Pensieve header: Testing lemmas 1,2,3 of the DoPeGDO handouts.

$$[F:\mathcal{E}]_B := \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}\partial_{z_i}\partial_{z_j}}\mathcal{E} \quad\text{and}\quad \langle F:\mathcal{E}\rangle_B := [F:\mathcal{E}]_B|_{z_B\to 0},$$

where $\mathcal{E}$ is a docile perturbed Gaussian. The following lemma allows us to restrict to the case where $\mathcal{E}$ has no B-B quadratic part:

Lemma 1. With convergences left to the reader,

$$\left\langle F:\mathcal{E}\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1}:\mathcal{E}\right\rangle_B.$$

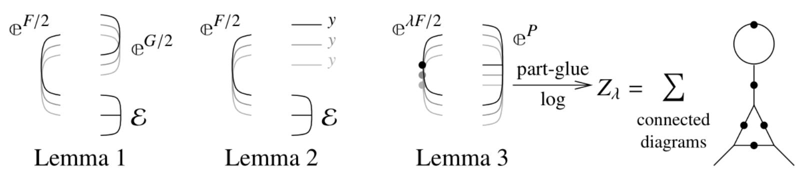
The next lemma dispatches the case where $\mathcal{E}$ has a B-linear part:

Lemma 2. $\left\langle F:\mathcal{E}\mathbb{e}^{\sum_{i\in B} y_iz_i}\right\rangle_B = \mathbb{e}^{\frac{1}{2}\sum_{i,j\in B} F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B$.

Finally, we deal with the docile perturbation case:

Lemma 3. With an extra variable λ, $Z_\lambda := \log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0 = P \quad\text{and}\quad \partial_\lambda Z_\lambda = \frac{1}{2}\sum_{i,j\in B} F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda + (\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$



Lemma 1 Lemma 2 Lemma 3

Goal:

Implement $\text{Act}_{k,B}[F,\mathcal{E}] := [F,\mathcal{E}]_B$ and $\text{Contract}_{k,B}[F,\mathcal{E}] := \langle F,\mathcal{E}\rangle_B$ as power series in $\hbar$, to degree k, and verify lemmas 1, 2, and 3. Inserting $\hbar$ in the appropriate places is user responsibility.

Generic Polynomials:

```
In[∘]:= GenericPolynomial[d_Integer, vars_List, gc_] := Total[Map[
          gc_{Sequence@@#} Times @@ (vars^#) &,
          Join @@ (Permutations /@ IntegerPartitions[d + Length@vars, {Length@vars}]) - 1
        ]];
        GenericPolynomial[specs_List, vars_List, gc_] :=
         Sum[GenericPolynomial[specs〚1〛, vars, gc], specs]
```

Variables and their duals:

```
In[∘]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
        {τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};  (u_i_)* := (u*)_i;  (B_List)* := #* & /@ B;
```

Act and Contract:

```
In[∘]:= Act_{k_,B_}[F_, ℰ_] := Total[
          CoefficientRules[Normal@Series[ℯ^(B*.F.B*/2), {ℏ, 0, k}], B*] /.
           (ps_ → c_) :→ c D[ℰ, Sequence @@ Thread[{B, ps}]]
        ] + O[ℏ]^(k+1);
        Contract_{k_,B_}[F_, ℰ_] := Act_{k,B}[F, ℰ] /. Alternatives @@ B → 0
```

In[∘]:= $\text{Act}_{2,\{x\}}\left[\hbar\,\{\{1\}\},\ e^{x\,x}\right]$

Out[∘]= $e^{x^2} + e^{x^2}\left(1 + 2x^2\right)\hbar + \frac{1}{2}e^{x^2}\left(3 + 12x^2 + 4x^4\right)\hbar^2 + O[\hbar]^3$

In[∘]:= $\text{Log@Contract}_{10,\{x\}}\left[\hbar\,\{\{1\}\},\ e^{x\,x}\right]$

Out[∘]= $\hbar + \hbar^2 + \frac{4\hbar^3}{3} + 2\hbar^4 + \frac{16\hbar^5}{5} + \frac{16\hbar^6}{3} + \frac{64\hbar^7}{7} + 16\hbar^8 + \frac{256\hbar^9}{9} + \frac{256\hbar^{10}}{5} + O[\hbar]^{11}$

In[∘]:= $\text{Act}_{2,\{x,y\}}\left[\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ e^{x\,y}\right]$

Out[∘]= $e^{x\,y} + e^{x\,y}\left(1 + x\,y\right)\hbar + \frac{1}{2}e^{x\,y}\left(2 + 4x\,y + x^2y^2\right)\hbar^2 + O[\hbar]^3$

In[∘]:= $\text{Contract}_{3,\{x,y\}}\left[\hbar\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ e^{3\,x\,y}\right]$

Out[∘]= $1 + 3\hbar + 9\hbar^2 + 27\hbar^3 + O[\hbar]^4$

Testing Lemma 1.

Lemma 1. With convergences left to the reader,

$$\left\langle F : \mathcal{E}\,e^{\frac{1}{2}\sum_{i,j\in B} G_{ij}z_iz_j}\right\rangle_B = \det(1-GF)^{-1/2}\left\langle F(1-GF)^{-1} : \mathcal{E}\right\rangle_B$$

In[∘]:= {p = 10, B = {x}, F = ℏ {{f}}, G = {{g}}, $\mathcal{E}$ = GenericPolynomial[{d, 0, 4}, B, c]}

Out[∘]= $\left\{10,\ \{x\},\ \{\{f\,\hbar\}\},\ \{\{g\}\},\ c_0 + x\,c_1 + x^2c_2 + x^3c_3 + x^4c_4\right\}$

In[∘]:= $\text{lhs = Simplify@Contract}_{p,\{x\}}\left[F,\ \mathcal{E}\,e^{B.G.B/2}\right]$

Out[∘]= $$c_0 + \left(\frac{1}{2}f\,g\,c_0 + f\,c_2\right)\hbar + \frac{3}{8}f^2\left(g^2c_0 + 4g\,c_2 + 8c_4\right)\hbar^2 + \frac{5}{16}f^3g\left(g^2c_0 + 6g\,c_2 + 24c_4\right)\hbar^3 + \frac{35}{128}f^4g^2\left(g^2c_0 + 8g\,c_2 + 48c_4\right)\hbar^4 + \frac{63}{256}f^5g^3\left(g^2c_0 + 10g\,c_2 + 80c_4\right)\hbar^5 + \frac{231f^6g^4\left(g^2c_0 + 12g\,c_2 + 120c_4\right)\hbar^6}{1024} + \frac{429f^7g^5\left(g^2c_0 + 14g\,c_2 + 168c_4\right)\hbar^7}{2048} + \frac{6435f^8g^6\left(g^2c_0 + 16g\,c_2 + 224c_4\right)\hbar^8}{32768} + \frac{12155f^9g^7\left(g^2c_0 + 18g\,c_2 + 288c_4\right)\hbar^9}{65536} + \frac{46189f^{10}g^8\left(g^2c_0 + 20g\,c_2 + 360c_4\right)\hbar^{10}}{262144} + O[\hbar]^{11}$$

In[∘]:=
```
rhs = Simplify[Series[
    Det[{{1}} - G.F]^-1/2 Contract_{p,{x}}[F.Inverse[{{1}} - G.F], ε],
    {ħ, 0, p}
  ]]
```

Out[∘]=
$$c_0 + \left(\frac{1}{2} f g c_0 + f c_2\right) \hbar + \frac{3}{8} f^2 \left(g^2 c_0 + 4 g c_2 + 8 c_4\right) \hbar^2 + \frac{5}{16} f^3 g \left(g^2 c_0 + 6 g c_2 + 24 c_4\right) \hbar^3 + \frac{35}{128} f^4 g^2 \left(g^2 c_0 + 8 g c_2 + 48 c_4\right) \hbar^4 + \frac{63}{256} f^5 g^3 \left(g^2 c_0 + 10 g c_2 + 80 c_4\right) \hbar^5 + \frac{231 f^6 g^4 \left(g^2 c_0 + 12 g c_2 + 120 c_4\right) \hbar^6}{1024} + \frac{429 f^7 g^5 \left(g^2 c_0 + 14 g c_2 + 168 c_4\right) \hbar^7}{2048} + \frac{6435 f^8 g^6 \left(g^2 c_0 + 16 g c_2 + 224 c_4\right) \hbar^8}{32768} + \frac{12155 f^9 g^7 \left(g^2 c_0 + 18 g c_2 + 288 c_4\right) \hbar^9}{65536} + \frac{46189 f^{10} g^8 \left(g^2 c_0 + 20 g c_2 + 360 c_4\right) \hbar^{10}}{262144} + O[\hbar]^{11}$$

In[∘]:=
```
Simplify[lhs / rhs]
```

Out[∘]= $1 + O[\hbar]^{11}$

In[∘]:=
$$\left\{p = 2,\ B = \{x, y\},\ F = \hbar \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix},\ G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix},\ \varepsilon = \text{GenericPolynomial}[\{d, 0, 2\}, B, c]\right\}$$

Out[∘]=
$$\{2, \{x, y\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, \{\{g_{11}, g_{12}\}, \{g_{12}, g_{22}\}\}, c_{0,0} + y\, c_{0,1} + y^2 c_{0,2} + x\, c_{1,0} + x\, y\, c_{1,1} + x^2 c_{2,0}\}$$

In[∘]:=
$$\text{lhs} = \text{Simplify@Contract}_{p,B}\left[F,\ \varepsilon\, e^{B.G.B/2}\right]$$

Out[∘]=
$$c_{0,0} + \frac{1}{2}\left(f_{22}\left(g_{22} c_{0,0} + 2 c_{0,2}\right) + 2 f_{12}\left(g_{12} c_{0,0} + c_{1,1}\right) + f_{11}\left(g_{11} c_{0,0} + 2 c_{2,0}\right)\right) \hbar + \frac{1}{8}\left(3 f_{22}^2 g_{22}\left(g_{22} c_{0,0} + 4 c_{0,2}\right) + 12 f_{12} f_{22}\left(g_{12}\left(g_{22} c_{0,0} + 2 c_{0,2}\right) + g_{22} c_{1,1}\right) + 3 f_{11}^2 g_{11}\left(g_{11} c_{0,0} + 4 c_{2,0}\right) + 12 f_{11} f_{12}\left(g_{11}\left(g_{12} c_{0,0} + c_{1,1}\right) + 2 g_{12} c_{2,0}\right) + 2\left(2 f_{12}^2 + f_{11} f_{22}\right)\left(2 g_{12}^2 c_{0,0} + g_{11}\left(g_{22} c_{0,0} + 2 c_{0,2}\right) + 4 g_{12} c_{1,1} + 2 g_{22} c_{2,0}\right)\right) \hbar^2 + O[\hbar]^3$$

In[∘]:=
$$\text{rhs} = \text{Simplify}\left[\text{Series}\left[\text{Det}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - G.F\right]^{-1/2} \text{Contract}_{p,B}\left[F.\text{Inverse}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - G.F\right], \varepsilon\right], \{\hbar, 0, p\}\right]\right]$$

Out[∘]=
$$c_{0,0} + \left(\frac{1}{2}\left(f_{11} g_{11} + 2 f_{12} g_{12} + f_{22} g_{22}\right) c_{0,0} + f_{22} c_{0,2} + f_{12} c_{1,1} + f_{11} c_{2,0}\right) \hbar + \frac{1}{8}\left(3 f_{22}^2 g_{22}\left(g_{22} c_{0,0} + 4 c_{0,2}\right) + 12 f_{12} f_{22}\left(g_{12}\left(g_{22} c_{0,0} + 2 c_{0,2}\right) + g_{22} c_{1,1}\right) + 3 f_{11}^2 g_{11}\left(g_{11} c_{0,0} + 4 c_{2,0}\right) + 4 f_{12}^2\left(2 g_{12}^2 c_{0,0} + g_{11}\left(g_{22} c_{0,0} + 2 c_{0,2}\right) + 4 g_{12} c_{1,1} + 2 g_{22} c_{2,0}\right) + 2 f_{11}\left(6 f_{12}\left(g_{11}\left(g_{12} c_{0,0} + c_{1,1}\right) + 2 g_{12} c_{2,0}\right) + f_{22}\left(2 g_{12}^2 c_{0,0} + g_{11}\left(g_{22} c_{0,0} + 2 c_{0,2}\right) + 4 g_{12} c_{1,1} + 2 g_{22} c_{2,0}\right)\right)\right) \hbar^2 + O[\hbar]^3$$

In[∘]:=
```
Simplify[lhs / rhs]
```

Out[∘]= $1 + O[\hbar]^3$

Testing Lemma 2.

Lemma 2. $\left\langle F:\mathcal{E}\,\mathbb{e}^{\sum_{i\in B}y_iz_i}\right\rangle_B=\mathbb{e}^{\frac{1}{2}\sum_{i,j\in B}F_{ij}y_iy_j}\left\langle F:\ \mathcal{E}|_{z_B\to z_B+Fy_B}\right\rangle_B.$

In[•]:=
```
{n = 2, p = 2, B = Table[z_i, {i, n}], Y = Table[y_i, {i, n}],
 F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], ℰ = GenericPolynomial[{d, 0, 3}, B, c]}
```

Out[•]= $\{2, 2, \{z_1, z_2\}, \{y_1, y_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\},$
$c_{0,0}+z_2c_{0,1}+z_2^2c_{0,2}+z_2^3c_{0,3}+z_1c_{1,0}+z_1z_2c_{1,1}+z_1z_2^2c_{1,2}+z_1^2c_{2,0}+z_1^2z_2c_{2,1}+z_1^3c_{3,0}\}$

In[•]:=
```
lhs = Simplify@Contract_{p,B}[F, ℰ ⅇ^{Y.B}]
```

Out[•]=
$$c_{0,0}+\frac{1}{2}\Big(f_{22}\left(y_2^2c_{0,0}+2y_2c_{0,1}+2c_{0,2}\right)+2f_{12}\left(y_1\left(y_2c_{0,0}+c_{0,1}\right)+y_2c_{1,0}+c_{1,1}\right)+f_{11}\left(y_1^2c_{0,0}+2y_1c_{1,0}+2c_{2,0}\right)\Big)\hbar+$$
$$\frac{1}{8}\Big(f_{22}^2y_2\left(y_2^3c_{0,0}+4y_2^2c_{0,1}+12y_2c_{0,2}+24c_{0,3}\right)+4f_{12}f_{22}\left(y_1\left(y_2^3c_{0,0}+3y_2^2c_{0,1}+6y_2c_{0,2}+6c_{0,3}\right)+y_2\left(y_2^2c_{1,0}+3y_2c_{1,1}+6c_{1,2}\right)\right)+2\left(2f_{12}^2+f_{11}f_{22}\right)\left(y_1^2\left(y_2^2c_{0,0}+2y_2c_{0,1}+2c_{0,2}\right)+2y_1\left(y_2^2c_{1,0}+2y_2c_{1,1}+2c_{1,2}\right)+2y_2\left(y_2c_{2,0}+2c_{2,1}\right)\right)+f_{11}^2y_1\left(y_1^3c_{0,0}+4y_1^2c_{1,0}+12y_1c_{2,0}+24c_{3,0}\right)+4f_{11}f_{12}\left(y_1^3\left(y_2c_{0,0}+c_{0,1}\right)+3y_1^2\left(y_2c_{1,0}+c_{1,1}\right)+6y_1\left(y_2c_{2,0}+c_{2,1}\right)+6y_2c_{3,0}\right)\Big)\hbar^2+O[\hbar]^3$$

In[•]:=
```
rhs = Simplify[Series[
   ⅇ^{Y.F.Y/2} Contract_{p,B}[F, ℰ /. Thread[B → B + F.Y]],
   {ℏ, 0, p}]]
```

Out[•]=
$$c_{0,0}+\left(\frac{1}{2}f_{22}\left(y_2^2c_{0,0}+2y_2c_{0,1}+2c_{0,2}\right)+f_{12}\left(y_1\left(y_2c_{0,0}+c_{0,1}\right)+y_2c_{1,0}+c_{1,1}\right)+f_{11}\left(\frac{1}{2}y_1^2c_{0,0}+y_1c_{1,0}+c_{2,0}\right)\right)\hbar+$$
$$\frac{1}{8}\Big(\left(f_{11}y_1^2+y_2\left(2f_{12}y_1+f_{22}y_2\right)\right)^2c_{0,0}+8\left(f_{12}y_1+f_{22}y_2\right)^2c_{0,2}+8\left(f_{11}y_1+f_{12}y_2\right)\left(f_{12}y_1+f_{22}y_2\right)c_{1,1}+4f_{22}\left(6\left(f_{12}y_1+f_{22}y_2\right)c_{0,3}+2\left(f_{11}y_1+f_{12}y_2\right)c_{1,2}\right)+8\left(f_{11}y_1+f_{12}y_2\right)^2c_{2,0}+4\left(f_{11}y_1^2+y_2\left(2f_{12}y_1+f_{22}y_2\right)\right)\left(f_{22}\left(y_2c_{0,1}+c_{0,2}\right)+f_{12}\left(y_1c_{0,1}+y_2c_{1,0}+c_{1,1}\right)+f_{11}\left(y_1c_{1,0}+c_{2,0}\right)\right)+8f_{12}\left(2\left(f_{12}y_1+f_{22}y_2\right)c_{1,2}+2\left(f_{11}y_1+f_{12}y_2\right)c_{2,1}\right)+4f_{11}\left(2\left(f_{12}y_1+f_{22}y_2\right)c_{2,1}+6\left(f_{11}y_1+f_{12}y_2\right)c_{3,0}\right)\Big)\hbar^2+O[\hbar]^3$$

In[•]:=
```
Simplify[lhs == rhs]
```

Out[•]= $O[\hbar]^3 == 0$

Testing Lemma 3.

Lemma 3. With an extra variable λ, $Z_\lambda:=\log[\lambda F:\mathbb{e}^P]_B$ satisfies and is determined by the following PDE / IVP:

$$Z_0=P\quad\text{and}\quad\partial_\lambda Z_\lambda=\frac{1}{2}\sum_{i,j\in B}F_{ij}\left(\partial_{z_i}\partial_{z_j}Z_\lambda+(\partial_{z_i}Z_\lambda)(\partial_{z_j}Z_\lambda)\right).$$

In[∘]:=
```
{n = 2, p = 3, B = Table[b_i, {i, n}],
 F = ℏ Table[f_{10,1}.Sort[{i,j}], {i, n}, {j, n}], P = GenericPolynomial[{d, 0, 2}, B, c]}
```

Out[∘]= $\{2, 3, \{b_1, b_2\}, \{\{\hbar f_{11}, \hbar f_{12}\}, \{\hbar f_{12}, \hbar f_{22}\}\}, c_{0,0} + b_2 c_{0,1} + b_2^2 c_{0,2} + b_1 c_{1,0} + b_1 b_2 c_{1,1} + b_1^2 c_{2,0}\}$

In[∘]:=
```
Short[Z = Simplify@PowerExpand@Simplify@Log[Act_{p,B}[λ F, e^P]], 5]
```

Out[∘]//Short=
$$\left(c_{0,0} + b_2^2 c_{0,2} + b_2 \left(c_{0,1} + b_1 c_{1,1}\right) + b_1 \left(c_{1,0} + b_1 c_{2,0}\right)\right) + \frac{1}{2} \lambda \left(f_{22} \left(c_{0,1}^2 + 4 b_2^2 c_{0,2}^2 + b_1^2 c_{1,1}^2 + 2 c_{0,1} \left(2 b_2 c_{0,2} + b_1 c_{1,1}\right) + c_{0,2} \left(2 + 4 b_1 b_2 c_{1,1}\right)\right) + f_{11} \left(c_{1,0}^2 + b_2^2 c_{1,1}^2 + 4 b_1 b_2 c_{1,1} c_{2,0} + 2 c_{1,0} \left(b_2 c_{1,1} + 2 b_1 c_{2,0}\right) + 2 c_{2,0} \left(1 + 2 b_1^2 c_{2,0}\right)\right) + 2 f_{12} \left(2 b_2^2 c_{0,2} c_{1,1} + c_{0,1} \left(c_{1,0} + b_2 c_{1,1} + 2 b_1 c_{2,0}\right) + c_{1,1} \left(1 + b_1 c_{1,0} + 2 b_1^2 c_{2,0}\right) + b_2 \left(b_1 c_{1,1}^2 + 2 c_{0,2} \left(c_{1,0} + 2 b_1 c_{2,0}\right)\right)\right)\right) \hbar + \frac{1}{2} \lambda^2 \left(\ll 1 \gg\right) \hbar^2 + \frac{1}{6} \lambda^3 \left(4 f_{22}^3 c_{0,2}^2 \left(3 c_{0,1}^2 + 12 b_2^2 c_{0,2}^2 + 3 b_1^2 c_{1,1}^2 + 6 c_{0,1} \left(2 b_2 c_{0,2} + b_1 c_{1,1}\right) + 2 c_{0,2} \left(1 + 6 b_1 b_2 c_{1,1}\right)\right) + \ll 5 \gg + 3 f_{22} \left(\ll 1 \gg\right)\right) \hbar^3 + O[\hbar]^4$$

In[∘]:=
```
Simplify[Z /. λ → 0]
```

Out[∘]= $\left(c_{0,0} + b_2^2 c_{0,2} + b_2 \left(c_{0,1} + b_1 c_{1,1}\right) + b_1 \left(c_{1,0} + b_1 c_{2,0}\right)\right) + O[\hbar]^4$

In[∘]:=
```
Simplify[(Z /. λ → 0) == P]
```

Out[∘]= $O[\hbar]^4 == 0$

In[∘]:=
```
lhs = ∂_λ Z
```

Out[∘]=

$$
\begin{aligned}
&\frac{1}{2}\left(f_{22}\left(c_{0,1}^2+4 b_2^2 c_{0,2}^2+b_1^2 c_{1,1}^2+2 c_{0,1}\left(2 b_2 c_{0,2}+b_1 c_{1,1}\right)+c_{0,2}\left(2+4 b_1 b_2 c_{1,1}\right)\right)+\right.\\
&\quad f_{11}\left(c_{1,0}^2+b_2^2 c_{1,1}^2+4 b_1 b_2 c_{1,1} c_{2,0}+2 c_{1,0}\left(b_2 c_{1,1}+2 b_1 c_{2,0}\right)+2 c_{2,0}\left(1+2 b_1^2 c_{2,0}\right)\right)+\\
&\quad 2 f_{12}\left(2 b_2^2 c_{0,2} c_{1,1}+c_{0,1}\left(c_{1,0}+b_2 c_{1,1}+2 b_1 c_{2,0}\right)+\right.\\
&\qquad \left.\left. c_{1,1}\left(1+b_1 c_{1,0}+2 b_1^2 c_{2,0}\right)+b_2\left(b_1 c_{1,1}^2+2 c_{0,2}\left(c_{1,0}+2 b_1 c_{2,0}\right)\right)\right)\right)\hbar+\\
&\lambda\left(2 f_{22}^2 c_{0,2}\left(c_{0,1}^2+4 b_2^2 c_{0,2}^2+b_1^2 c_{1,1}^2+2 c_{0,1}\left(2 b_2 c_{0,2}+b_1 c_{1,1}\right)+c_{0,2}\left(1+4 b_1 b_2 c_{1,1}\right)\right)+\right.\\
&\quad 2 f_{11}^2 c_{2,0}\left(c_{1,0}^2+b_2^2 c_{1,1}^2+c_{2,0}+4 b_1 b_2 c_{1,1} c_{2,0}+4 b_1^2 c_{2,0}^2+2 c_{1,0}\left(b_2 c_{1,1}+2 b_1 c_{2,0}\right)\right)+\\
&\quad f_{12}^2\left(2 c_{0,1}^2 c_{2,0}+8 b_2^2 c_{0,2}^2 c_{2,0}+2 c_{0,1} c_{1,1}\left(c_{1,0}+b_2 c_{1,1}+4 b_1 c_{2,0}\right)+\right.\\
&\qquad c_{1,1}^2\left(1+2 b_1\left(c_{1,0}+b_2 c_{1,1}\right)+6 b_1^2 c_{2,0}\right)+2 c_{0,2}\\
&\qquad\left.\left(c_{1,0}^2+3 b_2^2 c_{1,1}^2+4 b_2\left(c_{0,1}+3 b_1 c_{1,1}\right) c_{2,0}+4 c_{1,0}\left(b_2 c_{1,1}+b_1 c_{2,0}\right)+2 c_{2,0}\left(1+2 b_1^2 c_{2,0}\right)\right)\right)+\\
&\quad 2 f_{11} f_{12}\left(c_{1,0}^2 c_{1,1}+b_2^2\left(c_{1,1}^3+4 c_{0,2} c_{1,1} c_{2,0}\right)+2 c_{2,0}\left(2 b_1 c_{0,1} c_{2,0}+c_{1,1}\left(1+4 b_1^2 c_{2,0}\right)\right)+\right.\\
&\qquad 2 c_{1,0}\left(\left(c_{0,1}+3 b_1 c_{1,1}\right) c_{2,0}+b_2\left(c_{1,1}^2+2 c_{0,2} c_{2,0}\right)\right)+\\
&\qquad\left. 2 b_2 c_{2,0}\left(c_{0,1} c_{1,1}+b_1\left(3 c_{1,1}^2+4 c_{0,2} c_{2,0}\right)\right)\right)+\\
&\quad f_{22}\left(f_{11} c_{1,1}\left(4 b_2^2 c_{0,2} c_{1,1}+2 c_{0,1}\left(c_{1,0}+b_2 c_{1,1}+2 b_1 c_{2,0}\right)+c_{1,1}\left(1+2 b_1 c_{1,0}+4 b_1^2 c_{2,0}\right)+\right.\right.\\
&\qquad\quad\left. 2 b_2\left(b_1 c_{1,1}^2+2 c_{0,2}\left(c_{1,0}+2 b_1 c_{2,0}\right)\right)\right)+2 f_{12}\left(c_{0,1}^2 c_{1,1}+8 b_2^2 c_{0,2}^2 c_{1,1}+b_1^2 c_{1,1}^3+\right.\\
&\qquad\quad 2 c_{0,2} c_{1,1}\left(1+b_1 c_{1,0}+2 b_1^2 c_{2,0}\right)+2 b_2 c_{0,2}\left(3 b_1 c_{1,1}^2+2 c_{0,2}\left(c_{1,0}+2 b_1 c_{2,0}\right)\right)+\\
&\qquad\quad\left.\left.\left. 2 c_{0,1}\left(b_1 c_{1,1}^2+c_{0,2}\left(c_{1,0}+3 b_2 c_{1,1}+2 b_1 c_{2,0}\right)\right)\right)\right)\right)\hbar^2+\\
&\frac{1}{2}\lambda^2\left(4 f_{22}^3 c_{0,2}^2\left(3 c_{0,1}^2+12 b_2^2 c_{0,2}^2+3 b_1^2 c_{1,1}^2+6 c_{0,1}\left(2 b_2 c_{0,2}+b_1 c_{1,1}\right)+2 c_{0,2}\left(1+6 b_1 b_2 c_{1,1}\right)\right)+\right.\\
&\quad 4 f_{11}^3 c_{2,0}^2\left(3 c_{1,0}^2+3 b_2^2 c_{1,1}^2+12 b_1 b_2 c_{1,1} c_{2,0}+6 c_{1,0}\left(b_2 c_{1,1}+2 b_1 c_{2,0}\right)+2 c_{2,0}\left(1+6 b_1^2 c_{2,0}\right)\right)+\\
&\quad 24 f_{11}^2 f_{12} c_{2,0}\left(c_{1,0}^2 c_{1,1}+b_2^2\left(c_{1,1}^3+2 c_{0,2} c_{1,1} c_{2,0}\right)+c_{2,0}\left(2 b_1 c_{0,1} c_{2,0}+c_{1,1}\left(1+6 b_1^2 c_{2,0}\right)\right)+c_{1,0}\right.\\
&\qquad\left.\left(\left(c_{0,1}+5 b_1 c_{1,1}\right) c_{2,0}+2 b_2\left(c_{1,1}^2+c_{0,2} c_{2,0}\right)\right)+b_2 c_{2,0}\left(c_{0,1} c_{1,1}+b_1\left(5 c_{1,1}^2+4 c_{0,2} c_{2,0}\right)\right)\right)+\\
&\quad 3 f_{22}^2\left(f_{11} c_{1,1}\left(c_{0,1}^2 c_{1,1}+12 b_2^2 c_{0,2}^2 c_{1,1}+b_1^2 c_{1,1}^3+c_{0,2} c_{1,1}\left(2+4 b_1 c_{1,0}+8 b_1^2 c_{2,0}\right)+8 b_2\right.\right.\\
&\qquad\quad\left. c_{0,2}\left(b_1 c_{1,1}^2+c_{0,2}\left(c_{1,0}+2 b_1 c_{2,0}\right)\right)+2 c_{0,1}\left(b_1 c_{1,1}^2+2 c_{0,2}\left(c_{1,0}+2 b_2 c_{1,1}+2 b_1 c_{2,0}\right)\right)\right)+\\
&\qquad 8 f_{12} c_{0,2}\left(c_{0,1}^2 c_{1,1}+6 b_2^2 c_{0,2}^2 c_{1,1}+b_1^2 c_{1,1}^3+c_{0,2} c_{1,1}\left(1+b_1 c_{1,0}+2 b_1^2 c_{2,0}\right)+b_2 c_{0,2}\right.\\
&\qquad\quad\left.\left.\left(5 b_1 c_{1,1}^2+2 c_{0,2}\left(c_{1,0}+2 b_1 c_{2,0}\right)\right)+c_{0,1}\left(2 b_1 c_{1,1}^2+c_{0,2}\left(c_{1,0}+5 b_2 c_{1,1}+2 b_1 c_{2,0}\right)\right)\right)\right)+\\
&\quad 3 f_{11} f_{12}^2\left(c_{1,0}^2\left(3 c_{1,1}^2+8 c_{0,2} c_{2,0}\right)+b_2^2\left(3 c_{1,1}^4+32 c_{0,2} c_{1,1}^2 c_{2,0}+16 c_{0,2}^2 c_{2,0}^2\right)+\right.\\
&\qquad 4 b_2 c_{2,0}\left(6 b_1 c_{1,1}\left(c_{1,1}^2+4 c_{0,2} c_{2,0}\right)+c_{0,1}\left(3 c_{1,1}^2+4 c_{0,2} c_{2,0}\right)\right)+\\
&\qquad 2 c_{1,0}\left(2 c_{2,0}\left(3 c_{0,1} c_{1,1}+6 b_1 c_{1,1}^2+8 b_1 c_{0,2} c_{2,0}\right)+b_2\left(3 c_{1,1}^3+20 c_{0,2} c_{1,1} c_{2,0}\right)\right)+\\
&\qquad\left. 2 c_{2,0}\left(16 b_1 c_{0,1} c_{1,1} c_{2,0}+c_{1,1}^2\left(3+20 b_1^2 c_{2,0}\right)+2 c_{2,0}\left(c_{0,1}^2+2 c_{0,2}\left(1+4 b_1^2 c_{2,0}\right)\right)\right)\right)+\\
&\quad 2 f_{12}^3\left(24 b_2 c_{0,2}^2 c_{2,0}\left(c_{1,0}+2 b_2 c_{1,1}+2 b_1 c_{2,0}\right)+c_{1,1}\right.\\
&\qquad\left(6 c_{0,1}^2 c_{2,0}+3 c_{0,1} c_{1,1}\left(c_{1,0}+b_2 c_{1,1}+6 b_1 c_{2,0}\right)+c_{1,1}^2\left(1+3 b_1\left(c_{1,0}+b_2 c_{1,1}\right)+12 b_1^2 c_{2,0}\right)\right)+\\
&\qquad 6 c_{0,2}\left(c_{1,0}^2 c_{1,1}+c_{1,0}\left(3 b_2 c_{1,1}^2+2\left(c_{0,1}+3 b_1 c_{1,1}\right) c_{2,0}\right)+\right.\\
&\qquad\quad\left.\left. 2\left(b_2^2 c_{1,1}^3+3 b_2 c_{1,1}\left(c_{0,1}+2 b_1 c_{1,1}\right) c_{2,0}+c_{2,0}\left(2 b_1 c_{0,1} c_{2,0}+c_{1,1}\left(1+4 b_1^2 c_{2,0}\right)\right)\right)\right)\right)+\\
&\quad 3 f_{22}\left(f_{12}^2\left(3 c_{1,1}^2\left(c_{0,1}+b_1 c_{1,1}\right)^2+32 b_2^2 c_{0,2}^3 c_{2,0}+4 c_{0,2}^2\left(c_{1,0}^2+10 b_2^2 c_{1,1}^2+8 b_2\left(c_{0,1}+3 b_1 c_{1,1}\right)\right.\right.\right.\\
&\qquad\qquad\left. c_{2,0}+4 c_{1,0}\left(2 b_2 c_{1,1}+b_1 c_{2,0}\right)+2 c_{2,0}\left(1+2 b_1^2 c_{2,0}\right)\right)+2 c_{0,2}\left(4 c_{0,1}^2 c_{2,0}+2 c_{0,1}\right.\\
&\qquad\qquad\left.\left. c_{1,1}\left(3 c_{1,0}+6 b_2 c_{1,1}+10 b_1 c_{2,0}\right)+c_{1,1}^2\left(3+6 b_1\left(c_{1,0}+2 b_2 c_{1,1}\right)+16 b_1^2 c_{2,0}\right)\right)\right)+\\
&\qquad f_{11}^2 c_{1,1}\left(c_{1,0}^2 c_{1,1}+b_2^2\left(c_{1,1}^3+8 c_{0,2} c_{1,1} c_{2,0}\right)+2 c_{2,0}\left(4 b_1 c_{0,1} c_{2,0}+c_{1,1}\left(1+6 b_1^2 c_{2,0}\right)\right)+\right.\\
&\qquad\quad 4 b_2 c_{2,0}\left(c_{0,1} c_{1,1}+2 b_1\left(c_{1,1}^2+2 c_{0,2} c_{2,0}\right)\right)+\\
&\qquad\quad\left. 2 c_{1,0}\left(2\left(c_{0,1}+2 b_1 c_{1,1}\right) c_{2,0}+b_2\left(c_{1,1}^2+4 c_{0,2} c_{2,0}\right)\right)\right)+\\
&\qquad 2 f_{11} f_{12}\left(8 b_2 c_{0,2}^2 c_{2,0}\left(c_{1,0}+2 b_2 c_{1,1}+2 b_1 c_{2,0}\right)+c_{1,1}\left(2 c_{0,1}^2 c_{2,0}+\right.\right.\\
&\qquad\qquad\left. c_{0,1} c_{1,1}\left(3 c_{1,0}+3 b_2 c_{1,1}+10 b_1 c_{2,0}\right)+c_{1,1}^2\left(1+3 b_1\left(c_{1,0}+b_2 c_{1,1}\right)+8 b_1^2 c_{2,0}\right)\right)+\\
&\qquad\quad 2 c_{0,2}\left(c_{1,0}^2 c_{1,1}+c_{1,0}\left(5 b_2 c_{1,1}^2+2\left(c_{0,1}+3 b_1 c_{1,1}\right) c_{2,0}\right)+2\left(2 b_2^2 c_{1,1}^3+b_2 c_{1,1}\right.\right.\\
&\qquad\qquad\left.\left.\left.\left.\left.\left.\left.\left(3 c_{0,1}+8 b_1 c_{1,1}\right) c_{2,0}+c_{2,0}\left(2 b_1 c_{0,1} c_{2,0}+c_{1,1}\left(1+4 b_1^2 c_{2,0}\right)\right)\right)\right)\right)\right)\right)\right)\hbar^3+O[\hbar]^4
\end{aligned}
$$

In[∘]:= `B.F.B // Expand`

Out[∘]= $\hbar\, b_1^2\, f_{11} + 2\,\hbar\, b_1\, b_2\, f_{12} + \hbar\, b_2^2\, f_{22}$

In[∘]:= `Table[∂_{b1,b2}(B.F.B)/2, {b1, B}, {b2, B}] // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix} \hbar\, f_{11} & \hbar\, f_{12} \\ \hbar\, f_{12} & \hbar\, f_{22} \end{pmatrix}$$

In[∘]:= `Short[rhs = Simplify@Sum[(∂_{b1,b2}(B.F.B)) (∂_{b1,b2}Z + (∂_{b1}Z) (∂_{b2}Z)) / 4, {b1, B}, {b2, B}]]`

Out[∘]//Short=

$$\frac{1}{2}\left(f_{22}\left(2\, c_{0,2} + \left(c_{0,1} + 2\, b_2\, c_{0,2} + b_1\, c_{1,1}\right)^2\right) + 2\, f_{12}\left(c_{1,1} + \left(c_{0,1} + 2\, b_2\, c_{0,2} + b_1\, c_{1,1}\right)\left(c_{1,0} + b_2\, c_{\ll 1 \gg} + 2\, b_1\, c_{2,0}\right)\right) + f_{11}\left(2\, c_{2,0} + \left(c_{1,0} + b_2\, c_{1,1} + 2\, b_1\, c_{2,0}\right)^2\right)\right)\hbar + \ll 3 \gg + \ll 1 \gg$$

In[∘]:= `Simplify[lhs == rhs]`

Out[∘]= $O[\hbar]^4 == 0$