Pensieve header: A trace for GDO at $k=1$.

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2018-08"];
<< ../Projects/SL2Invariant/SL2Invariant.m
```

```
Loading KnotTheory` version of January 20, 2015, 10:42:19.1122.
Read more at http://katlas.org/wiki/KnotTheory.

This is Profile.m of http://www.drorbn.net/AcademicPensieve/Projects/Profile/.

This version: June 2018. Original version: July 1994.
```

In[•]:=
```
$k = 1; ħ = 1; γ = 1;
```

In[•]:=
```
Expand /@ dm_{i,j→m}
```

Out[•]=
$$\mathbb{E}_{\{i,j\}\to\{m\}}\Big[a_m\alpha_i+a_m\alpha_j+b_m\beta_i+b_m\beta_j,\ y_m\eta_i+\frac{y_m\eta_j}{\mathcal{A}_i}+\frac{x_m\xi_i}{\mathcal{A}_j}+\eta_j\xi_i-B_m\eta_j\xi_i+x_m\xi_j,\ 1+\frac{1}{4\mathcal{A}_i\mathcal{A}_j}\left(-4y_m\mathcal{A}_j\beta_i\eta_j-4x_m\mathcal{A}_i\beta_j\xi_i+4x_my_m\eta_j\xi_i+4a_mB_m\mathcal{A}_i\mathcal{A}_j\eta_j\xi_i+2y_m\mathcal{A}_j\eta_j^2\xi_i-6B_my_m\mathcal{A}_j\eta_j^2\xi_i+2x_m\mathcal{A}_i\eta_j\xi_i^2-6B_mx_m\mathcal{A}_i\eta_j\xi_i^2+\mathcal{A}_i\mathcal{A}_j\eta_j^2\xi_i^2-4B_m\mathcal{A}_i\mathcal{A}_j\eta_j^2\xi_i^2+3B_m^2\mathcal{A}_i\mathcal{A}_j\eta_j^2\xi_i^2\right)\epsilon+O[\epsilon]^2\Big]$$

In[•]:=
```
tr_k_ := 𝔼_{k}→{}[ β_k b_k, (𝒜_k (1 - B_k))/(-1 + 𝒜_k) ξ_k η_k / ħ,
    1 + ϵ (c + d β_k + e ξ_k η_k + f β_k^2 + m β_k ξ_k η_k + g ξ_k^2 η_k^2) ];
Simplify[ (dm_{i,j→k} // tr_k) ≡ (dm_{j,i→k} // tr_k) ]
```

Out[•]=
$$\frac{1}{\mathcal{A}_i\mathcal{A}_j\left(-1+\mathcal{A}_i\mathcal{A}_j\right)}\epsilon\,(\eta_j\xi_i-\eta_i\xi_j)\left(4g\left(\eta_i\xi_j+\eta_j\left(\xi_i+2\mathcal{A}_j\xi_j\right)\right)+4\mathcal{A}_i\left(2g\eta_i\xi_i+(-6g+mB_k)\mathcal{A}_j^2\eta_j\xi_j+\mathcal{A}_j\left(e+m\beta_i+m\beta_j-2g\eta_j\xi_i+mB_k\eta_j\xi_i-2g\eta_i\xi_j+mB_k\eta_i\xi_j\right)\right)+\mathcal{A}_i^4\mathcal{A}_j^3\left(2\left(1-4g+2\left(-2+m\right)B_k+3B_k^2\right)\eta_i\xi_i+\mathcal{A}_j\left(-4e-4m\beta_i-4m\beta_j+\eta_j\xi_i-4g\eta_j\xi_i+\eta_i\xi_j-4g\eta_i\xi_j+4B_k\left(d-f+2f\beta_i+2f\beta_j-\eta_j\xi_i-\eta_i\xi_j\right)+\left(3+4f\right)B_k^2\left(\eta_j\xi_i+\eta_i\xi_j\right)\right)\right)+\mathcal{A}_i^2\mathcal{A}_j\left(4\left(-6g+mB_k\right)\eta_i\xi_i+2\left(1+12g-2\left(1+2m\right)B_k+B_k^2\right)\mathcal{A}_j^2\eta_j\xi_j+\mathcal{A}_j\left(-4-12e+\left(4-12m\right)\beta_i+4\beta_j-12m\beta_j+\eta_j\xi_i+\eta_i\xi_j+\left(-1+4f\right)B_k^2\left(\eta_j\xi_i+\eta_i\xi_j\right)+4B_k\left(1+d-f+\left(-1+2f\right)\beta_i+\left(-1+2f\right)\beta_j-2m\eta_j\xi_i-2m\eta_i\xi_j\right)\right)+2\mathcal{A}_i^3\mathcal{A}_j^2\left(\left(1+12g-2\left(1+2m\right)B_k+B_k^2\right)\eta_i\xi_i+\left(1-4g+2\left(-2+m\right)B_k+3B_k^2\right)\mathcal{A}_j^2\eta_j\xi_j+\mathcal{A}_j\left(2+6e+\left(-2+6m\right)\beta_i-2\beta_j+6m\beta_j+\eta_j\xi_i+4g\eta_j\xi_i+\eta_i\xi_j+4g\eta_i\xi_j-\left(-3+4f\right)B_k^2\left(\eta_j\xi_i+\eta_i\xi_j\right)+2B_k\left(-1-2d+2f+\left(1-4f\right)\beta_i+\beta_j-4f\beta_j-2\eta_j\xi_i+m\eta_j\xi_i-2\eta_i\xi_j+m\eta_i\xi_j\right)\right)\right)\right)==0$$

```
In[•]:= eqn = Coefficient[(dm_{i,j→k} // tr_k)[[3]] - (dm_{j,i→k} // tr_k)[[3]], ϵ] // Simplify // Numerator
```

$$\begin{aligned}\text{Out[•]}= {}& (\eta_j\,\xi_i-\eta_i\,\xi_j)\,\big(4\,g\,(\eta_i\,\xi_j+\eta_j\,(\xi_i+2\,\mathcal{A}_j\,\xi_j))+4\,\mathcal{A}_i\,\big(2\,g\,\eta_i\,\xi_i+\\ &(-6\,g+m\,B_k)\,\mathcal{A}_j^2\,\eta_j\,\xi_j+\mathcal{A}_j\,(e+m\,\beta_i+m\,\beta_j-2\,g\,\eta_j\,\xi_i+m\,B_k\,\eta_j\,\xi_i-2\,g\,\eta_i\,\xi_j+m\,B_k\,\eta_i\,\xi_j)\big)+\\ &\mathcal{A}_i^4\,\mathcal{A}_j^3\,\big(2\,(1-4\,g+2\,(-2+m)\,B_k+3\,B_k^2)\,\eta_i\,\xi_i+\mathcal{A}_j\,(-4\,e-4\,m\,\beta_i-4\,m\,\beta_j+\eta_j\,\xi_i-4\,g\,\eta_j\,\xi_i+\\ &\eta_i\,\xi_j-4\,g\,\eta_i\,\xi_j+4\,B_k\,(d-f+2\,f\,\beta_i+2\,f\,\beta_j-\eta_j\,\xi_i-\eta_i\,\xi_j)+(3+4\,f)\,B_k^2\,(\eta_j\,\xi_i+\eta_i\,\xi_j))\big)+\\ &\mathcal{A}_i^2\,\mathcal{A}_j\,\big(4\,(-6\,g+m\,B_k)\,\eta_i\,\xi_i+2\,(1+12\,g-2\,(1+2\,m)\,B_k+B_k^2)\,\mathcal{A}_j^2\,\eta_j\,\xi_j+\\ &\mathcal{A}_j\,(-4-12\,e+(4-12\,m)\,\beta_i+4\,\beta_j-12\,m\,\beta_j+\eta_j\,\xi_i+\eta_i\,\xi_j+(-1+4\,f)\,B_k^2\,(\eta_j\,\xi_i+\eta_i\,\xi_j)+\\ &4\,B_k\,(1+d-f+(-1+2\,f)\,\beta_i+(-1+2\,f)\,\beta_j-2\,m\,\eta_j\,\xi_i-2\,m\,\eta_i\,\xi_j))\big)+\\ &2\,\mathcal{A}_i^3\,\mathcal{A}_j^2\,\big((1+12\,g-2\,(1+2\,m)\,B_k+B_k^2)\,\eta_i\,\xi_i+(1-4\,g+2\,(-2+m)\,B_k+3\,B_k^2)\,\mathcal{A}_j^2\,\eta_j\,\xi_j+\\ &\mathcal{A}_j\,(2+6\,e+(-2+6\,m)\,\beta_i-2\,\beta_j+6\,m\,\beta_j+\eta_j\,\xi_i+\\ &4\,g\,\eta_j\,\xi_i+\eta_i\,\xi_j+4\,g\,\eta_i\,\xi_j-(-3+4\,f)\,B_k^2\,(\eta_j\,\xi_i+\eta_i\,\xi_j)+\\ &2\,B_k\,(-1-2\,d+2\,f+(1-4\,f)\,\beta_i+\beta_j-4\,f\,\beta_j-2\,\eta_j\,\xi_i+m\,\eta_j\,\xi_i-2\,\eta_i\,\xi_j+m\,\eta_i\,\xi_j))\big)\big)\end{aligned}$$

```
In[•]:= sol = Last@SolveAlways[eqn == 0, {ξ_i, ξ_j, η_i, η_j, β_i, β_j(*,A_i,A_j,B_k*)}]
```

$$\begin{aligned}\text{Out[•]}= {}& \Big\{e\to\big(\mathcal{A}_i\,\mathcal{A}_j\,(1-B_k-d\,B_k+f\,B_k+d\,B_k\,\mathcal{A}_i\,\mathcal{A}_j-f\,B_k\,\mathcal{A}_i\,\mathcal{A}_j)\big)\Big/(-1+\mathcal{A}_i\,\mathcal{A}_j)^2,\\ &g\to\big(\mathcal{A}_i^2\,\mathcal{A}_j^2\,(1-4\,B_k+3\,B_k^2-4\,f\,B_k^2+\mathcal{A}_i\,\mathcal{A}_j-4\,B_k\,\mathcal{A}_i\,\mathcal{A}_j+3\,B_k^2\,\mathcal{A}_i\,\mathcal{A}_j+4\,f\,B_k^2\,\mathcal{A}_i\,\mathcal{A}_j)\big)\Big/\big(4\,(-1+\mathcal{A}_i\,\mathcal{A}_j)^3\big),\\ &m\to\frac{\mathcal{A}_i\,\mathcal{A}_j\,(-1+B_k-2\,f\,B_k+2\,f\,B_k\,\mathcal{A}_i\,\mathcal{A}_j)}{(-1+\mathcal{A}_i\,\mathcal{A}_j)^2}\Big\}\end{aligned}$$

```
In[•]:= Simplify[eqn /. sol]
```

$$\text{Out[•]}= 0$$

```
In[•]:= sol /. {c | d | f → 0}
```

$$\begin{aligned}\text{Out[•]}= {}& \Big\{e\to\frac{(1-B_k)\,\mathcal{A}_i\,\mathcal{A}_j}{(-1+\mathcal{A}_i\,\mathcal{A}_j)^2},\\ &g\to\big(\mathcal{A}_i^2\,\mathcal{A}_j^2\,(1-4\,B_k+3\,B_k^2+\mathcal{A}_i\,\mathcal{A}_j-4\,B_k\,\mathcal{A}_i\,\mathcal{A}_j+3\,B_k^2\,\mathcal{A}_i\,\mathcal{A}_j)\big)\Big/\big(4\,(-1+\mathcal{A}_i\,\mathcal{A}_j)^3\big),\ m\to\frac{(-1+B_k)\,\mathcal{A}_i\,\mathcal{A}_j}{(-1+\mathcal{A}_i\,\mathcal{A}_j)^2}\Big\}\end{aligned}$$

```
In[•]:= Tr_{k_} := 𝔼_{{k}→{}}[β_k b_k, A_k (1 - B_k)/(-1 + A_k) ξ_k η_k / ħ,
          1 + ϵ (c + d β_k + e ξ_k η_k + f β_k^2 + m β_k ξ_k η_k + g ξ_k^2 η_k^2)] /. sol /. {A_i → 1, A_j → A_k};
        Tr_k
```

$$\begin{aligned}\text{Out[•]}= {}& \mathbb{E}_{\{k\}\to\{\}}\Big[b_k\,\beta_k,\ \frac{(1-B_k)\,\mathcal{A}_k\,\eta_k\,\xi_k}{-1+\mathcal{A}_k},\\ &1+\epsilon\,\Big(c+d\,\beta_k+f\,\beta_k^2+\frac{1}{(-1+\mathcal{A}_k)^2}\mathcal{A}_k\,(1-B_k-d\,B_k+f\,B_k+d\,B_k\,\mathcal{A}_k-f\,B_k\,\mathcal{A}_k)\,\eta_k\,\xi_k+\\ &\frac{\mathcal{A}_k\,(-1+B_k-2\,f\,B_k+2\,f\,B_k\,\mathcal{A}_k)\,\beta_k\,\eta_k\,\xi_k}{(-1+\mathcal{A}_k)^2}+\\ &\big(\mathcal{A}_k^2\,(1-4\,B_k+3\,B_k^2-4\,f\,B_k^2+\mathcal{A}_k-4\,B_k\,\mathcal{A}_k+3\,B_k^2\,\mathcal{A}_k+4\,f\,B_k^2\,\mathcal{A}_k)\,\eta_k^2\,\xi_k^2\big)\Big/\big(4\,(-1+\mathcal{A}_k)^3\big)\Big)\Big]\end{aligned}$$

```
In[•]:= Simplify[(dm_{i,j→k} // Tr_k) ≡ (dm_{j,i→k} // Tr_k)]
Out[•]= True

In[•]:= Simplify /@ (Tr_k /. {c | d | f → 0})
```

$$\text{Out}[\bullet]= \mathbb{E}_{\{k\}\to\{\}}\left[b_k\,\beta_k,\ -\frac{\left(-1+B_k\right)\mathcal{A}_k\,\eta_k\,\xi_k}{-1+\mathcal{A}_k},\right.$$
$$\left.1+\left(\epsilon\left(-1+B_k\right)\mathcal{A}_k\,\eta_k\,\xi_k\left(4-4\,\beta_k+\left(-1+3\,B_k\right)\mathcal{A}_k^2\,\eta_k\,\xi_k+\mathcal{A}_k\left(-4+4\,\beta_k+\left(-1+3\,B_k\right)\eta_k\,\xi_k\right)\right)\right)\Big/\left(4\left(-1+\mathcal{A}_k\right)^3\right)\right]$$

```
In[•]:= TR_{k_} := E_{{k}→{}}[b_k β_k, -((-1 + B_k) A_k η_k ξ_k)/(-1 + A_k),
           1 + (ϵ (-1 + B_k) A_k η_k ξ_k (4 - 4 β_k + (-1 + 3 B_k) A_k^2 η_k ξ_k + A_k (-4 + 4 β_k + (-1 + 3 B_k) η_k ξ_k)))/
             (4 (-1 + A_k)^3)]

In[•]:= Simplify /@ ((R_{1,2} R_{3,4} C_5) // (dm_{1,4→1} dm_{2,3→2}) // (dm_{1,5→1}) // (TR_1))
```

$$\text{Out}[\bullet]= \mathbb{E}_{\{\}\to\{2\}}\left[a_2\,b_1,\ \frac{\left(-1+B_1\right)x_2\,y_2}{-1+B_2},\right.$$
$$\left.\frac{1}{\sqrt{B_1}}-\left(\left(\left(-1+B_1\right)x_2\,y_2\left(-4\,a_2\left(-1+B_2\right)B_2+\left(1-3\,B_2+B_1\left(1+5\,B_2\right)\right)x_2\,y_2\right)\right)\epsilon\right)\Big/\left(4\left(\sqrt{B_1}\ \left(-1+B_2\right)^3\right)\right)+O[\epsilon]^2\right]$$

```
In[•]:= Simplify /@ ((R_{1,2} R_{3,4} C_5) // (dm_{1,4→1} dm_{2,3→2}) // (dm_{1,5→1}) // (TR_1)) /. B_1 → 1
```

$$\text{Out}[\bullet]= \mathbb{E}_{\{\}\to\{2\}}\left[a_2\,b_1,\ 0,\ 1+O[\epsilon]^2\right]$$