

Pensieve header: A trace for GDO at $k=1$.

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\2018-08"];
<< ../Projects/SL2Invariant/SL2Invariant.m
```

Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.

Read more at <http://katlas.org/wiki/KnotTheory>.

This is Profile.m of <http://www.drorbn.net/AcademicPensieve/Projects/Profile/>.

This version: June 2018. Original version: July 1994.

```
In[*]:= $k = 1; h = 1; gamma = 1;
```

```
In[*]:= tr_m_ := E_{m} -> {} [ beta_m b_m, (A_m (1 - B_m)) / (A_m - 1) epsilon eta_m / h,
1 + Sum [ (epsilon^k / (A_m - 1)^(k+1)) p_{k,n,1} beta_m^n (epsilon eta_m)^1, {k, $k}, {n, 0, 2 k}, {1, 1, 2 k - n} ] ]_{k}
```

```
In[*]:= tr_m
```

$$Out[*] = E_{\{m\} \rightarrow \{ \}} \left[b_m \beta_m, \frac{(1 - B_m) A_m \eta_m \xi_m}{-1 + A_m}, 1 + \left(\frac{\eta_m \xi_m p_{1,0,1}}{(-1 + A_m)^2} + \frac{\eta_m^2 \xi_m^2 p_{1,0,2}}{(-1 + A_m)^3} + \frac{\beta_m \eta_m \xi_m p_{1,1,1}}{(-1 + A_m)^2} \right) \epsilon + O[\epsilon]^2 \right]$$

```
In[*]:= E_{kk_} := Block [ { $k = k k },
(-1 + A_i A_j) ^ {3 $k} Coefficient [ (dm_{i,j -> m} // tr_m) [[3]] - (dm_{j,i -> m} // tr_m) [[3]], epsilon, $k ] // Together ]
```

```
In[*]:= E_0
```

```
Out[*] = 0
```

In[*]:= **E₁**

$$\text{Out[*]} = -\frac{1}{4 \mathcal{A}_i^2 \mathcal{A}_j^2} (-1 + \mathcal{A}_i \mathcal{A}_j) \left(\begin{aligned} & 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j \xi_i - 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j \xi_i - 4 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_j \xi_i + 4 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_j \xi_i - 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_i \eta_j \xi_i + 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_i \eta_j \xi_i + \\ & 4 \mathcal{A}_i^3 \mathcal{A}_j^3 \beta_i \eta_j \xi_i - 4 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \beta_i \eta_j \xi_i - 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_j \eta_j \xi_i + 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_j \eta_j \xi_i + 4 \mathcal{A}_i^3 \mathcal{A}_j^3 \beta_j \eta_j \xi_i - \\ & 4 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \beta_j \eta_j \xi_i - 2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \eta_j \xi_i^2 + 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \eta_j \xi_i^2 - 2 B_m^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \eta_j \xi_i^2 - 2 \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i \eta_j \xi_i^2 + \\ & 8 B_m \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i \eta_j \xi_i^2 - 6 B_m^2 \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i \eta_j \xi_i^2 - \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i^2 + B_m^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i^2 - 2 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_j^2 \xi_i^2 + 8 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_j^2 \xi_i^2 - \\ & 6 B_m^2 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_j^2 \xi_i^2 - \mathcal{A}_i^4 \mathcal{A}_j^4 \eta_j^2 \xi_i^2 + 4 B_m \mathcal{A}_i^4 \mathcal{A}_j^4 \eta_j^2 \xi_i^2 - 3 B_m^2 \mathcal{A}_i^4 \mathcal{A}_j^4 \eta_j^2 \xi_i^2 - 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \xi_j + 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \xi_j + \\ & 4 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_i \xi_j - 4 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_i \xi_j + 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_j \eta_i \xi_j - 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_j \eta_i \xi_j - 4 \mathcal{A}_i^3 \mathcal{A}_j^3 \beta_i \eta_i \xi_j + \\ & 4 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \beta_i \eta_i \xi_j + 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_j \eta_i \xi_j - 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_j \eta_i \xi_j - 4 \mathcal{A}_i^3 \mathcal{A}_j^3 \beta_j \eta_i \xi_j + 4 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \beta_j \eta_i \xi_j + \\ & 2 \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i^2 \xi_i \xi_j - 4 B_m \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i^2 \xi_i \xi_j + 2 B_m^2 \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i^2 \xi_i \xi_j + 2 \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i^2 \xi_i \xi_j - 8 B_m \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i^2 \xi_i \xi_j + \\ & 6 B_m^2 \mathcal{A}_i^4 \mathcal{A}_j^3 \eta_i^2 \xi_i \xi_j - 2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i \xi_j + 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i \xi_j - 2 B_m^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j^2 \xi_i \xi_j - 2 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_j^2 \xi_i \xi_j + \\ & 8 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_j^2 \xi_i \xi_j - 6 B_m^2 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_j^2 \xi_i \xi_j + \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i^2 \xi_j^2 - B_m^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i^2 \xi_j^2 + 2 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_i^2 \xi_j^2 - 8 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_i^2 \xi_j^2 + \\ & 6 B_m^2 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_i^2 \xi_j^2 + \mathcal{A}_i^4 \mathcal{A}_j^4 \eta_i^2 \xi_j^2 - 4 B_m \mathcal{A}_i^4 \mathcal{A}_j^4 \eta_i^2 \xi_j^2 + 3 B_m^2 \mathcal{A}_i^4 \mathcal{A}_j^4 \eta_i^2 \xi_j^2 + 2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \eta_j \xi_j^2 - 4 B_m \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \eta_j \xi_j^2 + \\ & 2 B_m^2 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \eta_j \xi_j^2 + 2 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_i \eta_j \xi_j^2 - 8 B_m \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_i \eta_j \xi_j^2 + 6 B_m^2 \mathcal{A}_i^3 \mathcal{A}_j^3 \eta_i \eta_j \xi_j^2 - 4 \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i p_{1,0,1} + \\ & 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_j \xi_i p_{1,0,1} + 4 \mathcal{A}_i \mathcal{A}_j \eta_i \xi_j p_{1,0,1} - 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \eta_i \xi_j p_{1,0,1} + 8 \mathcal{A}_i \eta_i \eta_j \xi_i^2 p_{1,0,2} + \\ & 4 \eta_j^2 \xi_i^2 p_{1,0,2} + 4 \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 p_{1,0,2} - 8 \mathcal{A}_i \eta_i^2 \xi_i \xi_j p_{1,0,2} + 8 \mathcal{A}_j \eta_j^2 \xi_i \xi_j p_{1,0,2} - 4 \eta_i^2 \xi_j^2 p_{1,0,2} - \\ & 4 \mathcal{A}_i \mathcal{A}_j \eta_i^2 \xi_j^2 p_{1,0,2} - 8 \mathcal{A}_j \eta_i \eta_j \xi_j^2 p_{1,0,2} - 4 \mathcal{A}_i \mathcal{A}_j \beta_i \eta_j \xi_i p_{1,1,1} + 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_i \eta_j \xi_i p_{1,1,1} - \\ & 4 \mathcal{A}_i \mathcal{A}_j \beta_j \eta_j \xi_i p_{1,1,1} + 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_j \eta_j \xi_i p_{1,1,1} - 4 B_m \mathcal{A}_i^2 \mathcal{A}_j \eta_i \eta_j \xi_i^2 p_{1,1,1} - 4 B_m \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 p_{1,1,1} + \\ & 4 \mathcal{A}_i \mathcal{A}_j \beta_i \eta_i \xi_j p_{1,1,1} - 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_i \eta_i \xi_j p_{1,1,1} + 4 \mathcal{A}_i \mathcal{A}_j \beta_j \eta_i \xi_j p_{1,1,1} - 4 \mathcal{A}_i^2 \mathcal{A}_j^2 \beta_j \eta_i \xi_j p_{1,1,1} + \\ & 4 B_m \mathcal{A}_i^2 \mathcal{A}_j \eta_i^2 \xi_i \xi_j p_{1,1,1} - 4 B_m \mathcal{A}_i \mathcal{A}_j^2 \eta_j^2 \xi_i \xi_j p_{1,1,1} + 4 B_m \mathcal{A}_i \mathcal{A}_j \eta_i^2 \xi_j^2 p_{1,1,1} + 4 B_m \mathcal{A}_i \mathcal{A}_j^2 \eta_i \eta_j \xi_j^2 p_{1,1,1} \end{aligned} \right)$$

In[*]:= **SolveAlways**[**E₁** == 0, {**ξ_i**, **ξ_j**, **η_i**, **η_j**, **β_i**, **β_j** (*, **ℳ_i**, **ℳ_j**, **B_k***) }

$$\text{Out[*]} = \{ \{ p_{1,0,2} \rightarrow 0, \mathcal{A}_i \rightarrow 0 \}, \{ p_{1,0,2} \rightarrow 0, \mathcal{A}_j \rightarrow 0 \}, \{ p_{1,0,1} \rightarrow -(-1 + B_m) \mathcal{A}_i \mathcal{A}_j, \mathcal{A}_i \rightarrow \frac{1}{\mathcal{A}_j} \} \}$$

$$\text{In[*]} = \{ p_{1,0,1} = -(-1 + B_m) \mathcal{A}_m, p_{1,0,2} = \frac{1}{4} (1 - 4 B_m + 3 B_m^2) \mathcal{A}_m^2 (1 + \mathcal{A}_m), p_{1,1,1} = (-1 + B_m) \mathcal{A}_m \}$$

$$\text{Out[*]} = \{ (1 - B_m) \mathcal{A}_m, \frac{1}{4} (1 - 4 B_m + 3 B_m^2) \mathcal{A}_m^2 (1 + \mathcal{A}_m), (-1 + B_m) \mathcal{A}_m \}$$

In[*]:= **E₁**

Out[*]:= 0

In[*]:= **E₂**

$$\text{Out[*]} = -\frac{1}{288 \mathcal{A}_i^4 \mathcal{A}_j^4} (-1 + \mathcal{A}_i \mathcal{A}_j) \left(\begin{aligned} & 144 \mathcal{A}_i^4 \mathcal{A}_j^4 \eta_j \xi_i - 144 B_m \mathcal{A}_i^4 \mathcal{A}_j^4 \eta_j \xi_i - 288 \mathcal{A}_i^5 \mathcal{A}_j^5 \eta_j \xi_i + \dots 3357 \dots + 288 B_m^3 \mathcal{A}_i^5 \mathcal{A}_j^6 \eta_i^3 \eta_j \xi_j^4 p_{2,3,1} \end{aligned} \right)$$

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In[*]:= **SolveAlways**[**E₂** == 0, {**ξ_i**, **ξ_j**, **η_i**, **η_j**, **β_i**, **β_j** (*, **ℳ_i**, **ℳ_j**, **B_k***) }

In[*]:= **Expand** /@ **dm_{i,j→k}**

$$\text{Out[*]} = \mathbb{E}_{\{i,j\} \rightarrow \{k\}} \left[\mathbf{a}_k \alpha_i + \mathbf{a}_k \alpha_j + \mathbf{b}_k \beta_i + \mathbf{b}_k \beta_j, \mathbf{y}_k \eta_i + \frac{\mathbf{y}_k \eta_j}{\mathcal{A}_i} + \frac{\mathbf{x}_k \xi_i}{\mathcal{A}_j} + \eta_j \xi_i - \mathbf{B}_k \eta_j \xi_i + \mathbf{x}_k \xi_j, \mathbf{1} + \frac{1}{4 \mathcal{A}_i \mathcal{A}_j} \left(-4 \mathbf{y}_k \mathcal{A}_j \beta_i \eta_j - 4 \mathbf{x}_k \mathcal{A}_i \beta_j \xi_i + 4 \mathbf{x}_k \mathbf{y}_k \eta_j \xi_i + 4 \mathbf{a}_k \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j \eta_j \xi_i + 2 \mathbf{y}_k \mathcal{A}_j \eta_j^2 \xi_i - 6 \mathbf{B}_k \mathbf{y}_k \mathcal{A}_j \eta_j^2 \xi_i + 2 \mathbf{x}_k \mathcal{A}_i \eta_j \xi_i^2 - 6 \mathbf{B}_k \mathbf{x}_k \mathcal{A}_i \eta_j \xi_i^2 + \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 - 4 \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 + 3 \mathbf{B}_k^2 \mathcal{A}_i \mathcal{A}_j \eta_j^2 \xi_i^2 \right) \in + \mathbf{O}[\epsilon]^2 \right]$$

$$\text{In[*]} = \text{tr}_{\mathbf{r}_k} := \mathbb{E}_{\{k\} \rightarrow \{i\}} \left[\beta_k \mathbf{b}_k, \frac{\mathcal{A}_k (1 - \mathbf{B}_k)}{-1 + \mathcal{A}_k} \xi_k \eta_k / \hbar, \mathbf{1} + \epsilon \left(\mathbf{c} + \mathbf{d} \beta_k + \mathbf{e} \xi_k \eta_k + \mathbf{f} \beta_k^2 + \mathbf{m} \beta_k \xi_k \eta_k + \mathbf{g} \xi_k^2 \eta_k^2 \right) \right];$$

Simplify [**dm_{i,j→k}** // **tr_k**] ≡ **dm_{j,i→k}** // **tr_k**]

$$\text{Out[*]} = \frac{1}{\mathcal{A}_i \mathcal{A}_j (-1 + \mathcal{A}_i \mathcal{A}_j)} \in (\eta_j \xi_i - \eta_i \xi_j) \left(4 \mathbf{g} (\eta_i \xi_j + \eta_j (\xi_i + 2 \mathcal{A}_j \xi_j)) + 4 \mathcal{A}_i (2 \mathbf{g} \eta_i \xi_i + (-6 \mathbf{g} + \mathbf{m} \mathbf{B}_k) \mathcal{A}_j^2 \eta_j \xi_j + \mathcal{A}_j (\mathbf{e} + \mathbf{m} \beta_i + \mathbf{m} \beta_j - 2 \mathbf{g} \eta_j \xi_i + \mathbf{m} \mathbf{B}_k \eta_j \xi_i - 2 \mathbf{g} \eta_i \xi_j + \mathbf{m} \mathbf{B}_k \eta_i \xi_j)) + \mathcal{A}_i^4 \mathcal{A}_j^3 (2 (1 - 4 \mathbf{g} + 2 (-2 + \mathbf{m}) \mathbf{B}_k + 3 \mathbf{B}_k^2) \eta_i \xi_i + \mathcal{A}_j (-4 \mathbf{e} - 4 \mathbf{m} \beta_i - 4 \mathbf{m} \beta_j + \eta_j \xi_i - 4 \mathbf{g} \eta_j \xi_i + \eta_i \xi_j - 4 \mathbf{g} \eta_i \xi_j + 4 \mathbf{B}_k (\mathbf{d} - \mathbf{f} + 2 \mathbf{f} \beta_i + 2 \mathbf{f} \beta_j - \eta_j \xi_i - \eta_i \xi_j) + (3 + 4 \mathbf{f}) \mathbf{B}_k^2 (\eta_j \xi_i + \eta_i \xi_j)) \right) + \mathcal{A}_i^2 \mathcal{A}_j (4 (-6 \mathbf{g} + \mathbf{m} \mathbf{B}_k) \eta_i \xi_i + 2 (1 + 12 \mathbf{g} - 2 (1 + 2 \mathbf{m}) \mathbf{B}_k + \mathbf{B}_k^2) \mathcal{A}_j^2 \eta_j \xi_j + \mathcal{A}_j (-4 - 12 \mathbf{e} + (4 - 12 \mathbf{m}) \beta_i + 4 \beta_j - 12 \mathbf{m} \beta_j + \eta_j \xi_i + \eta_i \xi_j + (-1 + 4 \mathbf{f}) \mathbf{B}_k^2 (\eta_j \xi_i + \eta_i \xi_j) + 4 \mathbf{B}_k (1 + \mathbf{d} - \mathbf{f} + (-1 + 2 \mathbf{f}) \beta_i + (-1 + 2 \mathbf{f}) \beta_j - 2 \mathbf{m} \eta_j \xi_i - 2 \mathbf{m} \eta_i \xi_j)) \right) + 2 \mathcal{A}_i^3 \mathcal{A}_j^2 ((1 + 12 \mathbf{g} - 2 (1 + 2 \mathbf{m}) \mathbf{B}_k + \mathbf{B}_k^2) \eta_i \xi_i + (1 - 4 \mathbf{g} + 2 (-2 + \mathbf{m}) \mathbf{B}_k + 3 \mathbf{B}_k^2) \mathcal{A}_j^2 \eta_j \xi_j + \mathcal{A}_j (2 + 6 \mathbf{e} + (-2 + 6 \mathbf{m}) \beta_i - 2 \beta_j + 6 \mathbf{m} \beta_j + \eta_j \xi_i + 4 \mathbf{g} \eta_j \xi_i + \eta_i \xi_j + 4 \mathbf{g} \eta_i \xi_j - (-3 + 4 \mathbf{f}) \mathbf{B}_k^2 (\eta_j \xi_i + \eta_i \xi_j) + 2 \mathbf{B}_k (-1 - 2 \mathbf{d} + 2 \mathbf{f} + (1 - 4 \mathbf{f}) \beta_i + \beta_j - 4 \mathbf{f} \beta_j - 2 \eta_j \xi_i + \mathbf{m} \eta_j \xi_i - 2 \eta_i \xi_j + \mathbf{m} \eta_i \xi_j)) \right) \right) = 0$$

In[*]:= **eqn = Coefficient** [**dm_{i,j→k}** // **tr_k**] [[3]] - **dm_{j,i→k}** // **tr_k**] [[3]], **ε**] // **Simplify** // **Numerator**

$$\text{Out[*]} = (\eta_j \xi_i - \eta_i \xi_j) \left(4 \mathbf{g} (\eta_i \xi_j + \eta_j (\xi_i + 2 \mathcal{A}_j \xi_j)) + 4 \mathcal{A}_i (2 \mathbf{g} \eta_i \xi_i + (-6 \mathbf{g} + \mathbf{m} \mathbf{B}_k) \mathcal{A}_j^2 \eta_j \xi_j + \mathcal{A}_j (\mathbf{e} + \mathbf{m} \beta_i + \mathbf{m} \beta_j - 2 \mathbf{g} \eta_j \xi_i + \mathbf{m} \mathbf{B}_k \eta_j \xi_i - 2 \mathbf{g} \eta_i \xi_j + \mathbf{m} \mathbf{B}_k \eta_i \xi_j)) + \mathcal{A}_i^4 \mathcal{A}_j^3 (2 (1 - 4 \mathbf{g} + 2 (-2 + \mathbf{m}) \mathbf{B}_k + 3 \mathbf{B}_k^2) \eta_i \xi_i + \mathcal{A}_j (-4 \mathbf{e} - 4 \mathbf{m} \beta_i - 4 \mathbf{m} \beta_j + \eta_j \xi_i - 4 \mathbf{g} \eta_j \xi_i + \eta_i \xi_j - 4 \mathbf{g} \eta_i \xi_j + 4 \mathbf{B}_k (\mathbf{d} - \mathbf{f} + 2 \mathbf{f} \beta_i + 2 \mathbf{f} \beta_j - \eta_j \xi_i - \eta_i \xi_j) + (3 + 4 \mathbf{f}) \mathbf{B}_k^2 (\eta_j \xi_i + \eta_i \xi_j)) \right) + \mathcal{A}_i^2 \mathcal{A}_j (4 (-6 \mathbf{g} + \mathbf{m} \mathbf{B}_k) \eta_i \xi_i + 2 (1 + 12 \mathbf{g} - 2 (1 + 2 \mathbf{m}) \mathbf{B}_k + \mathbf{B}_k^2) \mathcal{A}_j^2 \eta_j \xi_j + \mathcal{A}_j (-4 - 12 \mathbf{e} + (4 - 12 \mathbf{m}) \beta_i + 4 \beta_j - 12 \mathbf{m} \beta_j + \eta_j \xi_i + \eta_i \xi_j + (-1 + 4 \mathbf{f}) \mathbf{B}_k^2 (\eta_j \xi_i + \eta_i \xi_j) + 4 \mathbf{B}_k (1 + \mathbf{d} - \mathbf{f} + (-1 + 2 \mathbf{f}) \beta_i + (-1 + 2 \mathbf{f}) \beta_j - 2 \mathbf{m} \eta_j \xi_i - 2 \mathbf{m} \eta_i \xi_j)) \right) + 2 \mathcal{A}_i^3 \mathcal{A}_j^2 ((1 + 12 \mathbf{g} - 2 (1 + 2 \mathbf{m}) \mathbf{B}_k + \mathbf{B}_k^2) \eta_i \xi_i + (1 - 4 \mathbf{g} + 2 (-2 + \mathbf{m}) \mathbf{B}_k + 3 \mathbf{B}_k^2) \mathcal{A}_j^2 \eta_j \xi_j + \mathcal{A}_j (2 + 6 \mathbf{e} + (-2 + 6 \mathbf{m}) \beta_i - 2 \beta_j + 6 \mathbf{m} \beta_j + \eta_j \xi_i + 4 \mathbf{g} \eta_j \xi_i + \eta_i \xi_j + 4 \mathbf{g} \eta_i \xi_j - (-3 + 4 \mathbf{f}) \mathbf{B}_k^2 (\eta_j \xi_i + \eta_i \xi_j) + 2 \mathbf{B}_k (-1 - 2 \mathbf{d} + 2 \mathbf{f} + (1 - 4 \mathbf{f}) \beta_i + \beta_j - 4 \mathbf{f} \beta_j - 2 \eta_j \xi_i + \mathbf{m} \eta_j \xi_i - 2 \eta_i \xi_j + \mathbf{m} \eta_i \xi_j)) \right) \right)$$

In[*]:= **sol = Last@SolveAlways** [**eqn == 0**, { **ξ_i**, **ξ_j**, **η_i**, **η_j**, **β_i**, **β_j** (*, **ξ_i**, **ξ_j**, **β_k***) }]

$$\text{Out[*]} = \left\{ \begin{aligned} \mathbf{e} &\rightarrow (\mathcal{A}_i \mathcal{A}_j (1 - \mathbf{B}_k - \mathbf{d} \mathbf{B}_k + \mathbf{f} \mathbf{B}_k + \mathbf{d} \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j - \mathbf{f} \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j)) / (-1 + \mathcal{A}_i \mathcal{A}_j)^2, \\ \mathbf{g} &\rightarrow (\mathcal{A}_i^2 \mathcal{A}_j^2 (1 - 4 \mathbf{B}_k + 3 \mathbf{B}_k^2 - 4 \mathbf{f} \mathbf{B}_k^2 + \mathcal{A}_i \mathcal{A}_j - 4 \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j + 3 \mathbf{B}_k^2 \mathcal{A}_i \mathcal{A}_j + 4 \mathbf{f} \mathbf{B}_k^2 \mathcal{A}_i \mathcal{A}_j)) / (4 (-1 + \mathcal{A}_i \mathcal{A}_j)^3), \\ \mathbf{m} &\rightarrow \frac{\mathcal{A}_i \mathcal{A}_j (-1 + \mathbf{B}_k - 2 \mathbf{f} \mathbf{B}_k + 2 \mathbf{f} \mathbf{B}_k \mathcal{A}_i \mathcal{A}_j)}{(-1 + \mathcal{A}_i \mathcal{A}_j)^2} \end{aligned} \right\}$$

In[*]:= Simplify[eqn /. sol]

Out[*]= 0

In[*]:= sol /. {c | d | f -> 0}

$$\text{Out[*]} = \left\{ e \rightarrow \frac{(1 - B_k) \mathcal{A}_i \mathcal{A}_j}{(-1 + \mathcal{A}_i \mathcal{A}_j)^2}, \right. \\ \left. g \rightarrow \left(\mathcal{A}_i^2 \mathcal{A}_j^2 (1 - 4 B_k + 3 B_k^2 + \mathcal{A}_i \mathcal{A}_j - 4 B_k \mathcal{A}_i \mathcal{A}_j + 3 B_k^2 \mathcal{A}_i \mathcal{A}_j) \right) / \left(4 (-1 + \mathcal{A}_i \mathcal{A}_j)^3 \right), m \rightarrow \frac{(-1 + B_k) \mathcal{A}_i \mathcal{A}_j}{(-1 + \mathcal{A}_i \mathcal{A}_j)^2} \right\}$$

In[*]:= $\text{Tr}_{R_-} := \mathbb{E}_{\{k\} \rightarrow \{i\}} \left[\beta_k b_k, \frac{\mathcal{A}_k (1 - B_k)}{-1 + \mathcal{A}_k} \xi_k \eta_k / \hbar, \right. \\ \left. 1 + \epsilon (c + d \beta_k + e \xi_k \eta_k + f \beta_k^2 + m \beta_k \xi_k \eta_k + g \xi_k^2 \eta_k^2) \right] /. \text{sol} /. \{\mathcal{A}_i \rightarrow 1, \mathcal{A}_j \rightarrow \mathcal{A}_k\};$
 Tr_k

$$\text{Out[*]} = \mathbb{E}_{\{k\} \rightarrow \{i\}} \left[b_k \beta_k, \frac{(1 - B_k) \mathcal{A}_k \eta_k \xi_k}{-1 + \mathcal{A}_k}, \right. \\ \left. 1 + \epsilon \left(c + d \beta_k + f \beta_k^2 + \frac{1}{(-1 + \mathcal{A}_k)^2} \mathcal{A}_k (1 - B_k - d B_k + f B_k + d B_k \mathcal{A}_k - f B_k \mathcal{A}_k) \eta_k \xi_k + \right. \right. \\ \left. \frac{\mathcal{A}_k (-1 + B_k - 2 f B_k + 2 f B_k \mathcal{A}_k) \beta_k \eta_k \xi_k}{(-1 + \mathcal{A}_k)^2} + \right. \\ \left. \left. \left(\mathcal{A}_k^2 (1 - 4 B_k + 3 B_k^2 - 4 f B_k^2 + \mathcal{A}_k - 4 B_k \mathcal{A}_k + 3 B_k^2 \mathcal{A}_k + 4 f B_k^2 \mathcal{A}_k) \eta_k^2 \xi_k^2 \right) / \left(4 (-1 + \mathcal{A}_k)^3 \right) \right) \right]$$

In[*]:= Simplify[(dm_{i,j→k} // Tr_k) ≡ (dm_{j,i→k} // Tr_k)]

Out[*]= True

In[*]:= Simplify /@ (Tr_k /. {c | d | f -> 0})

$$\text{Out[*]} = \mathbb{E}_{\{k\} \rightarrow \{i\}} \left[b_k \beta_k, - \frac{(-1 + B_k) \mathcal{A}_k \eta_k \xi_k}{-1 + \mathcal{A}_k}, \right. \\ \left. 1 + \left(\epsilon (-1 + B_k) \mathcal{A}_k \eta_k \xi_k (4 - 4 \beta_k + (-1 + 3 B_k) \mathcal{A}_k^2 \eta_k \xi_k + \mathcal{A}_k (-4 + 4 \beta_k + (-1 + 3 B_k) \eta_k \xi_k)) \right) / \right. \\ \left. \left(4 (-1 + \mathcal{A}_k)^3 \right) \right]$$

In[*]:= $\text{Tr}_{R_-} := \mathbb{E}_{\{k\} \rightarrow \{i\}} \left[b_k \beta_k, - \frac{(-1 + B_k) \mathcal{A}_k \eta_k \xi_k}{-1 + \mathcal{A}_k}, \right. \\ \left. 1 + \left(\epsilon (-1 + B_k) \mathcal{A}_k \eta_k \xi_k (4 - 4 \beta_k + (-1 + 3 B_k) \mathcal{A}_k^2 \eta_k \xi_k + \mathcal{A}_k (-4 + 4 \beta_k + (-1 + 3 B_k) \eta_k \xi_k)) \right) / \right. \\ \left. \left(4 (-1 + \mathcal{A}_k)^3 \right) \right]$

In[*]:= **Simplify** /@ ((**R**_{1,2} **R**_{3,4} **C**₅) // (**dm**_{1,4→1} **dm**_{2,3→2}) // (**dm**_{1,5→1}) // (**TR**₁))

$$\text{Out[*]} = \mathbb{E}_{\{\} \rightarrow \{2\}} \left[a_2 b_1, \frac{(-1 + B_1) x_2 y_2}{-1 + B_2}, \right. \\ \left. \frac{1}{\sqrt{B_1}} - \left(\left((-1 + B_1) x_2 y_2 (-4 a_2 (-1 + B_2) B_2 + (1 - 3 B_2 + B_1 (1 + 5 B_2)) x_2 y_2) \right) \epsilon \right) / \right. \\ \left. \left(4 \left(\sqrt{B_1} (-1 + B_2)^3 \right) \right) + O[\epsilon]^2 \right]$$

In[*]:= **Simplify** /@ ((**R**_{1,2} **R**_{3,4} **C**₅) // (**dm**_{1,4→1} **dm**_{2,3→2}) // (**dm**_{1,5→1}) // (**TR**₁)) /. **B**₁ → **1**

$$\text{Out[*]} = \mathbb{E}_{\{\} \rightarrow \{2\}} \left[a_2 b_1, 0, 1 + O[\epsilon]^2 \right]$$