Alekseev@FI: Double (Poisson) Brackets, 2

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http://www.fields.utoronto.ca/activities/18-19/poisson-summer-school

k - field of char 0

$A = k\langle x_1 \dots x_m\rangle$ ring of nc polys

$= k\langle$ words in letters $x_1 \dots x_m\rangle$

$=$ free assoc. alg $/k$ w/ gens $x_1 \dots x_m$

Sometimes $A \rightsquigarrow k\langle\langle x_1 \dots x_m\rangle\rangle$: formal power series.

Example. $\log e^{x_1} e^{x_2} = x_1 + x_2 + \frac{1}{2}[x_1, x_2] + \dots$

__Def__ A map $\{,\}: A\otimes A \circlearrowleft$ is a double bracket if

$$\{ab, c\} = (1\otimes a)\{b,c\} + \{a,c\}(b\otimes 1)$$

$$\{a, bc\} = (b\otimes 1)\{a,c\} + \{a,b\}(1\otimes c)$$

__Def__ Assuming the above, $\{\}$ is "skew-symmetric" if

$$\{a,b\}^{21} = -\{b,a\}$$

__Remarks__ 1. often notation is $\{\{\ ,\ \}\}$

2. A double bracket is completely determined by its values on generators:

$$\{x_i, x_j\} =: \pi_{ij} \in A\otimes A$$

3. Sweedler notation

$$A\otimes A \ni \alpha = \sum_i \alpha_i' \otimes \alpha_i'' \rightsquigarrow \alpha'\otimes\alpha''$$

W/ this notation,

$$\pi_{ij} = \pi_{ij}' \otimes \pi_{ij}''$$

4. $\hat{A} = k\langle x_1 \dots x_m, s_1, s_2, \dots \rangle$ "spectators"

Any double bracket on A extends to $\hat{A}$ by

$$\{x_i, x_j\}^\wedge = \{x_i, x_j\}$$

$$\{s_i, -\}^\wedge = 0$$

Example $m=1$, $A = k[x]$

$$\{x, x\} = x\otimes 1 - 1 \otimes x$$

Example $m=2$, $A = k\langle x, y\rangle$ "NC symplectic"

$$\{x,x\} = \{y,y\} = 0 \qquad \{x,y\} = \{1\otimes 1\}$$

or just double the first example:

$$\{x,y\} = 0 \qquad \text{"NC KKS"}$$

$$\{x,x\} = x\otimes 1 - 1\otimes x \qquad \{y,y\} = y\otimes 1 - 1\otimes y$$

$|A| = A/[A,A]$ = cyclic words in $x_1 \dots x_m$

vector space

Prop $\{\ \}$ induces the following ops

1. $A\otimes |A| \longrightarrow A \qquad \{a, |b|\} := \{a,b\}'\{a,b\}''$

$= m_1^{12}\{a, b\}$

2. $|A| \otimes A \to A \qquad \{|a|, b\} = \{a,b\}''\{a,b\}'$

3. $|A| \otimes |A| \to |A|$ by $\{|a|, |b|\} = |\{|a|, b\}|$
$= |\{a, |b|\}|$

Remarks Different double bracket may reduce to the same ops on cyclic words; but not if spectators are included!

$\{|s_1 a|, |s_2 b|\}^{\wedge} = |\{s_1 a, s_2 b\}^{\wedge\prime}\{s_1 a, s_2 b\}^{\wedge\prime\prime}|$

$= |s_2\{a, b\}' s_1\{a,b\}''| \leftarrow$ use the s's as "cut marks"

Def $\{\ \}$ is a double poisson bracket if

$\{|s_1 a|, \{|s_2 b|, |s_3 c|\}\} +$ cyc perms of $s_1 a, s_2 b, s_3 c$ $= 0$

Prop Let $\{\ \}$ be a DPB on A, then $(|A|, \{\ \})$ is a Lie algebra.

Proof Put $s_1 = s_2 = s_3$ in def of double Jacobi, get Jacobi.
For skew symmetry use direct computation.

Prop For $A = k\langle p_1 \ldots p_n, q_1 \ldots q_n\rangle$

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{ij}\, 1 \otimes 1$$

is a DPB.

Also $A = k\langle x_1 \ldots x_m\rangle$

$$\{x_i, x_j\} = \delta_{ij}(x_i \otimes 1 - 1 \otimes x_i)$$

"Kontsevich Principle":

"For a property of the non-commutative ring A to have a geometric meaning it should induce std geometric properties on $\mathrm{Rep}(A, M_{N\times N})$".

$$\mathrm{Rep}(A, M_{N\times N}) = (M_{N\times N}(k))^m$$ in our case.

Notation

$$a \in A \longrightarrow \rho(a) \in M_{N\times N} \qquad \underbrace{(\rho(a))_{st}}_{\substack{\text{Think of this as a fctn}\\ \text{of } \rho \text{ parametrized by}\\ a,\, s,\, t.}} \in k$$

matrix entry

Thm (van den Bergh) Let $\{\ \}$ be a DPB on $A = k\langle x_1 \ldots x_m\rangle$. Then $\forall N$

$$\{\rho(a)_{st}, \rho(b)_{uv}\} := \rho(\{a,b\}')_{ut} \cdot \rho(\{a,b\}'')_{sv}$$

as functions of ρ

defines a PB on $(Mat_n)^m$!

Examples 1&2 of today reproduce examples 1&2 of yesterday.