Pensieve header: Verifying the zipping theorem. Continues ZipBindDemo.nb in pensieve://Talks/Matemale-1804/.

$$\left\langle P(z_i,\zeta^j)\mathrm{e}^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}|\left\langle P(z_i,\zeta^j)\mathrm{e}^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|\mathrm{e}^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

In[•]:=
```
HL[ℰ_] := Style[ℰ, Background → Yellow];
```

In[•]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
{z*, x*, y*} = {ζ, ξ, η}; {ζ*, ξ*, η*} = {z, x, y};
(u_i_)* := (u*)_i;
```

In[•]:=
```
𝔼 /: 𝔼[Q1_, P1_] ≡ 𝔼[Q2_, P2_] := Simplify[Q1 == Q2] ∧ Simplify[Normal[P1 - P2] == 0];
```

Zip

In[•]:=
```
Zip_{}[P_] := P; Zip_{ξ_,ξs___}[P_] := (Expand[P // Zip_{ξs}] /. f_. ξ^d_. :↦ ∂_{ξ*,d} f) /. ξ* → 0
```

In[•]:=
```
Zip_{ζ}[(a ζ^6 + ζ + 3) (z^5 ℯ^z + 7 z) + 99 b]
```
Out[•]=
$$7 + 720\,a + 99\,b$$

In[•]:=
```
Zip_{ξ,η}[ξ^3 η^3 ℯ^(a x + b y + c x y)]
```
Out[•]=
$$a^3 b^3 + 9 a^2 b^2 c + 18 a b c^2 + 6 c^3$$

In[•]:=
```
(* 𝔼[Q,P] means ℯ^Q P *)
𝔼 /: Zip_{ξs_List}@𝔼[Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, Q1, Q2},
   zs = Table[ζ*, {ζ, ξs}];
   c = Q /. Alternatives @@ (ξs ⋃ zs) → 0;
   ys = Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ζ, ξs}];
   ηs = Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}];
   qt = Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ζ} Q, {ζ, ξs}, {z, zs}];
   zrule = Thread[zs → qt.(zs + ys)];
   Q1 = c + ηs.zs /. zrule;
   Q2 = Q1 /. Alternatives @@ zs → 0;
   Simplify /@ 𝔼[Q2, Det[qt] ℯ^(-Q2) Zip_ξs[ℯ^Q1 (P /. zrule)]]];
```

In[•]:=
```
Eh = 𝔼[h Sum_{i=1}^3 Sum_{j=1}^3 a_{10 i+j} x_i ξ_j, Sum_{i=1}^3 f_i[x_1, x_2, x_3] ξ_i]; E1 = Eh /. h → 1
```
Out[•]=
$$\mathbb{E}[a_{11}x_1\xi_1 + a_{21}x_2\xi_1 + a_{31}x_3\xi_1 + a_{12}x_1\xi_2 + a_{22}x_2\xi_2 + a_{32}x_3\xi_2 + a_{13}x_1\xi_3 + a_{23}x_2\xi_3 + a_{33}x_3\xi_3,$$
$$\xi_1 f_1[x_1,x_2,x_3] + \xi_2 f_2[x_1,x_2,x_3] + \xi_3 f_3[x_1,x_2,x_3]]$$

```
In[•]:= Short[lhs = Zip_{ξ1,ξ2}@E1, 5]
```

Out[•]//Short=
$$\mathbb{E}\Big[\Big(\Big(a_{13}\ \Big(\big(-1+a_{22}\big)\ a_{31}-a_{21}\ a_{32}\Big)+a_{12}\ \big(-a_{23}\ a_{31}+a_{21}\ a_{33}\big)+\big(-1+a_{11}\big)\ \big(a_{23}\ a_{32}-\big(-1+a_{22}\big)\ a_{33}\big)\Big)\ x_3\ \xi_3\Big)\Big/\Big(-1+a_{12}\ a_{21}-a_{11}\ \big(-1+a_{22}\big)+a_{22}\Big),\ \frac{\text{<<17>>}+a_{21}\ \text{<<1>>}}{\big(-1+a_{12}\ a_{21}-a_{11}\ \big(-1+a_{22}\big)+a_{22}\big)^2}\Big]$$

```
In[•]:= HL[lhs == Zip_{ξ1}@Zip_{ξ2}@E1 == Zip_{ξ2}@Zip_{ξ1}@E1]
```

```
Out[•]= True
```

```
In[•]:= Short[lhs = Normal[Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]] // Zip_{ξ1,ξ2}, 5]
```

Out[•]//Short=
$$h\ a_{13}\ \xi_3\ f_1[0,0,x_3]+2\ h^2\ a_{11}\ a_{13}\ \xi_3\ f_1[0,0,x_3]+3\ h^3\ a_{11}^2\ a_{13}\ \xi_3\ f_1[0,0,x_3]+2\ h^3\ a_{12}\ a_{13}\ a_{21}\ \xi_3\ f_1[0,0,x_3]+h^2\ a_{13}\ a_{22}\ \xi_3\ f_1[0,0,x_3]+2\ h^3\ a_{11}\ a_{13}\ a_{22}\ \xi_3\ f_1[0,0,x_3]+\text{<<334>>}+\frac{1}{2}\ h^3\ a_{31}^2\ a_{33}\ x_3^3\ \xi_3\ f_1^{(3,0,0)}[0,0,x_3]+\frac{1}{2}\ h^3\ a_{21}\ a_{31}^2\ x_3^2\ f_2^{(3,0,0)}[0,0,x_3]+\frac{1}{6}\ h^3\ a_{31}^3\ x_3^3\ \xi_3\ f_3^{(3,0,0)}[0,0,x_3]+\frac{1}{2}\ h^3\ a_{31}^2\ a_{32}\ x_3^3\ f_1^{(3,1,0)}[0,0,x_3]+\frac{1}{6}\ h^3\ a_{31}^3\ x_3^3\ f_2^{(3,1,0)}[0,0,x_3]+\frac{1}{6}\ h^3\ a_{31}^3\ x_3^3\ f_1^{(4,0,0)}[0,0,x_3]$$

```
In[•]:= rhs = Normal[Zip_{ξ1,ξ2}@Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]];
        HL@Simplify[lhs == rhs]
```

```
Out[•]= True
```

Zip2

```
In[•]:= 𝔼 /: Zip2_{ξs_List}@𝔼[Q_, P_] := Module[{ξ, z, zs, c, ys, ηs, qt, zrule, ξrule},
           zs = Table[ξ*, {ξ, ξs}];
           c = Q /. Alternatives @@ (ξs ⋃ zs) → 0;
           ys = Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ξ, ξs}];
           ηs = Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}];
           qt = Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ξ}Q, {ξ, ξs}, {z, zs}];
           zrule = Thread[zs → qt.(zs + ys)];
           ξrule = Thread[ξs → ξs + ηs.qt];
           Simplify /@ 𝔼[c + ηs.qt.ys, Det[qt] Zip_ξs[P /. (zrule ⋃ ξrule)]]];
```

```
In[•]:= Short[Zip2_{ξ1,ξ2}@E1, 5]
```

Out[•]//Short=
$$\mathbb{E}\Big[\Big(\Big(a_{13}\ \Big(\big(-1+a_{22}\big)\ a_{31}-a_{21}\ a_{32}\Big)+a_{12}\ \big(-a_{23}\ a_{31}+a_{21}\ a_{33}\big)+\big(-1+a_{11}\big)\ \big(a_{23}\ a_{32}-\big(-1+a_{22}\big)\ a_{33}\big)\Big)\ x_3\ \xi_3\Big)\Big/\Big(-1+a_{12}\ a_{21}-a_{11}\ \big(-1+a_{22}\big)+a_{22}\Big),\ \frac{\text{<<17>>}+a_{21}\ \text{<<1>>}}{\big(-1+a_{12}\ a_{21}-a_{11}\ \big(-1+a_{22}\big)+a_{22}\big)^2}\Big]$$

```
In[•]:= HL[Zip_{ξ1,ξ2}@E1 ≡ Zip2_{ξ1,ξ2}@E1]
```

```
Out[•]= True
```

In[•]:=
```
Eh2 = 𝔼[h Sum[a10 i+j x_i ξ_j, {i,1,3},{j,1,3}], Sum[f10 i+j[x1,x2,x3] ξ_i ξ_j, {i,1,3},{j,1,3}]]; E2 = Eh2 /. h → 1
```

Out[•]=
$$\mathbb{E}\Big[a_{11}\,x_1\,\xi_1 + a_{21}\,x_2\,\xi_1 + a_{31}\,x_3\,\xi_1 + a_{12}\,x_1\,\xi_2 + a_{22}\,x_2\,\xi_2 + a_{32}\,x_3\,\xi_2 + a_{13}\,x_1\,\xi_3 + a_{23}\,x_2\,\xi_3 + a_{33}\,x_3\,\xi_3,$$
$$\xi_1^2 f_{11}[x_1,x_2,x_3] + \xi_1\,\xi_2\,f_{12}[x_1,x_2,x_3] + \xi_1\,\xi_3\,f_{13}[x_1,x_2,x_3] +$$
$$\xi_1\,\xi_2\,f_{21}[x_1,x_2,x_3] + \xi_2^2 f_{22}[x_1,x_2,x_3] + \xi_2\,\xi_3\,f_{23}[x_1,x_2,x_3] +$$
$$\xi_1\,\xi_3\,f_{31}[x_1,x_2,x_3] + \xi_2\,\xi_3\,f_{32}[x_1,x_2,x_3] + \xi_3^2 f_{33}[x_1,x_2,x_3]\Big]$$

In[•]:=
```
Timing[lhs = Zip_{ξ1,ξ2}@E2]
```

Out[•]=
$$\Big\{9.90625,\ \mathbb{E}\Big[\Big(\big(a_{13}\,\big((-1+a_{22})\,a_{31} - a_{21}\,a_{32}\big) + a_{12}\,(-a_{23}\,a_{31} + a_{21}\,a_{33}) + (-1+a_{11})\,\big(a_{23}\,a_{32} - (-1+a_{22})\,a_{33}\big)\big)$$
$$x_3\,\xi_3\Big) \Big/ \big(-1 + a_{12}\,a_{21} - a_{11}\,(-1+a_{22}) + a_{22}\big),\ \frac{\langle\!\langle 1\rangle\!\rangle}{\big(1 - a_{12}\,a_{21} + a_{11}\,(\langle\!\langle 1\rangle\!\rangle) - a_{22}\big)^3}\Big]\Big\}$$

large output | show less | show more | show all | set size limit...

In[•]:=
```
Timing[rhs = Zip2_{ξ1,ξ2}@E2]
```

Out[•]=
$$\Big\{5.84375,\ \mathbb{E}\Big[\Big(\big(a_{13}\,\big((-1+a_{22})\,a_{31} - a_{21}\,a_{32}\big) + a_{12}\,(-a_{23}\,a_{31} + a_{21}\,a_{33}) + (-1+a_{11})\,\big(a_{23}\,a_{32} - (-1+a_{22})\,a_{33}\big)\big)$$
$$x_3\,\xi_3\Big) \Big/ \big(-1 + a_{12}\,a_{21} - a_{11}\,(-1+a_{22}) + a_{22}\big),\ \frac{\langle\!\langle 1\rangle\!\rangle}{\big(1 - a_{12}\,a_{21} + a_{11}\,(\langle\!\langle 1\rangle\!\rangle) - a_{22}\big)^3}\Big]\Big\}$$

large output | show less | show more | show all | set size limit...

In[•]:=
```
Timing@HL[lhs ≡ rhs]
```

Out[•]=
```
{0., True}
```

In[•]:=
```
Short[lhs = Zip2_{ξ1,ξ2}@E1, 5]
```

Out[•]//Short=
$$\mathbb{E}\Big[\Big(\big(a_{13}\,\big((-1+a_{22})\,a_{31} - a_{21}\,a_{32}\big) + a_{12}\,(-a_{23}\,a_{31} + a_{21}\,a_{33}) + (-1+a_{11})\,\big(a_{23}\,a_{32} - (-1+a_{22})\,a_{33}\big)\big)$$
$$x_3\,\xi_3\Big) \Big/ \big(-1 + a_{12}\,a_{21} - a_{11}\,(-1+a_{22}) + a_{22}\big),\ \frac{\langle\langle 17\rangle\rangle + a_{21}\ \langle\langle 1\rangle\rangle}{\big(-1 + a_{12}\,a_{21} - a_{11}\,(-1+a_{22}) + a_{22}\big)^2}\Big]$$

In[•]:=
```
HL[lhs == Zip2_{ξ1}@Zip2_{ξ2}@E1 == Zip2_{ξ2}@Zip2_{ξ1}@E1]
```

Out[•]=
```
True
```

In[•]:=
```
Short[lhs = Normal[Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]] // Zip_{ξ1,ξ2}, 5]
```

Out[•]//Short=
$$h\,a_{13}\,\xi_3\,f_1[0,0,x_3] + 2\,h^2\,a_{11}\,a_{13}\,\xi_3\,f_1[0,0,x_3] + 3\,h^3\,a_{11}^2\,a_{13}\,\xi_3\,f_1[0,0,x_3] +$$
$$2\,h^3\,a_{12}\,a_{13}\,a_{21}\,\xi_3\,f_1[0,0,x_3] + h^2\,a_{13}\,a_{22}\,\xi_3\,f_1[0,0,x_3] + 2\,h^3\,a_{11}\,a_{13}\,a_{22}\,\xi_3\,f_1[0,0,x_3] +$$
$$\langle\langle 334\rangle\rangle + \frac{1}{2}\,h^3\,a_{31}^2\,a_{33}\,x_3^3\,\xi_3\,f_1^{(3,0,0)}[0,0,x_3] + \frac{1}{2}\,h^3\,a_{21}\,a_{31}^2\,x_3^2\,f_2^{(3,0,0)}[0,0,x_3] +$$
$$\frac{1}{6}\,h^3\,a_{31}^3\,x_3^3\,\xi_3\,f_3^{(3,0,0)}[0,0,x_3] + \frac{1}{2}\,h^3\,a_{31}^2\,a_{32}\,x_3^3\,f_1^{(3,1,0)}[0,0,x_3] +$$
$$\frac{1}{6}\,h^3\,a_{31}^3\,x_3^3\,f_2^{(3,1,0)}[0,0,x_3] + \frac{1}{6}\,h^3\,a_{31}^3\,x_3^3\,f_1^{(4,0,0)}[0,0,x_3]$$

```
In[•]:= rhs = Normal[Zip2_{ξ1,ξ2}@Eh /. 𝔼[Q_, P_] :> Series[P e^Q, {h, 0, 3}]];
        HL@Simplify[lhs == rhs]
Out[•]= True
```