Alekseev@FI: Double (Poisson) Brackets, 1

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http://www.fields.utoronto.ca/activities/18-19/poisson-summer-school

Idea: Define poisson structures in a non-commutative context

[van den Bergh, Crawley-Boevey, Etingof, Ginzburg, Kontsevich]

Consider families of (commutative) poisson mflds with poisson structures and functions of a very special type.

Lecture 1 Poisson brackets on matrix spaces & moment maps.

Lecture 2 Double brackets of free algebras.

Lecture 3 Examples: • necklace double brackets (⇝ quivers)

• Goldman brackets & upgrades ($\pi_1(\Sigma)$, moduli of flat connections)

Lecture 4 • Knizhnik-Zamolodchikov connection

• Hitchin's theorem

• non-commutative integrability

Recall Poisson mflds: $A = C^\infty(M)$

$\{\,\}: A \otimes A \longrightarrow A$ s.t.

derivation: $\{fg, h\} = \ldots$

$$\{f, gh\} = \ldots$$

+ skew symmetry + Jacobi.

Eg 1: $M = \mathbb{R}^2_{x,y}$ $\quad \{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$

$\rightsquigarrow$ bivector $\partial_x \wedge \partial_y = \pi$

<u>Example 2</u> $\quad M = \underset{X}{Mat_{N\times N}} \times \underset{Y}{Mat_{N\times N}}$ (over $\mathbb{K}$ or another char 0 field)

bivector:

$$\pi = \sum \frac{\partial}{\partial X_{ij}} \wedge \frac{\partial}{\partial Y_{ji}}$$

or

$$\{X_{ij}, X_{kl}\} = 0 = \{Y_{ij}, Y_{kl}\}$$

$$\{X_{ij}, Y_{kl}\} = \delta_{il}\delta_{kl}$$

<u>Notation</u> $\quad (\partial_X)_{ij} = \left(\frac{\partial}{\partial X_{ji}}\right)$

so $\pi = \sum (\partial_X)_{ji} \wedge (\partial_Y)_{ij} = Tr(\partial_X \wedge \partial_Y)$

$= -Tr(\partial_Y \wedge \partial_X)$

This formula "hides" N.

<u>Example</u> $\mathfrak{g}$ a f.d. Lie algebra w/ basis

$\{\xi_i\}$, $\quad [\xi_i, \xi_j] = \sum_k C^k_{ij}\, \xi_k$

$\mathfrak{g}^*$ is poisson by

$\xi \in \mathfrak{g} \rightsquigarrow f_\xi(x) = \langle x, \xi \rangle$ a function on $\mathfrak{g}^*$

π_{KKS} is uniquely defined by the property

Kirillov-Kostant-Souriau

$$\{f_\xi, f_\eta\} = f_{[\xi,\eta]}$$

using basis, w/ $x_i = f_{\xi_i}$, get

$$\{x_i, x_j\} = \sum c_{ij}^k x_k$$

For $\mathfrak{g} = \mathfrak{gl}(N)$, $\mathfrak{g}^* \cong \mathrm{Mat}_{N\times N}$, with

$$\langle x, \xi \rangle = \mathrm{Tr}(X \cdot \xi) \qquad \begin{matrix}\cup\!\!\!\!\cup \\ X\end{matrix}$$

$$E_{ij} = \begin{array}{c} j \\ \left(\begin{matrix} & 1 & \end{matrix} \right) i \end{array} \qquad [E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}$$

$$\pi = \mathrm{Tr}(\partial_X \wedge X \partial_X) = -\mathrm{Tr}(X \partial_X \wedge \partial_X)$$

$$= \frac{1}{2} \mathrm{tr}([\partial_X, X] \partial_X)$$

<u>Ex 1</u> Verify this

<u>Setup</u> $M = \underbrace{\mathrm{Mat}_N \times \dots \times \mathrm{Mat}_N}_{m \text{ times}}$

$$\pi = \sum_{i,j,k,L} c_{i,j,k,L} \mathrm{Tr}(\partial_i X_k \partial_j X_L)$$

Notation: $\partial_i = \frac{\partial}{\partial x_i}$ $\quad K = (k_1 \dots k_s)$

$L = (l_1 \dots l_t)$

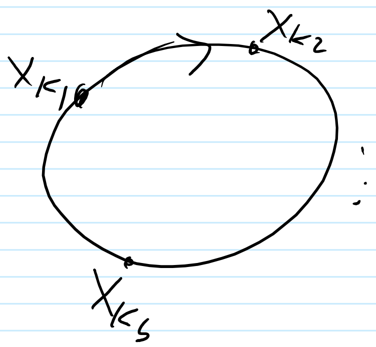
$X_K = X_{k_1} \dots X_{k_s}$

Generalizes both of the last two examples!

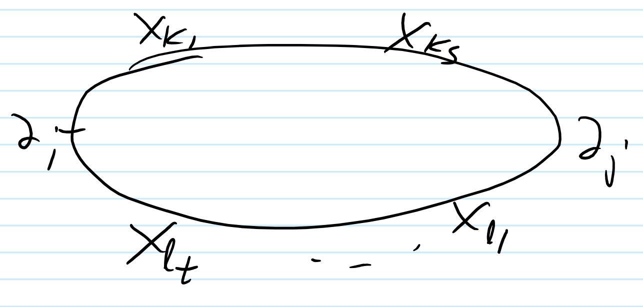
<u>Remark</u> Sums can be infinite if working in an appropriate formal setting.

<u>Functions</u> $\sum_K c_K \operatorname{tr} X_K \qquad K = (k_1 \dots k_s)$

Pictorial presentation $\operatorname{tr} X_K \rightsquigarrow$

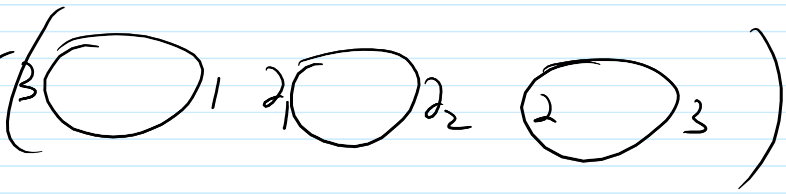


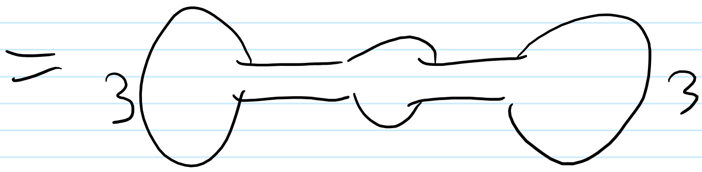
$\operatorname{tr}(\partial_i X_K \partial_j X_L) \rightsquigarrow$



E.g. $\Pi = \operatorname{Tr}(\partial_1 \partial_2) \qquad f = \operatorname{Tr}(X_1 X_3)$

$g = \operatorname{Tr}(X_2 X_3)$

$\{F,g\}_\pi = \mathrm{Tr}\Big($  $\Big)$

$=$  $= \mathrm{Tr}(X_3^2)$

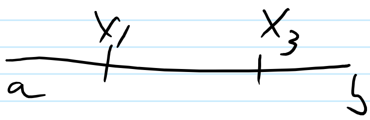
<u>Proposition</u> $g_\pi = k\langle$ functions of the type $\mathrm{tr}(X_k)\rangle$
field

then under $\{\ \}_\pi$, g_π is a Lie algebra "independent" of N.

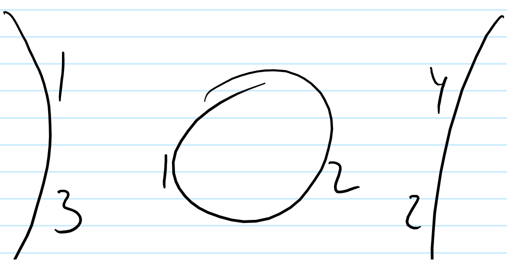
[really, g_π depends on N, but the structural computations do not]

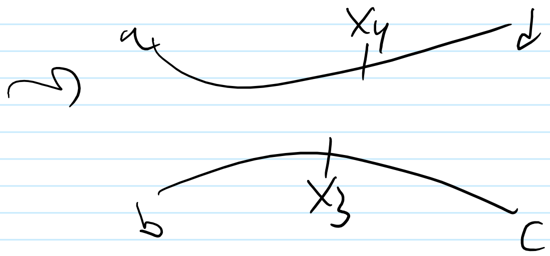
More brackets:

$\{(X_k)_{ab}, (X_L)_{cd}\} =$ [illegible]

E.g. $(X_1X_3)_{ab} \longrightarrow$ 

$\{(X_1X_3)_{ab}, (X_2X_4)_{cd}\}_\pi = \mathrm{tr}(\partial_1 1 \partial_2)$

$\longrightarrow$ 



$= (X_4)_{ad} (X_3)_{cb}$

"double bracket"

<u>Questions</u> 1. N independence?

2. When does π yield a poisson bracket?