Gaussian Zipping

March 3, 2019 6:38 AM

Let $F_\lambda = e^{\lambda\partial_\zeta\partial_z} e^{q\zeta z}$. Then $F_0 = e^{q\zeta z}$ and

$\partial_\lambda F_\lambda = \partial_\lambda e^{\lambda\partial_\zeta\partial_z} e^{q\zeta z} = \partial_\zeta\partial_z F$ Also,

$$\partial_\lambda F_\lambda = e^{\lambda\partial_\zeta\partial_z}\partial_\zeta\partial_z e^{q\zeta z} =$$
$$= e^{\lambda\partial_\zeta\partial_z}\partial_\zeta(q\zeta e^{q\zeta z})$$
$$= e^{\lambda\partial_\zeta\partial_z}(q+q^2\zeta z)e^{q\zeta z}$$

Claim soln is

$$F_\lambda = \frac{1}{1-\lambda q} e^{\frac{q\zeta z}{1-\lambda q}}$$

Proof

$$\partial_\zeta\partial_z F_\lambda = \frac{q}{(1-\lambda q)^2}\partial_\zeta\left(\zeta e^{q\zeta z/1-\lambda q}\right)$$

$$= \frac{q}{(1-\lambda q)^3}\left((1-\lambda q)+q\zeta z\right)e^{q\zeta z/1-\lambda q}$$

$$\partial_\lambda F_\lambda = \frac{q}{(1-\lambda q)^2} e^{q\zeta z/1-\lambda q} + \frac{q^2\zeta z}{(1-\lambda q)^3} e^{q\zeta z/1-\lambda q}$$

$$= \frac{q}{(1-\lambda q)^3}\left((1-\lambda q)+q\zeta z\right) e^{q\zeta z/1-q}$$

Precise:

In[1]:= $\mathrm{zip}_{q\_,\lambda\_}[\eta\_, y\_] := \dfrac{e^{\lambda\eta y/(1-\lambda q)}}{1-\lambda q}$;

$F[q\_, \lambda\_] := \mathrm{Simplify}[\mathrm{zip}_{q,\lambda}[q\,\xi+\eta, q\,z+y]\, e^{q\,\xi\,z+y\,\xi+\eta\,z}]$;

$\{F[q,\lambda], F[q,0], F[q,1], F[q,1] \,/.\, \{z \mid \xi \to 0\}\}$ // Simplify

Out[3]= $\left\{-\dfrac{e^{\frac{y\xi+qz\xi+z\eta+y\eta\lambda}{1-q\lambda}}}{-1+q\lambda},\ e^{y\xi+qz\xi+z\eta},\ -\dfrac{e^{-\frac{y(\xi+\eta)+z(q\xi+\eta)}{-1+q}}}{-1+q},\ -\dfrac{e^{-\frac{y\eta}{-1+q}}}{-1+q}\right\}$

In[4]:= $\{\mathrm{lhs} = \partial_\lambda F[q,\lambda], \mathrm{rhs} = \partial_\xi\partial_z F[q,\lambda], \mathrm{lhs} == \mathrm{rhs}\}$ // Simplify

Out[4]= $\left\{-\dfrac{e^{\frac{y\xi+qz\xi+z\eta+y\eta\lambda}{1-q\lambda}}\left(y\eta+q(1+y\xi+z\eta)+q^2(z\xi-\lambda)\right)}{(-1+q\lambda)^3},\right.$

$\left.-\dfrac{e^{\frac{y\xi+qz\xi+z\eta+y\eta\lambda}{1-q\lambda}}\left(y\eta+q(1+y\xi+z\eta)+q^2(z\xi-\lambda)\right)}{(-1+q\lambda)^3},\ \mathrm{True}\right\}$