Auroux @ FI: an invitation to mirror symmetry

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Mirror Symmetry
Late 80's physics $\longrightarrow$ math.

* Algebraic (or analytic geometry)
  Alg varieties, intersections of alg cycles,
  Functions
  sections of (bundles / sheaves)
* Symplectic Geometry

$$(X,\omega) \underset{loc}{\cong} \left(\mathbb{R}^n_{x_i,y_i},\ \omega=\sum dx_i\wedge dy_i\right)$$

e.g. surface + area form.

Homological Mirror Symmetry (Kontsevitch 1994)

Coherent sheaves on alg. V: Generalize 123:

1. The structure sheaf $\mathcal{O}_V$: regular fncts on any set.
2. Structures sheaves on subvarieties.
3. Line bundles / vector bundles.

Example for grown-ups $D^b Coh(V)$

$$\text{HMS}\updownarrow$$

$$D^\pi \mathcal{F}(X,\omega)$$

$\uparrow$ Fukaya category.

Fukaya Category of (X,ω)

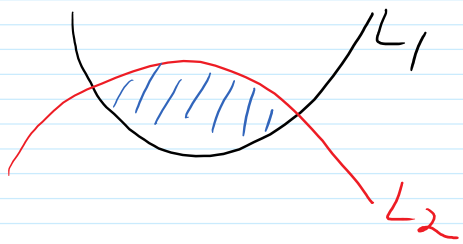
Objects: Lagrangian submanifolds –

$L^n \subset X^{2n}$ s.t. $\omega|_L \equiv 0$

E.g. $\mathbb{R}^n_{x_i} \subset \mathbb{R}^{2n}_{x_i, y_i}$

curve $\subset$ surface

Ask about Lagrangian intersections



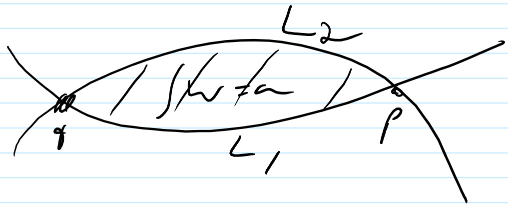
topologically can cancel these intersections. But are they stable under Lagrangian isotopies?

$\rightsquigarrow$ Floer Homology

$$CF^*(L_1, L_2) = \mathbb{K}^{|L_1 \cap L_2|}$$

a Field

w/ differential



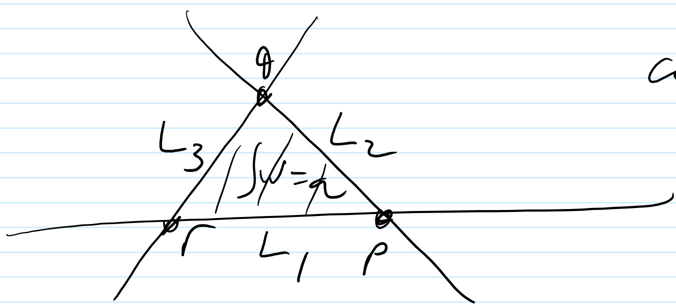
$\partial p = T^a q$

(in good cases $\partial^2 = 0$)

$$HF^*(L_1, L_2) = \frac{\ker \partial}{\operatorname{im} \partial}$$

inv. under isotopies.

Product structure:



contributes $\pm T^a$ to coeff of r in $q \cdot p$

$$CF^*(L_2,L_3)\otimes CF^*(L_1,L_2)\longrightarrow CF^*(L_1,L_3)$$

$\underbrace{\text{Fukaya Cat}}_{A_\infty}$: Obj: Lagrangian
mor $= CF, \partial$
composition of mor's: Floer product.

<u>Example</u> Elliptic curve E / T^2
alg. geom. sympl. geom.

Polishchuk-Zaslow 1998

$E = \mathbb{C}_z/\mathbb{Z}+\tau\mathbb{Z}$ w/ $\operatorname{Im}\tau>0$

The only global holomorphic fncts are consts.

Instead look at sections of line bundles:

$$\mathcal{L} = \mathbb{C}^2_{(z,v)} \Big/ (z,v)\sim \begin{matrix}(z+1,v)\\ (z+\tau, e^{-\pi i\tau-2\pi i z}v)\end{matrix}$$

A section of $\mathcal{L}$ is a holo fnctn w/

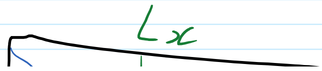
$$s(z+1)=s(z) \quad s(z+\tau)=e^{\cdots}s(z)$$

"The Jacobi θ-function"

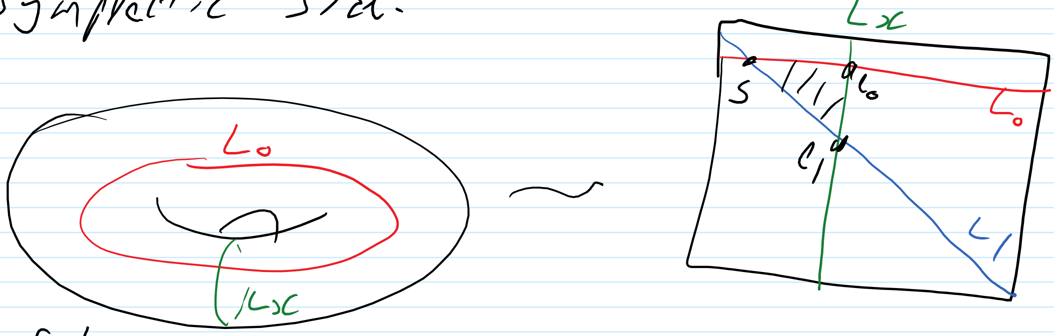
$$s(z)=\theta(\tau,z)=\sum_{n\in\mathbb{Z}}\exp(\pi i n^2\tau+2\pi i n z)$$

The only section!

Symplectic side:



symplectic side:



in Fukaya:

$$L_0 \xrightarrow{s} L_1 \xrightarrow{e_1} L_x$$

$$e_1 \circ s = \boxed{?}\, e_0$$

$$\boxed{?} = \sum_n T^{(x+n)^2/2} = T^{x^2/2} \sum_{n \in \mathbb{Z}} T^{\frac{1}{2}n^2 + nx}$$

under $T = e^{2\pi i \tau}$ $\quad \tau x = z$,

$$= e^{\pi i \tau x^2} \theta(\tau, z)$$

on alg geom side:

$$\mathcal{O} \xrightarrow{s} \mathcal{L} \xrightarrow{ev_z} \mathcal{O}_z$$

$$s(z)\, ev_z$$

skyscraper at z

Goals

1. How does one find a mirror space to something

(Strominger-Yau-Zaslow ~1996)

2. How to prove mirror symmetry not by brute force?

SYZ conjecture: Mirror pairs come out of Lagrangian torus fibrations.