_{2,0} + 6 \lambda b_1^2 f_{12}^2 c_{1,1}^2 c_{2,0} + 4 \lambda b_1^2 f_{11} f_{22} c_{1,1}^2 c_{2,0} + 2 \lambda f_{11}^2 c_{2,0}^2 + 8 \lambda b_1 f_{11} f_{12} c_{0,1} c_{2,0}^2 + 16 \lambda b_1 b_2 f_{11} f_{12} c_{0,2} c_{2,0}^2 + 8 \lambda b_1^2 f_{12}^2 c_{0,2} c_{2,0}^2 + 8 \lambda b_1 f_{11}^2 c_{1,0} c_{2,0}^2 + 8 \lambda b_1 b_2 f_{11}^2 c_{1,1} c_{2,0}^2 + 16 \lambda b_1^2 f_{11} f_{12} c_{1,1} c_{2,0}^2 + 8 \lambda b_1^2 f_{11}^2 c_{2,0}^3 \right) \hbar^2 + O[\hbar]^3$$

In[*]:= **Short[rhs = Expand@Sum[($\partial_{b_1, b_2}(\mathbf{B} \cdot \mathbf{F} \cdot \mathbf{B})$) ($\partial_{b_1, b_2} Z$) + ($\partial_{b_1} Z$) ($\partial_{b_2} Z$)] / 4, {b1, B}, {b2, B}]**

Out[*]//Short=
$$\left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + 2 b_2 f_{22} c_{0,1} c_{0,2} + \ll 18 \gg + 2 b_1 b_2 f_{11} c_{1,1} c_{2,0} + 2 b_1^2 f_{12} c_{1,1} c_{2,0} + 2 b_1^2 f_{11} c_{2,0}^2 \right) \hbar + (\ll 1 \gg) \ll 1 \gg + \ll 1 \gg + O[\hbar]^4$$

In[*]:= **Expand[lhs - rhs]**

Out[*]:= $O[\hbar]^3$

In[*]:= **Z /. $\lambda \rightarrow 1$ /. Alternatives @@ B $\rightarrow \theta$**

Out[*]:=
$$c_{0,0} + \left(\frac{1}{2} f_{22} c_{0,1}^2 + f_{22} c_{0,2} + f_{12} c_{0,1} c_{1,0} + \frac{1}{2} f_{11} c_{1,0}^2 + f_{12} c_{1,1} + f_{11} c_{2,0} \right) \hbar +$$

$$\left(f_{22}^2 c_{0,1}^2 c_{0,2} + f_{22}^2 c_{0,2}^2 + 2 f_{12} f_{22} c_{0,1} c_{0,2} c_{1,0} + f_{12}^2 c_{0,2} c_{1,0}^2 + f_{12} f_{22} c_{0,1}^2 c_{1,1} + 2 f_{12} f_{22} c_{0,2} c_{1,1} + f_{12}^2 c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{22} c_{0,1} c_{1,0} c_{1,1} + f_{11} f_{12} c_{1,0}^2 c_{1,1} + \frac{1}{2} f_{12}^2 c_{1,1}^2 + \frac{1}{2} f_{11} f_{22} c_{1,1}^2 + f_{12}^2 c_{0,1}^2 c_{2,0} + 2 f_{12}^2 c_{0,2} c_{2,0} + 2 f_{11} f_{12} c_{0,1} c_{1,0} c_{2,0} + f_{11}^2 c_{1,0}^2 c_{2,0} + 2 f_{11} f_{12} c_{1,1} c_{2,0} + f_{11}^2 c_{2,0}^2 \right) \hbar^2 + O[\hbar]^3$$