Quantum torus computations, I

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Note QT is bigraded w/ $\deg_x$ & $\deg_y$!

Problem Under $yx=qxy$ find $P_1=P_1(\alpha,\beta,q)$ and $P_2=P_2(\gamma,q)$ s.t. [$P_{1,2}$ are power series in $q-1$]

1. $e^{\alpha x+\beta y}=\mathbb{O}_{xy}(e^{\alpha x+\beta y}P_1)$
2. $e^{\gamma xy}=\mathbb{O}_{xy}(e^{\gamma xy}P_2)$

Aside: $y^n x = q^n x y^n$ so $f(y)x = xf(qy)$

2. At $\gamma=0$, $P_2=1$. Compute ∂_γ & switch sides:

$$\mathbb{O}_{xy}\left(e^{\gamma xy}(xyP_2+\partial_\gamma P_2)\right)=e^{\gamma xy}xy=$$

$$=\mathbb{O}_{xy}(e^{\gamma xy}P_2)xy=\mathbb{O}_{xy}\left(xe^{\gamma qxy}P_2|_{y\to qy}\,y\right)$$

I'm retarded, See below. Better in green!

~~So~~

$$\cancel{e^{\gamma xy}(xyP_2+\partial_\gamma P_2)=xy\,e^{\gamma qxy}P_2|_{y\to qy}}$$

$$\cancel{xyP_2+\partial_\gamma P_2=xy\,e^{\gamma(q-1)xy}P_2|_{y\to qy}}$$

Perhaps better: Find Q s.t.

$$e^{xy}=\mathbb{O}_{xy}(Q)$$

$E(e^x)=xe^x$

Sol'n: Apply E_y & switch sides

$$\mathbb{O}_{xy}(y\partial_y Q)=e^{xy}xy=\mathbb{O}(Q)xy=\mathbb{O}(xyQ(qy))$$

So

$$\partial_y Q = xQ(qy)$$

write $Q=\sum a_k y^k$ & get looking at coeff of y^k

$$(k+1)a_{k+1}=x\,q^k a_k \qquad w/\ a_0=e^x$$

so $$a_{k+1}=\frac{x q^k}{k+1}a_k$$

$$e^{xy}=\mathbb{O}\sum_{n=0}^{\infty}\frac{1}{n!}x^n y^n q^{\binom{n}{2}}$$

$q-1=\epsilon$

$q=1+\epsilon$

$$=\mathbb{O}\sum_{n=0}^{\infty}\frac{1}{n!}x^n y^n (1+\epsilon)^{\frac{n(n-1)}{2}}$$

$$=\mathbb{O}\left(e^{xy}\left(1+\epsilon\ldots\right)\right)$$

$$=\mathbb{O}\sum_{n=0}^{\infty}\frac{x^n y^n}{n!}\sum_{k=0}^{n(n-1)/2}\binom{n(n-1)/2}{k}\epsilon^k$$

$$=\mathbb{O}\sum_{k=0}^{\infty}\epsilon^k\sum_{n\sim\sqrt{k}}^{\infty}\frac{x^n y^n}{n!}\binom{n(n-1)/2}{k}$$

Aside.

$$\sum\frac{x^n}{n!}n^p=E^p e^x$$

$$\sum\frac{x^n}{n!}\binom{n}{k}=\frac{x^k}{k!}e^x$$

$$\binom{\binom{n}{2}}{k}=\frac{1}{2^k k!}n(n-1)(\text{messy})$$

$$n(n-1)-2p=n^2-n-2p$$

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under $q=e^\epsilon$,

$$e^{xy}=\mathbb{O}\sum\frac{1}{n!}x^n y^n e^{\binom{n}{2}\epsilon}$$

$$=\mathbb{O}\sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\sum_{n=0}^{\infty}\frac{x^n y^n}{n!}\binom{n}{2}^k$$

$$=\mathbb{O}\sum_{k=0}^{\infty}\frac{\epsilon^k}{2^k k!}\left(x^2\partial_x^2\right)^k e^x$$