

```
In[*]:= m0_{i_,j_>k_} := {A_k -> A_i A_j, xi_k -> A_j xi_i + xi_j, eta_k -> A_i eta_j + eta_i, beta_k -> beta_i + beta_j - xi_i eta_j};
sigma_{i_>j_} := {A_j -> A_i, xi_j -> xi_i, eta_j -> eta_i, beta_j -> beta_i}
```

```
In[*]:= d1[f_] := Plus[
  f /. m0_{1,2->0},
  -f /. m0_{2,1->0}
]
```

```
In[*]:= d2[g_] := Plus[
  g /. (sigma_{a->1} U m0_{b,c->2}),
  g /. (sigma_{b->1} U m0_{c,a->2}),
  g /. (sigma_{c->1} U m0_{a,b->2})
]
```

```
In[*]:= Simplify[(f[A_0, beta_0, xi_0, eta_0] /. m0_{1,2->0} /. (sigma_{a->1} U m0_{b,c->2})) -
  (f[A_0, beta_0, xi_0, eta_0] /. m0_{2,1->0} /. (sigma_{c->1} U m0_{a,b->2}))]
```

```
Out[*]= 0
```

```
In[*]:= Expand/@(f[A_0, beta_0, xi_0, eta_0] /. m0_{1,2->0} /. (sigma_{a->1} U m0_{b,c->2}))
```

```
Out[*]= f[A_a A_b A_c, beta_a + beta_b + beta_c - eta_b xi_a - A_b eta_c xi_a - eta_c xi_b, A_b A_c xi_a + A_c xi_b + xi_c, eta_a + A_a eta_b + A_a A_b eta_c]
```

```
In[*]:= Expand/@(f[A_0, beta_0, xi_0, eta_0] /. m0_{2,1->0} /. (sigma_{c->1} U m0_{a,b->2}))
```

```
Out[*]= f[A_a A_b A_c, beta_a + beta_b + beta_c - eta_b xi_a - A_b eta_c xi_a - eta_c xi_b, A_b A_c xi_a + A_c xi_b + xi_c, eta_a + A_a eta_b + A_a A_b eta_c]
```

```
In[*]:= f[A_0, beta_0, xi_0, eta_0] // d1
```

```
Out[*]= f[A_1 A_2, beta_1 + beta_2 - eta_2 xi_1, A_2 xi_1 + xi_2, eta_1 + A_1 eta_2] - f[A_1 A_2, beta_1 + beta_2 - eta_1 xi_2, xi_1 + A_1 xi_2, A_2 eta_1 + eta_2]
```

```
In[*]:= f[A_0, beta_0, xi_0, eta_0] // d1 // d2
```

```
Out[*]= f[A_a A_b A_c, beta_a + beta_b + beta_c - (eta_b + A_b eta_c) xi_a - eta_c xi_b, A_b A_c xi_a + A_c xi_b + xi_c, eta_a + A_a (eta_b + A_b eta_c)] -
  f[A_a A_b A_c, beta_a + beta_b + beta_c - eta_b xi_a - eta_c (A_b xi_a + xi_b), A_c (A_b xi_a + xi_b) + xi_c, eta_a + A_a eta_b + A_a A_b eta_c] +
  f[A_a A_b A_c, beta_a + beta_b + beta_c - (A_c eta_a + eta_c) xi_b - eta_a xi_c, xi_a + A_a A_c xi_b + A_a xi_c, eta_b + A_b (A_c eta_a + eta_c)] +
  f[A_a A_b A_c, beta_a + beta_b + beta_c - eta_b xi_a - (eta_a + A_a eta_b) xi_c, A_b xi_a + xi_b + A_a A_b xi_c, A_c (eta_a + A_a eta_b) + eta_c] -
  f[A_a A_b A_c, beta_a + beta_b + beta_c - eta_c xi_b - eta_a (A_c xi_b + xi_c), xi_a + A_a (A_c xi_b + xi_c), A_b A_c eta_a + eta_b + A_b eta_c] -
  f[A_a A_b A_c, beta_a + beta_b + beta_c - eta_a xi_c - eta_b (xi_a + A_a xi_c), xi_b + A_b (xi_a + A_a xi_c), A_c eta_a + A_a A_c eta_b + eta_c]
```

```
In[*]:= Simplify[f[A_0, beta_0, xi_0, eta_0] // d1 // d2]
```

```
Out[*]= 0
```

```
In[*]:= f[A_0, xi_0, eta_0] // d1
```

```
Out[*]= f[A_1 A_2, A_2 xi_1 + xi_2, eta_1 + A_1 eta_2] - f[A_1 A_2, xi_1 + A_1 xi_2, A_2 eta_1 + eta_2]
```

```
In[*]:= f[A_1, xi_1, eta_1, A_2, xi_2, eta_2] // d2
```

```
Out[*]= f[A_a, xi_a, eta_a, A_b A_c, A_c xi_b + xi_c, eta_b + A_b eta_c] +
  f[A_b, xi_b, eta_b, A_a A_c, xi_a + A_a xi_c, A_c eta_a + eta_c] + f[A_c, xi_c, eta_c, A_a A_b, A_b xi_a + xi_b, eta_a + A_a eta_b]
```

In[*]:= Simplify[$\beta_0 - \frac{\mathcal{A}_0^{-1}}{\mathcal{A}_0^{-1} - 1} \xi_0 \eta_0$ // d1]

Out[*]= 0

In[*]:= Flatten@Table[{ $\xi_i \eta_j$ }, {i, 2}, {j, 2}]

Out[*]= { $\eta_1 \xi_1, \eta_2 \xi_1, \eta_1 \xi_2, \eta_2 \xi_2$ }

In[*]:= Simplify@d1[$\xi_0 \eta_0$]

Out[*]= $(-1 + \mathcal{A}_1 \mathcal{A}_2) (\eta_2 \xi_1 - \eta_1 \xi_2)$

In[*]:= Column@Flatten@Table[{ $\xi_i \eta_j \rightarrow$ Expand@d2[$\xi_i \eta_j$]}, {i, 2}, {j, 2}]

$\eta_1 \xi_1 \rightarrow \eta_a \xi_a + \eta_b \xi_b + \eta_c \xi_c$

$\eta_2 \xi_1 \rightarrow \eta_b \xi_a + \mathcal{A}_b \eta_c \xi_a + \mathcal{A}_c \eta_a \xi_b + \eta_c \xi_b + \eta_a \xi_c + \mathcal{A}_a \eta_b \xi_c$

Out[*]= $\eta_1 \xi_2 \rightarrow \eta_b \xi_a + \mathcal{A}_b \eta_c \xi_a + \mathcal{A}_c \eta_a \xi_b + \eta_c \xi_b + \eta_a \xi_c + \mathcal{A}_a \eta_b \xi_c$

$\eta_2 \xi_2 \rightarrow \mathcal{A}_b \eta_a \xi_a + \mathcal{A}_c \eta_a \xi_a + \mathcal{A}_a \mathcal{A}_b \eta_b \xi_a + \eta_c \xi_a + \eta_a \xi_b +$

$\mathcal{A}_a \eta_b \xi_b + \mathcal{A}_c \eta_b \xi_b + \mathcal{A}_b \mathcal{A}_c \eta_c \xi_b + \mathcal{A}_a \mathcal{A}_c \eta_a \xi_c + \eta_b \xi_c + \mathcal{A}_a \eta_c \xi_c + \mathcal{A}_b \eta_c \xi_c$

In[*]:= BP_n[λ List] := Union@Flatten@Table[
 Product[ξ_i , {i, p1}] Product[η_i , {i, p2}],
 {p1, Tuples[λ , n]}, {p2, Tuples[λ , n]}
]

In[*]:= BP₃[{1, 2}] // Length

Out[*]= 16

In[*]:= BP₂@{0}

Out[*]= { $\eta_0^2 \xi_0^2$ }

In[*]:= md1_n := Module[{dz}, Table[
 dz = d1[z];
 Table[Expand@Coefficient[dz, w], {w, BP_n@{1, 2}}],
 {z, BP_n@{0}}]
]];
 md2_n := Module[{dz}, Table[
 dz = d2[z];
 Table[Expand@Coefficient[dz, w], {w, BP_n@{a, b, c}}],
 {z, BP_n@{1, 2}}]
]]

In[*]:= md2₁ // MatrixForm

Out[*]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & \mathcal{A}_b & \mathcal{A}_c & 0 & 1 & 1 & \mathcal{A}_a & 0 \\ 0 & 1 & \mathcal{A}_b & \mathcal{A}_c & 0 & 1 & 1 & \mathcal{A}_a & 0 \\ \mathcal{A}_b + \mathcal{A}_c & \mathcal{A}_a \mathcal{A}_b & 1 & 1 & \mathcal{A}_a + \mathcal{A}_c & \mathcal{A}_b \mathcal{A}_c & \mathcal{A}_a \mathcal{A}_c & 1 & \mathcal{A}_a + \mathcal{A}_b \end{pmatrix}$$

In[*]:= MatrixRank[md2₁]

Out[*]= 3

In[]:= MatrixRank[md2₂]

Out[]:= 9

In[]:= MatrixRank[md1₂ /. A_ -> 1]

Out[]:= 0

In[]:= MatrixRank[md2₂ /. A_ -> 1]

Out[]:= 7

In[]:= Dimensions[md2₂]

Out[]:= {9, 36}

In[]:= md1₂ // MatrixForm

Out[]//MatrixForm=

$$\left(\begin{matrix} 0 & -2A_2 + 2A_1A_2^2 & -1 + A_1^2A_2^2 & 2A_2 - 2A_1A_2^2 & 0 & -2A_1 + 2A_1^2A_2 & 1 - A_1^2A_2^2 & 2A_1 - 2A_1^2A_2 & 0 \end{matrix} \right)$$

In[]:= Expand[md1₂.md2₂]

$$\begin{aligned} \text{Out[]} = \{ & \left\{ \begin{aligned} & \left\{ 0, -2A_2 + 2A_1A_2^2 + 2A_1A_2C - 2A_1^2A_2A_2C, \right. \\ & \left. 0, -2A_2A_2B + 2A_1A_2^2A_2B + 2A_1A_2^2 - 2A_1^2A_2A_2^2, \right. \\ & \left. 2A_1 - 2A_1^2A_2 - 2A_2B + 2A_1^2A_2^2A_2B + 2A_1A_2A_2^2 - 2A_1^2A_2A_2A_2^2, \right. \\ & \left. 0, 2A_2A_2C - 2A_1A_2^2A_2C - 2A_1A_2^2C + 2A_1^2A_2A_2^2C, \right. \\ & \left. 0, 2A_2 - 2A_1A_2^2 - 2A_1A_2C + 2A_1^2A_2A_2C, \right. \\ & \left. 4A_1A_2B - 4A_1^2A_2A_2B - 4A_1A_2C + 4A_1^2A_2A_2^2C, \right. \\ & \left. 4A_1A_2A_2B - 4A_1^2A_2A_2A_2B - 4A_1A_2B^2C + 4A_1^2A_2A_2B^2C, \right. \\ & \left. -2A_1 + 2A_1^2A_2 + 2A_2B - 2A_1^2A_2^2A_2B - 2A_1A_2^2A_2C + 2A_1^2A_2A_2^2A_2C, \right. \\ & \left. -2A_2A_2C + 2A_1A_2^2A_2C + 2A_1A_2^2C - 2A_1^2A_2A_2^2C, \right. \\ & \left. 0, 2A_1 - 2A_1^2A_2 - 2A_2C + 2A_1^2A_2^2A_2C + 2A_1A_2B^2C - 2A_1^2A_2A_2B^2C, \right. \\ & \left. -2A_2 + 2A_1A_2^2 + 2A_1A_2A - 2A_1^2A_2A_2A, \right. \\ & \left. 0, 2A_2 - 2A_1A_2^2 - 2A_1A_2B + 2A_1^2A_2A_2B, \right. \\ & \left. -4A_1A_2A_2B + 4A_1^2A_2A_2A_2B + 4A_1A_2A_2C - 4A_1^2A_2A_2A_2C, \right. \\ & \left. -2A_1 + 2A_1^2A_2 + 2A_2A - 2A_1^2A_2^2A - 2A_1A_2^2A_2B + 2A_1^2A_2A_2^2A_2B, \right. \\ & \left. 0, 4A_1A_2A - 4A_1^2A_2A_2A - 4A_1A_2B + 4A_1^2A_2A_2B, \right. \\ & \left. 2A_2A_2B - 2A_1A_2^2A_2B - 2A_1A_2^2B + 2A_1^2A_2A_2^2B, \right. \\ & \left. -2A_1 + 2A_1^2A_2 + 2A_2C - 2A_1^2A_2^2A_2C - 2A_1A_2A_2^2C + 2A_1^2A_2A_2A_2^2C, \right. \\ & \left. -4A_1A_2A + 4A_1^2A_2A_2A + 4A_1A_2C - 4A_1^2A_2A_2^2C, \right. \\ & \left. 2A_2A_2A - 2A_1A_2^2A_2A - 2A_1A_2^2A + 2A_1^2A_2A_2^2A, \right. \\ & \left. -4A_1A_2A_2C + 4A_1^2A_2A_2A_2C + 4A_1A_2B^2C - 4A_1^2A_2A_2B^2C, \right. \\ & \left. 0, 2A_2 - 2A_1A_2^2 - 2A_1A_2A + 2A_1^2A_2A_2A, \right. \\ & \left. 0, 2A_1 - 2A_1^2A_2 - 2A_2A + 2A_1^2A_2^2A + 2A_1A_2^2A_2C - 2A_1^2A_2A_2^2A_2C, \right. \\ & \left. 0, \right. \\ & \left. -2A_2 + 2A_1A_2^2 + 2A_1A_2B - 2A_1^2A_2A_2B, \right. \\ & \left. -2A_2A_2A + 2A_1A_2^2A_2A + 2A_1A_2^2A - 2A_1^2A_2A_2^2A, \right. \\ & \left. 0 \right\} \} \end{aligned}$$

In[]:= $\xi_0^2 \eta_0^2$ // d1 // Expand // d2 // Expand

Out[]:= 0

In[]:= Expand[md1₂.md2₂] /. A_ -> 1

$$\text{Out[]} = \{ \left\{ \begin{aligned} & \left\{ 0, \right. \\ & \left. 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 \right\} \} \end{aligned} \right.$$